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Essays on fair allocation in the presence of indivisibilities

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Essays on Fair Allocation in the Presence of Indivisibilities

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Contents

Preface

In this collection of essays, we study allocation problems in economic environments. Our primary objective is to find solutions to these problems. We may of course not consider any solution as satisfactory. The purpose of this study is hence to find solutions that are well-behaved with respect to desirable properties. Therefore, we need to determine which properties solutions for such problems should satisfy. Then, based on these properties, we need to justify solutions.

Our motivation is normative, by opposition to positive. We study economic environments not only as they are, but also as they could and should be if a certain number of norms were satisfied. Above all, we consider norms of efficiency and equity. Establishing our work on ethical grounds, our ultimate goal is to clarify the trade-offs faced by social decision makers having to solve such allocation problems.

At any scale of today's world, we are confronted to a large variety of allocation problems. Thus, we cannot represent reality by a unique economic model. Furthermore, fair allocation has been widely studied in classical economic environments, in which the notion of what is to be allocated is extremely abstract.[1] Our secondary objective is hence to determine how the nature of specific problems influences the identification of desirable solutions, in particular whether it hardens or on the contrary, eases this identification.

The domain under study is rich. In this collection of essays, we thus focus on allocation problems in one specific kind of non-classical economic environments, namely those including indivisibilities. Such problems are frequent. Think of divorce cases, inheritance problems, tasks and responsibilities sharing among the members of an institution, organs donations, social housing allocation problems, time blocks sharing among the users of a facility, or allotment problems of public schools or universities among students.

We consider three specific questions: The allocation of indivisible goods when monetary transfers are impossible or not customary, the organization of queues when several individuals simultaneously need a facility for the

[1] For a extended survey on fair allocation in economic environments, see Thomson, W. (2005): "Fair allocation rules," mimeo, University of Rochester, Rochester, NY, USA.

same service, and the matching of two kinds of individuals under terms of contracts. For each of these questions, we use a method that is now standard in the literature.

First, following an axiomatic approach, we search for allocation rules that systematically provide us with lists of bundles for each individual. As just explained, these rules should satisfy axioms embodying basic and appropriate efficiency and equity requirements.

Second, if needed and possible, following an approach focused on the decentralization problem, we search for allocation rules that satisfy axioms embodying incentive compatibility requirements. Indeed, social decision makers may not be able to force agents into some allocation and agents may be able reallocate their bundles among themselves. Furthermore, social decision makers may not know agents individual characteristics, as their preferences or their opportunity sets. Agents being aware of these facts, may behave strategically when receiving their allocated bundles or when announcing their characteristics. Thus, efficiency nor equity may be attained.

Chapter 1

Fair allocation of indivisible goods

Abstract: A finite set of indivisible goods must be allocated without monetary compensation among a finite set of agents. We impose Pareto-efficiency, anonymity, a weak notion of no-envy, a welfare lower bound based on each agent's ranking of the subsets of goods, and a monotonicity property relative to changes in agents' preferences. We prove that in two-agent economies, there is a rule that satisfies these axioms. Moreover, if there are three goods, it is the only rule, together with one of its subcorrespondences, that satisfies each equity axiom and that does not discriminate between goods. Then, we prove that there is a clear gap between these economies and economies with more than two agents.

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1.1 Introduction

We study allocation problems of indivisible goods when monetary compensations are impossible or not customary. Such problems are frequent. Think of managers who must assign tasks or responsibilities among the direction board of their firm, family members who must allocate objects inherited from relatives (as handkerchiefs, chairs, tools...) among themselves, or city councils that must share time blocks between users of a facility. We assume that there are more agents than goods. Agents may receive more than one good. Preferences over subsets are strict and additively separable. Goods are desirable.

In line with the theory of fair allocation, our objective is to identify allocation rules that satisfy *Pareto-efficiency* and axioms embodying basic equity properties. The agents' names should not matter. No agent should prefer another agent's bundle to her own. Each agent should be guaranteed with a minimal welfare level. As the consumption of each good is private, differences in preferences generate welfare surplus. Thus, if an agent's preferences become less similar to other agents', none of the latter should be worse off.

We first prove that in two-agent economies, there is such a rule. Moreover, if there are three goods, it is the only rule, together with one of its subcorrespondences, that is desirable according to each equity property and that does not discriminate between goods.

To come to these central results, we use *anonymity* that embodies the first equity property and we introduce three axioms that embody the other three equity properties respectively. First, *conditional no-envy*, i.e., when possible and not implying inefficiency, a rule should select *envy-free* allocations. Second, the *identical-preferences lower bound*, i.e., each agent should find her bundle at least as desirable as the worst bundle *equal treatment of equals* recommends when the other agents have her preferences. Third, *preference-monotonicity*, i.e., if an agent's preferences change such that she now disagrees with another agent on at least one more pair of subsets in addition to the ones they previously disagreed on, then the latter should not find herself worse off on average.

We identify this rule. For each problem, the *Maximin rule* maximizes the minimal rank of an allocation, where the rank of an agent's bundle is its position in her preferences starting from the least to the most preferred of all subsets. *Maximin divisions* maximize the minimal sum of points given to goods included in agents' bundles with $|K|$ (number of goods), ..., 1 point(s) given to their most, ..., least preferred of all goods respectively (Brams, Edelman and Fishburn, 2000). *Maximin allocations* maximize the minimal rank of the least preferred goods in agents' bundles (Brams and King, 2001). These concepts are based on the well-known idea that we should first care for the least fortunates. Yet as the Maximin rule uses ordinal information on preferences over subsets, it does not assume welfare comparability and it takes each good it allocates into account. Allocations it selects can be obtained as solutions of procedures (Herreiner and Puppe, 2002).

Second, we prove that there is a clear gap between economies with two agents and economies with more than two agents. It is in the latter case that there may be no *Pareto-efficient* and *envy-free* allocation. In the former case, the *identical-preferences lower bound* guarantees each agent with a minimal welfare level that corresponds to her $2^{|K|-1}th$ least preferred subset, whereas in the latter case, it may depend on her preferences. Most importantly, in the latter case, the Maximin rule violates *Pareto-efficiency*. Also, this rule and each of its subcorrespondences violate *conditional no-envy*.

The literature on fair allocation problems of several indivisible goods per agents without monetary compensation has focused on *no-envy*. When agents have strict preferences over goods, there are necessary and sufficient conditions for *envy-free* as well as *Pareto-efficient* and *envy-free* allocations to exist (Brams and Fishburn, 2000, Edelman and Fishburn, 2001, Brams, Edelman and Fishburn, 2000, and Brams and King, 2001). When agents have strict preferences over subsets, there is a procedure that may yield *envy-free* allocations when they exist (Herreiner and Puppe, 2002).

This literature has also studied other equity properties, in particular, solidarity properties relative to changes in the set of goods, the set of agents, or preferences (Klaus and Miyagawa, 2001 and Ehlers and Klaus, 2003). However, these properties may impose selecting allocations that are either not *Pareto-efficient* nor *envy-free* when these exist.

Thus, this literature is mute when it comes to fairly solve problems in which preventing agents from envying others is impossible. It has not studied properties of welfare lower bounds nor monotonicity due to changes in preferences. The former has been much studied in classical economies, as well as in economies with both perfectly divisible and indivisible goods (Steinhaus, 1948, Moulin, 1990, 1991, 1992, and Beviá, 1996). The latter has been studied in economies with public goods (Sprumont, 1993). Yet, their formulation crucially depends on the problem they have been applied to.

We draw three lessons from our study. First, as any large number of indivisible goods will never replace money as a compensating means, the search for *Pareto-efficient* and fair allocations is even harder than when money is available. In particular, since axioms based on *no-envy* are of limited scope, it is all the more important to study properties of welfare lower bounds and monotonicity due to changes in preferences. Second, the Maximin rule is a desirable solution to allocation problems of indivisible goods among two agents when monetary transfers are not available. Third, there is a clear gap between economies with two agents and economies with more than two agents. It might imply an incompatibility between *Pareto-efficiency* and the equity properties in the latter economies.

In Section 1.2, we formally introduce the model. In Section 1.3, we define the axioms we impose on rules. In Section 1.4, we prove that the Maximin rule is a desirable rule. In Section 1.5, we prove that there is a clear gap between economies with two agents and economies with more than two agents. Finally, we give concluding remarks.

1.2 Model

There is a finite set of indivisible goods or *objects* K to allocate among a finite set of agents N with $|K| > |N| \geq 2$. Each agent $i \in N$ has a complete and transitive preference relation R_i over the set of all subsets of objects

$\mathcal{S}$ that satisfies the following assumptions.[1] First, R_i is *strict*, i.e., for each $S, S' \in \mathcal{S}$ with $S \neq S'$, either $S\ P_i\ S'$ or $S'\ P_i\ S$. Second, R_i is *additively separable*, i.e., there is a function $u_i : K \cup \emptyset \to \mathbb{R}$ such that for each $S, S' \in \mathcal{S}$, we have $\sum_{k\in S} u_i(k) \geq \sum_{k\in S'} u_i(k)$ if and only if $S\ R_i\ S'$, with the convention that $u_i(\emptyset) = 0$.[2] Third, each object is *desirable*, i.e., for each $k \in K$, we have $\{k\}\ R_i\ \emptyset$.

Let $\mathcal{R}$ be the set of all admissible preferences. Let $\mathcal{U}$ be the set of all numerical representations of $\mathcal{R}$. Let $\mathcal{R}^N \equiv \times_{i\in N}\mathcal{R}$ be the set of all preference profiles. We do not study effects of changes in the set of objects nor of agents. Thus, for simplicity, an *economy* is a list $R \equiv (R_i)_{i\in N} \in \mathcal{R}^N$.

An *allocation* is a list of bundles $x \equiv (x_i)_{i\in N}$ such that: (i) it is *feasible*, i.e., for each $i \in N$, we have $x_i \subseteq K$ and for each $j \in N\backslash\{i\}$, we have $x_i \cap x_j = \emptyset$, and (ii) there is *no free disposal*, i.e., $\cup_{i\in N} x_i = K$.[3] Let $\mathcal{X}$ be the set of all allocations. An (allocation) *rule* φ is a correspondence that associates with each economy $R \in \mathcal{R}^N$ a non-empty subset of allocations $\varphi(R) \subseteq \mathcal{X}$.

For $S \in \mathcal{S}$, let $(S)_c$ be its *complement*, i.e., $(S)_c \equiv K\backslash S$. For $i \in N$, $R_i \in \mathcal{R}$, and $S \in \mathcal{S}$, let $r(S, R_i)$ be the *rank* of subset S in preferences R_i defined from the least to the best of all subsets. Formally, there is a bijection $r : \mathcal{S} \times \mathcal{R} \leftrightarrow \{1, 2, ..., 2^{|K|}\}$ such that for each $S, S' \in \mathcal{S}$ with $S \neq S'$, we have $r(S, R_i) > r(S', R_i)$ if and only if $S\ P_i\ S'$. For each $S \in \mathcal{S}$, we use the following notational shortcut. For $R_i \in \mathcal{R}$, let $r_i(S) \equiv r(S, R_i)$; for $R'_i \in \mathcal{R}$, let $r_i(S) \equiv r(S, R'_i)$; for $R''_i \in \mathcal{R}$, let $r_i(S) \equiv r(S, R''_i)$; and so on. Such a bijection has the following properties.[4]

Lemma 1 *For each $R \in \mathcal{R}^N$ and each $i \in N$,*

1. $r_i(K) = 2^{|K|}$;
2. $r_i(\emptyset) = 1$;
3. *for each $S \in \mathcal{S}$, we have $r_i(S) + r_i((S)_c) = 2^{|K|} + 1$;*

[1] Let P_i and I_i be the strict preference relation and the indifference relation associated with R_i respectively.

[2] Additive separability implies ***separability***, i.e., for each $S \in \mathcal{S}$ and each $k \in K\backslash S$, we have $S \cup \{k\}\ P_i\ S$ if and only if $\{k\}\ P_i\ \emptyset$, and ***responsiveness***, i.e., for each $S \in \mathcal{S}$ and each $k, k' \in K\backslash S$, we have $S \cup \{k\}\ P_i\ S \cup \{k'\}$ if and only if $\{k\}\ P_i\ \{k'\}$. The converse is not true with more than four objects.

[3] As we are concerned with efficiency and each object is desirable, the assumption of no free disposal has no influence on the results' generality. We introduce it for simplicity.

[4] The proofs are simple and omitted. Point 4 is a corollary of Point 3. It is a well-known property of complements due to the assumptions that preferences are additively separable and objects are desirable. By definition of ranks, it can be formulated in terms of preference relation: $S\ P_i\ S'$ if and only if $(S')_c\ P_i\ (S)_c$.

4. for each $S, S' \in \mathcal{S}$, we have $r_i(S) > r_i(S')$ if and only if $r_i((S)_c) < r_i((S')_c)$.

By Lemma 1.3 and 1.4, for each $i, j \in N$ and each $S, S' \in \mathcal{S}$, we have $r_i(S) > r_j(S')$ if and only if $r_i((S)_c) < r_j((S')_c)$.

We end this section with an example. Three brothers inherit from their father. They have to share five objects: an aquarium, a bed, a coat, a dining table, and an encyclopedia, denoted by a, b, c, d, and e respectively. Their preferences can be represented as follows:

	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
1	2	5	2.3	5.2	1
2	180	175	120	110	100
3	10	12	21.5	10.2	50

There is one row for each son and one column for each object. For instance, the first son assigns a value of 5 to the bed, the second son assigns a value of 230 ($= 120 + 110$) to the subset consisting of the coat and the dining table. As preferences are strict, each agent assigns different values to all subsets. We can order each subset as in Figure 1.1. One's way to rank subsets does not depend on any other's. For instance, $r_1(\{a,b\}) = 13$ and $r_3(\{a,b\}) = 7$, i.e., 1 ranks $\{a,b\}$ as her thirteenth least preferred subset and 3 ranks it seventh. Finally, observe what Lemma 1 implies. For instance, $r_1((\{c,d,e\})_c) = 32 - 20 + 1$ and $r_3((\{c,d,e\})_c) = 32 - 26 + 1$.

1.3 Properties of rules

In this section, we define the axioms we impose on rules. Let φ be a rule.

Efficiency is standard. There should be no allocation that each agent finds at least as desirable as a selected allocation and at least one agent prefers. Formally, $x \in \mathcal{X}$ is *Pareto-efficient for* $R \in \mathcal{R}^N$ if there is no $x' \in \mathcal{X}$ such that for each $i \in N$, we have $x'_i \ R_i \ x_i$ and for at least one $j \in N$, we have $x'_j \ P_j \ x_j$. Let $P(R)$ be the set of all *Pareto-efficient* allocations for $R \in \mathcal{R}^N$.

Pareto-efficiency: *For each $R \in \mathcal{R}^N$, we have $\varphi(R) \subseteq P(R)$.*

Equity is as follows. First, the agents' names should not matter. Thus, if we permute agents' preferences, we should permute the selected bundles accordingly. Formally, let Σ be the set of all permutations on N. For $\sigma \in \Sigma$, $R \in \mathcal{R}^N$, and $x \in \varphi(R)$, let $\sigma(R) \equiv (R_{\sigma(i)})_{i \in N}$ and $\sigma(x) \equiv (x_{\sigma(i)})_{i \in N}$.

Anonymity: *For each $R \in \mathcal{R}^N$, each $x \in \varphi(R)$, and each $\sigma \in \Sigma$, we have $\sigma(x) \in \varphi(\sigma(R))$.*

	R_1	R_2	R_3
32	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$
31	$\{a,b,c,d\}$	$\{a,b,c,e\}$	$\{b,c,d,e\}$
30	$\{b,c,d,e\}$	$\{a,b,c,e\}$	$\{a,b,c,e\}$
29	$\{a,b,d,e\}$	$\{a,b,d,e\}$	$\{a,c,d,e\}$
28	$\{b,c,d\}$	$\{a,c,d,e\}$	$\{b,c,e\}$
27	$\{a,b,d\}$	$\{b,c,d,e\}$	$\{a,b,d,e\}$
26	$\{b,d,e\}$	$\{a,b,c\}$	$\{c,d,e\}$
25	$\{a,c,d,e\}$	$\{a,b,d\}$	$\{a,c,e\}$
24	$\{a,b,c,e\}$	$\{a,b,e\}$	$\{b,d,e\}$
23	$\{b,d\}$	$\{a,c,d\}$	$\{a,b,e\}$
22	$\{a,c,d\}$	$\{b,c,d\}$	$\{c,e\}$
21	$\{a,d,c\}$	$\{a,c,e\}$	$\{a,d,e\}$
20	$\{c,d,e\}$	$\{b,c,e\}$	$\{b,e\}$
19	$\{b,c,e\}$	$\{a,d,e\}$	$\{d,e\}$
18	$\{a,d,e\}$	$\{b,d,e\}$	$\{a,e\}$
17	$\{a,b,e\}$	$\{a,b\}$	$\{a,b,c,d\}$
16	$\{c,d\}$	$\{c,d,e\}$	$\{e\}$
15	$\{b,c\}$	$\{a,c\}$	$\{b,c,d\}$
14	$\{a,d\}$	$\{b,c\}$	$\{a,b,c\}$
13	$\{a,b\}$	$\{a,d\}$	$\{a,c,d\}$
12	$\{d,e\}$	$\{b,d\}$	$\{b,c\}$
11	$\{b,e\}$	$\{a,e\}$	$\{a,b,d\}$
10	$\{a,c,e\}$	$\{b,e\}$	$\{c,d\}$
9	$\{d\}$	$\{c,d\}$	$\{a,c\}$
8	$\{b\}$	$\{c,e\}$	$\{b,d\}$
7	$\{a,c\}$	$\{d,e\}$	$\{a,b\}$
6	$\{c,e\}$	$\{a\}$	$\{c\}$
5	$\{a,e\}$	$\{b\}$	$\{a,d\}$
4	$\{c\}$	$\{c\}$	$\{b\}$
3	$\{a\}$	$\{d\}$	$\{d\}$
2	$\{e\}$	$\{e\}$	$\{a\}$
1	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.1: *Typical three-agent and five-object economy.*

Second, no agent should prefer another agent's bundle to her own. Formally, $x \in \mathcal{X}$ is *envy-free for* $R \in \mathcal{R}^N$ if there is no $i \in N$ such that for $j \in N \backslash \{i\}$, we have $x_j \; P_i \; x_i$. Let $F(R)$ be the set of all *envy-free* allocations for $R \in \mathcal{R}^N$ and $PF(R) \equiv P(R) \cap F(R)$ be the set of all *Pareto-efficient* and *envy-free* allocations for $R \in \mathcal{R}^N$. When e.g. agents have the same most preferred object and prefer it to the set consisting of all the other objects, no allocation is *envy-free*. In two-agent economies, if there are *envy-free* allocations, then at least one is *Pareto-efficient.* In economies with one more object than agents, each *envy-free* allocation is *Pareto-efficient.* However, in any other case, even if there are *envy-free* allocations, none may be *Pareto-efficient.* We prove these results in what follows.

Theorem 1

1. *There may be no* envy-free *allocation.*
2. *Suppose* $|N| = 2$. *If there are* envy-free *allocations, then at least one is* Pareto-efficient.
3. *Suppose* $|K| = |N| + 1$. *Then, each* envy-free *allocation is* Pareto-efficient.
4. *Suppose* $|N| > 2$ *and* $|K| > |N| + 1$. *Even if there are* envy-free *allocations, none may be* Pareto-efficient.

Proof.

Statement 1: *See last paragraph.*

Statement 2: *Let* $N = \{1, 2\}$ *and* $R \in \mathcal{R}^N$ *be such that there is* $x \in F(R)$. *If* $x \notin P(R)$, *then as preferences are strict, there is* $x' \in P(R)$ *such that* **(*i*)** $x_1' \; P_1 \; x_1$ *and* $x_2' \; P_2 \; x_1$. *As* $x \in F(R)$, *we have* $x_1 R_1 x_2$ *and* $x_2 R_2 x_2$. *By Lemma 1,* **(*i*)** *implies* $(x_1)_c \; P_1 \; (x_1')_c$ *and* $(x_2)_c \; P_2 \; (x_2')_c$. *As* $|N| = 2$, *we have* $(x_1)_c = x_2$ *and* $(x_1')_c = x_2'$, *and* $(x_2)_c = x_1$ *and* $(x_2')_c = x_1'$. *Altogether,* $x_1' P_1 x_2'$ *and* $x_2' P_2 x_1'$ *respectively. Thus,* $x' \in PF(R)$.

Statement 3: *Let* $R \in \mathcal{R}^N$ *be such that there is* $x \in F(R)$. *By contradiction, suppose* $x \notin P(R)$. *Then,*

- *There is* $x' \in P(R)$ *such that* **(*i*)** *for each* $i \in N$, *we have* $x_i' \; R_i \; x_i$ *and* **(*ii*)** *for* $j \in N$, *we have* $x_j' \; P_j \; x_j$. *Thus, as preferences are strict,* **(*iii*)** $x_j' \neq x_j$.
- *As* $|K| = |N| + 1$ *and objects are desirable, there is* $j^* \in N$ *such that* $|x_{j^*}| = 2$ *and for each* $i \in N \backslash \{j^*\}$, *we have* $|x_i| = 1$. *Thus, as* **(*i*)** *holds and objects are desirable,* **(*iv*)** $|x_{j^*}'| = 2$ *implying that for each* $i \in N \backslash \{j^*\}$, *we have* $|x_i'| = 1$.
- *As* **(*iii*)** *and* **(*iv*)** *hold and objects are desirable,* **(*v*)** $j \neq j^*$.

By **(*iii*)**, **(*iv*)**, *and* **(*v*)**, *there is* $j^{**} \in N$ *such that* $x_{j^{**}} \supseteq x'_j$. *As objects are desirable,* $x_{j^{**}} \supseteq x'_j$ *implies* $x^{**}_j \; R_j \; x'_j$. *Thus, by* **(*ii*)**, $x^{**}_j \; P_j \; x_j$, *contradicting* $x \in F(R)$.

Statement 4: *Let* $N = \{1,2,3\}$, $K = \{a,b,c,d,e\}$, *and* $R \in \mathcal{R}^N$ *be as follows:*

	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
1	20	12	16	14	7
2	2	4	29	32	16
3	20	8	24	10	1

Let $x = (\{a,e\},\{b,d\},\{c\}) \in \mathcal{X}$ *and* $x' = (\{a,b\},\{d,e\},\{c\}) \in \mathcal{X}$. *It can be shown that* $F(R) = \{x\}$.[5] *Also,* $u_1(x'_1) = 32 > 27 = u_1(x_1)$, $u_2(x'_2) = 48 > 36 = u_2(x_2)$, *and* $u_3(x'_3) = u_3(x_3)$. *Thus,* $x \notin P(R)$. ■

Thus, we define a weaker notion of *no-envy.* When possible and not implying inefficiency, a rule should select *envy-free* allocations. Formally,

Conditional no-envy: *For each* $R \in \mathcal{R}^N$ *such that* $PF(R) \neq \emptyset$, *we have* $\varphi(R) \subseteq F(R)$.

Third, each agent should be guaranteed with a minimal welfare level. For instance, let $N = \{1,2,3\}$, $K = \{a,b,c,d,e\}$, and $R = (R_1,R_2,R_3) \in \mathcal{R}$ be as in Figure 1.1. Consider agent 1. As consumption of each object is private, the more 2 and 3 differ from her, the more welfare can we simultaneously secure for each of the three. Thus, 1 should receive a bundle at least as desirable as any bundle she should receive when 2 and 3 have her preferences. Suppose that were the case. As goods are indivisible and preferences are strict, it is impossible to give these equal agents equal bundles or bundles between which they are indifferent. To treat them as equally as possible, we should allocate objects minimizing disparities in their welfare. We should hence give them bundles $\{b\}$, $\{d\}$, and $\{a,c,e\}$. The worst that can happen to 1 is to receive $\{b\}$. Thus, in R, agent 1 should find her bundle at least as desirable as $\{b\}$. Also, by the same logic, agents 2 and 3 should find their bundle at least as desirable as $\{a\}$ and $\{b,d\}$ respectively.

We require that each agent should find her bundle at least as desirable as the worst bundle *equal treatment of equals* recommends when the other agents have her preferences. Formally, let $\underline{r}(x,R) \equiv \min_{i \in N}\{r_i(x_i)\}$ be the minimal rank of $x = (x_i)_{i \in N}$ for $R = (R_i)_{i \in N} \in \mathcal{R}^N$. For $i \in N$ and $R_i \in \mathcal{R}$, let $R_{(i)} \equiv (R_j = R_i)_{j \in N}$ be *i's identical-agent economy for* R_i and let $x^{PE}(R_{(i)})$ be an allocation that *equal treatment of equals* recommends this

[5] Proof available to the reader upon request to the author.

economy, i.e., there is no $x \in \mathcal{X}$ such that $\underline{r}(x, R_{(i)}) > \underline{r}(x^{PE}(R_{(i)}), R_{(i)})$.[6]

Identical-preferences lower bound: *For each $R \in \mathcal{R}^N$, each $x \in \varphi(R)$, and each $i \in N$, we have $r_i(x_i) \geq \underline{r}(x^{PE}(R_{(i)}), R_{(i)})$.*

This axiom guarantees each agent with a minimal welfare level that does not depend on others' preferences. As it sets this level in terms of welfare associated to a subset of objects, it applies to economies without compensating means. At the same time, as it measures this level in terms of ranks, it only requires ordinal information on preferences.

Furthermore, as it requires a minimal welfare level, it is compatible with *Pareto-efficiency*. Let $B(R)$ be the set of all allocations meeting the *identical-preferences lower bound* for $R \in \mathcal{R}^N$. Let $PB(R) \equiv P(R) \cap B(R)$ be the set of all *Pareto-efficient* allocations meeting the *identical-preferences lower bound* for $R \in \mathcal{R}^N$. In the economy of Figure 1.1, if $x = (\{d\}, \{b, c\}, \{a, e\}) \in \mathcal{X}$, then $\underline{r}(x^{PE}(R_{(1)}), R_{(1)}) = 8 < 9 = r_1(x_1)$, $\underline{r}(x^{PE}(R_{(2)}), R_{(2)}) = 6 < 14 = r_2(x_2)$, and $\underline{r}(x^{PE}(R_{(3)}), R_{(3)}) = 8 < 12 = r_3(x_3)$. Thus, $x \in B(R) \cap PB(R)$.

Finally, the minimal welfare it requires depends on the number of objects, the number of agents, and the agent's own preferences. Thus, for given sets of objects and agents, it need not be equal across agents. In the economy of Figure 1.1, agent 1, 2, and 3's level corresponds to her eighth, sixth, and eighth least preferred subset respectively. However, in what follows, we prove that in two-agent economies and in economies with one more object than agents, this level does not depend on the agent's own preferences and corresponds to her $2^{|K|-1}th$ and $4th$ least preferred subset respectively.[7]

[6] Agent i's minimal welfare level corresponds to the welfare she associates to the worst bundle that *equal treatment of equals* recommends in her identical-agent economy. Thus, however we treat equal agents as equally as possible, we only need to care for minimal ranks of allocations.

[7] This level depends on the agent's own preferences if and only if $|K| - 1 > |N| > 2$. It is at its lowest when the agent's preferences are *quantity-monotonic*. Formally, for $i \in N$ and $R_i \in \mathcal{R}$, let R_i be such that for each $S, S' \in \mathcal{S}$ with $|S| \neq |S'|$, we have $S \; P_i \; S'$ if and only if $|S| > |S'|$. Let $[|K|/|N|]$ be the integer part of $|K|/|N|]$. The worst bundle *equal treatment of equals* recommends in i's identical-agent economy for R_i contains at least $[|K|/|N|] - 1$ objects. Thus, its rank is greater than the rank of all subsets including less than $[|K|/|N|] - 1$ objects. Also, if K and N are such that only one agent can be allocated $[|K|/|N|]$ objects, its rank corresponds to her $|N| - 2th$ most preferred subset including $[|K|/|N|] - 1$ objects, ..., if K and N are such that each agent but one can be allocated $[|K|/|N|]$ objects, its rank corresponds to her first most preferred subset including $[|K|/|N|] - 1$ objects, if K and N are such that each agent can be allocated $[|K|/|N|]$ objects, its rank corresponds to her $|N| - 1th$ most preferred subset including $|K|/|N|$ objects. Thus, for each $R \in \mathcal{R}^N$ and each $i \in N$, if $|K| - 1 > |N| > 2$, then $\underline{r}(x^{PE}(R_{(i)}), R_{(i)}) \geq 1 + |K| + |K|!/(|K|-2)!2! + ... + |K|!/(|K| - ([|K|/|N|] - 1))!([|K|/|N|] - 1)! + (|K| - [|K|/|N|](|N|([|K|/|N|] + 1) - |K| - 1))!/(|K| - [|K|/|N|](|N|([|K|/|N|] + 1) - |K| - 1) - [|K|/|N|])![|K|/|N|]!$.

Theorem 2 *For each $R \in \mathcal{R}^N$ and each $i \in N$,*

1. *if $|N| = 2$, then $\underline{r}(x^{PE}(R_{(i)}), R_{(i)}) = 2^{|K|-1}$;*

2. *if $|N| = |K| - 1$, then $\underline{r}(x^{PE}(R_{(i)}), R_{(i)}) = 4$.*

Proof.

Statement 1: *Let $N = \{1, 2\}$ and $(R_1, R_2) \in \mathcal{R}^N$. Without loss of generality, consider 1. Let $R'_2 \in \mathcal{R}$ be such that $R'_2 = R_1$. Then,*

- *Let $x = (x_1, x_2) \in \mathcal{X}$ be such that $r_1(x_1) = 2^{|K|-1}$. By Lemma 1, $r_1((x_1)_c) = 2^{|K|-1}+1$. As $|N| = 2$, we have $(x_1)_c = x_2$. By assumption, $R'_2 = R_1$. Thus, **(i)** there is $x \in \mathcal{X}$ such that $r_1(x_1) = 2^{|K|-1}$ and $r'_2(x_2) = 2^{|K|-1} + 1$.*

- *Let $x' = (x'_1, x'_2) \in \mathcal{X}$. Suppose $r_1(x'_1) \geq 2^{|K|-1} + 1$. By Lemma 1, $r_1((x'_1)_c) \leq 2^{|K|-1}$. As $|N| = 2$, we have $(x'_1)_c = x'_2$. By assumption, $R'_2 = R_1$. Thus, **(ii)** if $x' = (x'_1, x'_2) \in \mathcal{X}$ is such that $r_1(x'_1) \geq 2^{|K|-1} + 1$, then $r'_2(x'_2) \leq 2^{|K|-1}$.*

*By **(i)** and **(ii)**, $\underline{r}(x^{PE}(R_{(1)}), R_{(1)}) = 2^{|K|-1}$.*

Statement 2: *Let $|N| = |K| - 1$, $R \in \mathcal{R}^N$, and $i \in N$. Let $k, k', k'' \in K$ be i's first, second, and third least preferred object respectively. For each $j \in N \backslash \{i\}$, let $R'_j \in \mathcal{R}$ be such that $R'_j = R_i$. Then,*

- *Let $x_i \subset K$ be such that $r_i(x_i) = 4$. Clearly, either $x_i = \{k, k'\}$ or $x_i = \{k''\}$. If $x_i = \{k, k'\}$, then let $(x_j)_{j \in \backslash \{i\}}$ be such that: for each $j \in N \backslash \{i\}$, we have $x_j \neq \emptyset$, $x_j \subseteq K \backslash \{k, k'\}$, and for each $j' \in N \backslash \{i, j\}$, we have $x_j \cap x_{j'} = \emptyset$; and $\cup_{j \in N \backslash \{i\}} x_j = K \backslash \{k, k'\}$. Otherwise, if $x_i = \{k''\}$, then let $(x_j)_{j \in \backslash \{i\}}$ be such that: there is $i' \in N \backslash \{i\}$ with $x_{i'} = \{k, k'\}$; for each $j \in N \backslash \{i, i'\}$, we have $x_j \neq \emptyset$, $x_j \subseteq K \backslash \{k, k', k''\}$, and for each $j' \in N \backslash \{i, i', j\}$, we have $x_j \cap x_{j'} = \emptyset$; and $\cup_{j \in N \backslash \{i, i'\}} x_j = K \backslash \{k, k', k''\}$. For each $j \in N \backslash \{i\}$, as $|N| = |K| - 1$ and by assumption, $R'_j = R_i$, we have $r'_j(x_j) > 4$. Thus, **(i)** there is $(x_i, (x_j)_{j \in \backslash \{i\}}) \in \mathcal{X}$ such that $r_i(x_i) = 4$ and for each $j \in N \backslash \{i\}$, we have $r'_j(x_j) > 4$.*

- *Let $x' = (x'_j)_{j \in N} \in \mathcal{X}$. Suppose $r_i(x'_i) > 4$. As $|N| = |K| - 1$, there is $i' \in \backslash \{i\}$ such that $x_{i'} \subseteq \{k, k'\}$. As by assumption, $R'_{i'} = R_i$, we have $r_{i'}(x_{i'}) \leq 4$. Thus, **(ii)** if $x' = (x'_j)_{j \in N} \in \mathcal{X}$ such that $r_i(x'_i) > 4$, then there is $i' \in \backslash \{i\}$ such that $r_{i'}(x_{i'}) \leq 4$.*

R_1	R_2	R_1	R'_2	R_1	R''_2	R_1	R'''_2
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{b,c\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,c\}$	$\{b,c\}$
$\{b,c\}$	$\{b\}$	$\{b,c\}$	$\{a,b\}$	$\{b,c\}$	$\{b\}$	$\{b,c\}$	$\{a,c\}$
$\{a\}$	$\{a,c\}$	$\{a\}$	$\{c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$	$\{b\}$
$\{b\}$	$\{c\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a\}$
$\{c\}$	$\{a\}$	$\{c\}$	$\{b\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.2: *Preference-monotonicity.*

By ***(i)*** *and* ***(ii)***, $\underline{r}(x^{PE}(R_{(i)}), R_{(i)}) = 4$. ∎

Fourth, as consumption of each object is private, differences in preferences generate welfare surplus. Thus, if an agent's preferences become less similar to other agents', none of the latter should be worse off and if an agent's preferences become more similar to other agents', none of the latter should be better off.

However, consider the economies of Figure 1.2, where $N = \{1,2\}$, $K = \{a,b,c\}$, and $R = (R_1,R_2), R' = (R_1,R'_2), R'' = (R_1,R''_2), R''' = (R_1,R'''_2) \in \mathcal{R}^N$ respectively. Let φ be a rule such that $\varphi(R) = \{(\{a,b,c\},\emptyset),(\emptyset,\{a,d,c\})\}$, $\varphi(R') = \{(\{a,b\},\{c\}),(\{a\},\{b,c\})\}$, $\varphi(R'') = \{(\{a,b,c\},\emptyset),(\emptyset,\{a,d,c\})\}$, and $\varphi(R''') = \{(\{a,c\},\{b\}),(\{a\},\{b,c\})\}$. First, we cannot determine which of R_2 and R'_2 is more similar to R_1. Indeed, as agents 1 and 2 disagree on pairs of subsets in R' they agree on in R (e.g. $\{b\}$ and $\{c\}$), we should say that R'_2 is not more similar to R_1 than R_2. Also, as agents 1 and 2 disagree on pairs of subsets in R they agree on in R' (e.g. $\{b\}$ and $\{a,c\}$), we should say that R_2 is not more similar to R_1 than R'_2. As opposite movements take place when 2's preferences change from R'_2 to R_2 and from R_2 to R'_2, these preferences are incomparable with respect to R_1.

On the contrary, as in addition to the ones agents 1 and 2 agree on in R, they agree on two more pairs of subsets in R'' (i.e. $\{a\}$ and $\{c\}$, $\{a,b\}$ and $\{c,b\}$), we should say that R''_2 is more similar to R_1 than R_2. As in addition to the ones agents 1 and 2 agree on in R'', they agree on one more pair of subsets in R''' (i.e. $\{b\}$ and $\{a,c\}$), we should say that R'''_2 is more similar to R_1 than R''_2. Thus, we should also say that R'''_2 is more similar to R_1 than R_2. Formally, for $i,j \in N$ and $R_i, R_j, R'_j \in \mathcal{R}$, we say that *$R'_j$ is closer to R_i than R_j* if in addition to the ones i and j agree on in (R_i,R_j), they agree on at least one more pair of subsets in (R_i,R'_j), i.e., $\{(S,S') \in \mathcal{S}^2 : S\ R_i\ S'$ and $S\ R'_j\ S'\} \supset \{(S,S') \in \mathcal{S}^2 : S\ R_i\ S'$ and $S\ R_j\ S'\}$. For each $i,j \in N$ and each $R_i \in \mathcal{R}$, agent j's preferences *become closer* to R_i if they change from $R_j \in \mathcal{R}$ to $R'_j \in \mathcal{R}$ such that *R'_j is closer to R_i than R_j*. Thus, R''_2 and R'''_2 are closer to R_1 than R_2. Also, if agent 2's preferences change from R''_2 to R'''_2, they become closer to R_1.

Second, as φ is multi-valued, we cannot determine the overall "sign"of welfare variations without any refinements. Indeed, if 2's preferences change from R'''_2 to R'_2 (or R_2), 1 is worse off with respect to $(\{a,b,c\},\emptyset)$ and better off with respect to $(\emptyset,\{a,d,c\})$.

We require that if an agent's preferences become closer to other agents', none of the latter should be better off on average, i.e., when she puts equal weight the selected allocations. Formally, for $i \in N$, let $R_{-i} \equiv (R_j)_{j \in N \setminus \{i\}}$.

Preference-monotonicity: *For each $R \in \mathcal{R}^N$, each $i,j \in N$ with $i \neq j$, each $R'_j \in \mathcal{R}$ such that R'_j is closer to R_i than R_j, and each $u_i \in \mathcal{U}$ representing R_i, we have*
$\frac{1}{|\varphi(R_j,R_{-j})|}\sum_{x\in\varphi(R_j,R_{-j})} u_i(x_i) \geq \frac{1}{|\varphi(R'_j,R_{-j})|}\sum_{x'\in\varphi(R'_j,R_{-j})} u_i(x'_i)$.

Let us come back on the definition of closer. Let Δ be the $|K|-1$ dimensional simplex, i.e., $\Delta \equiv \{v \in \mathbb{R}^K_+ : \sum_{k\in K} v_k = 1\}$. Identifying each vertex as an object, each point in Δ gives a ranking of the subsets of objects according to how it is located with respect to each vertex. Thus, each agent's preferences can be represented as a point in Δ.[8] Furthermore, for each $S, S' \in \mathcal{S}$ with $S \cap S' = \emptyset$, let $H(S,S')$ be the separating hyperplane between S and S', i.e., $H(S,S') \equiv \{v \in \mathbb{R}^K_+ : \sum_{k\in S} v_k = \sum_{k \in S'} v_k\}$. The separating hyperplanes define polyhedrons with each point in their interior representing the same ranking.

For instance, any three-object economy can be depicted in an equilateral triangle as in Figure 1.3, where $N=\{1,2\}$ and $K=\{a,b,c\}$. As the point u_1 is located such that $u_1(a) > u_1(b)$, $u_1(a) > u_1(c)$, $u_1(b) > u_1(c)$, $u_1(a) < u_1(b)+u_1(c)$, $u_1(b) < u_1(a)+u_1(c)$, and $u_1(c) < u_1(a)+u_1(b)$, it represents R_1 of Figure 1.2. As each point in the interior of the smallest triangle including u_1 is located as u_1 with respect to each separating hyperplane, it also represents R_1. Each point in the interior of the smallest triangle including u_2, u'_2, u''_2, and u'''_2 represents R_2, R'_2, R''_2, and R'''_2 of Figure 1.2 respectively.

This geometric representation allows us to reinterpret the definition of closer and gives us a complete understanding of what happens in terms of ranks when an agent's preferences become closer to another agent's. For

[8]As preferences are strict, the converse is not true, i.e., each point in Δ does not represent admissible preferences. Indeed, each point in a separating hyperplane represent preferences that admit indifferences and thus that are not in $\mathcal{R}$.

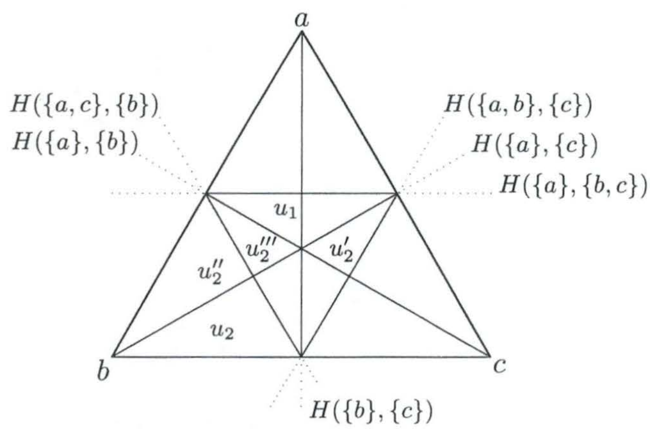

Figure 1.3: *Geometric representation of the definition of closer.*

$i, j \in N$ and $R_i, R_j, R'_j \in \mathcal{R}$, we have R'_j closer to R_i than R_j if and only if the set of hyperplanes crossed from R'_j to R_i is properly included in the one from R_j to R_i. Thus, a change from R_j to R'_j implies a sequence of consecutively crossed hyperplanes. That is, it implies a sequence of consecutive switches between adjacent bundles. Thus, there is a sequence of preferences from R_j to R'_j with each closer to R_i than the preceding one because of switches between adjacent bundles. For instance, in Figure 1.2, as R_2''' is closer to R_1 than R_2, there is (R_2, R_2'', R_2'''). Formally,

Lemma 2 *For each $R_i, R_j, R'_j \in \mathcal{R}$, if R'_j is closer to R_i than R_j, then there is a sequence $(R_j^t)^{t\in\{1,\dots,T\}}$ with $R_j^1 = R_j$ and $R_j^T = R'_j$ such that for each $t \in \{1, \dots, T-1\}$:*

1. *R_j^{t+1} closer to R_i than R_j^t;*

2. *for each $S, S' \in \mathcal{S}$, we have $r_j^t(S) > r_j^t(S')$ and $r_j^{t+1}(S) < r_j^{t+1}(S')$ if and only if $r_j^t(S) = r_j^{t+1}(S') = r_j^t(S') + 1$ and $r_j^{t+1}(S) = r_j^t(S') = r_j^{t+1}(S') - 1$.*

1.4 Two-agent economies

In this section, we prove that in two-agent economies, there is a rule that satisfies *Pareto-efficiency*, *anonymity*, *conditional no-envy*, the *identical-preferences lower bound*, and *preference-monotonicity*. Moreover, if there are three objects, it is the only rule, together with one of its subcorrespondences, that satisfies each equity axiom and that does not discriminate between objects. Therefore, suppose $|N| = 2$.

$\mathcal{X}$	$r_1(.)$	$r_2(.)$
$(\emptyset, \{a,b,c\})$	$\underline{1}$	8
$(\{a\}, \{b,c\})$	$\boxed{\underline{4}}$	6
$(\{b\}, \{a,c\})$	$\underline{3}$	7
$(\{c\}, \{a,b\})$	$\underline{2}$	5
$(\{a,b\}, \{c\})$	7	$\boxed{\underline{4}}$
$(\{a,c\}, \{b\})$	6	$\underline{2}$
$(\{b,c\}, \{a\})$	5	$\underline{3}$
$(\{a,b,c\}, \emptyset)$	8	$\underline{1}$

Figure 1.4: *The Maximin rule and the Maximin-minimax rule.*

This rule embodies the well-known idea according to which we should first care for the least fortunates, using only ordinal information on preferences. For each preference profile, it selects the allocations with the maximal minimal rank. For $R' = (R_1, R_2') \in \mathcal{R}^N$ of Figure 1.2, this rule selects $(\{a\}, \{b,c\})$ and $(\{a,b\}, \{c\})$. Indeed, in Figure 1.4 (where there is a row for each allocation with the associated ranks of agents 1 and 2 in the second and third column respectively), we have first underlined the minimal rank of each allocation and then going across these, we have surrounded the maximal ones that correspond to $(\{a\}, \{b,c\})$ and $(\{a,b\}, \{c\})$. Formally,

The Maximin rule, φ^M: *For each $R \in R^N$, we have $\varphi^M(R) = \arg\max_{x \in \mathcal{X}} \underline{r}(x, R)$.*

This rule has two particular subcorrespondences based on the following two distinct ideas.[9] First, conditional on "helping the worst off", if preventing each agent from envying the other is impossible, then each agent should have a chance to be the one not envying and if preventing each agent from envying the other is possible, then we should minimize disparities in welfare. Second, conditional on "helping the worst off", we should also "help the best off". In our last example, as $PF(R') = \{x, x'\}$, the former subcorrespondence selects $\{(\{a\}, \{b,c\})\}$ and the latter subcorrespondence selects $\{(\{a,b\}, \{c\})\}$. Formally,

[9] These subcorrespondences' informal and formal definitions are given for two-agent economies. Their extension to economies with more than two agents is not immediate. However, as its name makes it implicit, we define the second subcorrespondence as the lexicographical application the Maximin rule. It first selects the allocations with the maximal minimal rank. Then, among these allocations, it selects the allocations with the maximal second minimal rank. It is done until no further distinction is possible. This idea has been first introduced by Sen, 1970. It has been as much discussed, most of all, in d'Aspremont and Gevers, 1978.

The Maximin-minimax rule, φ^{Mm}: *For each $R \in R^N$, if $PF(R) = \emptyset$, then $\varphi^{Mm}(R) = \varphi^M(R)$ and if $PF(R) \neq \emptyset$, then $\varphi^{Mm}(R) = \arg\min_{x \in \varphi^M(R)}\{\max_{i \in N}\{r(x_i, R_i)\}\}$.*

The Leximin rule, φ^L: *For each $R \in R^N$, we have $\varphi^L(R) = \arg\max_{x \in \varphi^M(R)}\{\max_{i \in N}\{r(x_i, R_i)\}\}$.*

To come to our main results, we distinguish rules that do not discriminate between objects. Thus, if all agents reverse their preferences over a pair of objects, we should permute the selected allocations accordingly. Formally, let Γ be the set of all permutations on K. For $\gamma \in \Gamma$, $R_i \in \mathcal{R}$, and $R \in \mathcal{R}^N$, let $\gamma(R_i)$ be such that for each $S, S' \in \mathcal{S}$, we have $\cup_{k \in S}\gamma(k)\ \gamma(R_i)\ \cup_{k \in S'}\gamma(k)$ if and only if SR_iS'. Let $\gamma(R) \equiv (\gamma(R_i))_{i \in N}$ and $\gamma(x) \equiv (\gamma(x_i))_{i \in N}$. Let φ be a rule.

Neutrality: *For each $R \in \mathcal{R}^N$, each $x \in \varphi(R)$, and each $\gamma \in \Gamma$, we have $\gamma(x) \in \varphi(\gamma(R))$.*

We are now ready to state and prove our main result. The Maximin rule satisfies each axiom of the previous section. Moreover, if there are three objects, it is the only rule, together with the Maximin-minimax rule, that satisfies each equity axiom and *neutrality*.

Theorem 3

1. *The Maximin rule satisfies* Pareto-efficiency, anonymity, conditional no-envy, *the* identical-preferences lower bound, *and* preference-monotonicity.
2. *If $|K| = 3$, then a rule satisfies* anonymity, conditional no-envy, *the* identical-preferences lower bound, preference-monotonicity, *and* neutrality *if and only if it is the Maximin rule or the Maximin-minimax rule.*

Proof.

Statement 1:

Pareto-efficiency: *Let $R \in \mathcal{R}$ and $x \in \varphi^M(R)$. By contradiction, suppose that there is $x' \in \mathcal{X}$ such that for each $i \in N$, we have $r_i(x'_i) \geq r_i(x_i)$ and for $j \in N$, we have $r_j(x'_j) > r_j(x_j)$. Then, for each $i \in N$, as R_i is strict and $|N| = 2$, we have $r_i(x'_i) > r_i(x_i)$. Thus, by definition $\underline{r}(x', R) > \underline{r}(x, R)$, contradicting $x \in \varphi^M(R)$.*

Anonymity: *As φ^M never uses agents' names, it satisfies* anonymity.

Conditional no-envy: *Let* $N = \{1,2\}$ *and* $R \in \mathcal{R}^N$ *be such that there is* $x \in F(R)$. *By contradiction, suppose that there is* $x' \in \varphi^M(R)$ *such that* $x' \notin F(R)$. *Without loss of generality, suppose* $\underline{r}(x,R) = r_1(x_1)$. *Then, for each* $i \in N$,

- *By definition of* φ^M, *we have* $r_i(x'_i) \geq \underline{r}(x',R)$. *As* $x' \in \varphi^M(R)$, *we have* $\underline{r}(x',R) \geq \underline{r}(x,R)$. *By assumption,* $\underline{r}(x,R) = r_1(x_1)$. *Thus,* ***(i)*** $r_i(x'_i) \geq r_1(x_1)$.
- *By assumption,* $x \in F(R)$. *Also,* R_i *is strict. Thus,* ***(ii)*** $r_1(x_1) > r_1(x_2)$.
- *By Lemma 1,* ***(i)*** *implies* $r_i((x'_i)_c) \leq r_1((x_1)_c)$. *As* $|N| = 2$, *we have* $(x_1)_c = x_2$ *and for* $j \in N\backslash\{i\}$, *we have* $(x'_i)_c = x'_j$. *Thus,* ***(iii)*** $r_1(x_2) \geq r_i(x'_j)$.

By ***(i)***, ***(ii)***, *and* ***(iii)***, $r_1(x'_1) > r_1(x'_2)$ *and* $r_2(x'_2) > r_2(x'_1)$, *contradicting* $x' \notin F(R)$.

Identical-preferences lower bound: *Let* $R \in \mathcal{R}$ *and* $x \in \varphi^M(R)$. *By contradiction, suppose that for* $i \in N$, *we have* $r_i(x_i) < \underline{r}(x^{PE}(R_{(i)}), R_{(i)})$. *By Theorem 2,* $r_i(x_i) < 2^{|K|-1}$. *Thus, by definition,* $\underline{r}(x,R) < 2^{|K|-1}$. *By Theorem 2, there is* $x' \in \mathcal{X}$ *such that for each* $i \in N$, *we have* $r_i(x'_i) \geq 2^{|K|-1}$. *Thus, by definition,* $\underline{r}(x',R) \geq 2^{|K|-1}$. *Altogether,* $\underline{r}(x,R) < 2^{|K|-1} \leq \underline{r}(x',R)$, *contradicting* $x \in \varphi^M(R)$.

Preference-monotonicity: *Let* $N = \{1,2\}$. *Suppose that 2's preferences become closer to 1's,* $R_1 \in \mathcal{R}$. *By contradiction, suppose that 1 is better off on average after the change. By Lemma 2, there is a sequence of consecutive switches between adjacent bundles such that after one at least, 1 finds herself better off on average. Formally, there are* $R_2, R'_2 \in \mathcal{R}$ *with*

(1) R'_2 *is closer to* R_1 *than* R_2;
(2) *for each* $S, S' \in \mathcal{S}$, *we have* $r_2(S) > r_2(S')$ *and* $r'_2(S) < r'_2(S')$ *if and only if* $r_2(S) = r'_2(S') = r_2(S') + 1$ *and* $r'_2(S) = r_2(S') = r'_2(S') - 1$;
(3) *for* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $\sum_{x\in\varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) < \sum_{x'\in\varphi^M(R_1,R'_2)} \frac{1}{|\varphi^M(R_1,R'_2)|} u_1(x'_1)$.

Thus, there are $x = (x_1,x_2) \in \varphi^M(R_1,R_2)$ *and* $x' = (x'_1,x'_2) \in \varphi^M(R_1,R'_2)$ *such that* $r(x'_1) > r_1(x_1)$. *In what follows, we prove that these assumptions lead to a contradiction.*

Step 1 (Identification of the minimal rank of x' ***in R):*** $\underline{r}(x',(R_1,R_2)) = r_2(x'_2) < r_1(x'_1)$.
By assumption, $r_1(x'_1) > r_1(x_1)$. *By definition,* $r_1(x_1) \geq \underline{r}(x,(R_1,R_2))$. *As*

$x \in \varphi^M(R_1, R_2)$, we have $\underline{r}(x,(R_1,R_2)) \geq \underline{r}(x',(R_1,R_2))$. Thus, $r_1(x'_1) > \underline{r}(x',(R_1,R_2))$ implying $\underline{r}(x',(R_1,R_2)) = r_2(x'_2)$.

Step 2 (Identification of the minimal rank of x in R'): *$\underline{r}(x,(R_1,R'_2)) = r_1(x_1)$.*
By contradiction, suppose $\underline{r}(x,(R_1,R'_2)) \neq r_1(x_1)$. Then,

- *By definition, $r_1(x_1) > \underline{r}(x,(R_1,R'_2))$. Thus, **(i)** $\underline{r}(x,(R_1,R'_2)) = r'_2(x_2)$.*
- *By assumption, $r_1(x'_1) > r_1(x_1)$. By Lemma 1, $r_1((x_1)_c) > r_1((x'_1)_c)$. As $|N| = 2$, we have $(x_1)_c = x_2$ and $(x'_1)_c = x'_2$. Thus, **(ii)** $r_1(x_2) > r_1(x'_2)$.*
- *By definition, $r_2(x_2) \geq \underline{r}(x,(R_1,R_2))$. As $x \in \varphi^M(R_1,R_2)$, we have $\underline{r}(x,(R_1,R_2)) \geq \underline{r}(x',(R_1,R_2))$. By Step 1, $\underline{r}(x',(R_1,R_2)) = r_2(x'_2)$. As R_2 is strict and $x \neq x'$, we have $r_2(x_2) \neq r_2(x'_2)$. Thus, **(iii)** $r_2(x_2) > r_2(x'_2)$.*
- *By **(1)**, **(ii)**, and **(iii)**, **(iv)** $r'_2(x_2) > r'_2(x'_2)$.*
- *By definition, **(v)** $r'_2(x'_2) \geq \underline{r}(x',(R_1,R'_2))$.*

*By **(i)**, **(iv)**, and **(v)**, $\underline{r}(x,(R_1,R'_2)) > \underline{r}(x',(R_1,R'_2))$, contradicting $x' \in \varphi^M(R_1,R'_2)$.*

Step 3 (Identification of a switch between adjacent bundles when 2's preferences change from R'_2 to R'_2): *there is $y \in \mathcal{X}$ with $r_2(y_2) > r_2(x'_2)$ and $r'_2(y_2) < r'_2(x'_2)$ such that $r_2(x'_2) = r'_2(y_2) = r_2(y_2) - 1$ and $r'_2(x'_2) = r_2(y_2) = r'_2(y_2) + 1$.*
The following holds.

- ***(i)** $r'_2(x'_2) > r_2(x'_2)$. By contradiction, suppose $r'_2(x'_2) \leq r_2(x'_2)$. As $x' \in \varphi^M(R_1,R'_2)$, we have $\underline{r}(x',(R_1,R'_2)) \geq \underline{r}(x,(R_1,R'_2))$. By Step 2, $\underline{r}(x,(R_1,R'_2)) = r_1(x_1)$. By definition, $r_1(x_1) \geq \underline{r}(x,(R_1,R_2))$. As $x \in \varphi^M(R_1,R_2)$, we have $\underline{r}(x,(R_1,R_2)) \geq \underline{r}(x',(R_1,R_2))$. By Step 1, $\underline{r}(x',(R_1,R_2)) = r_2(x'_2)$. By assumption, $r_2(x'_2) \geq r'_2(x'_2)$. By definition, $r'_2(x'_2) \geq \underline{r}(x',(R_1,R'_2))$. Thus, $\underline{r}(x',(R_1,R'_2)) = \underline{r}(x,(R_1,R'_2)) = \underline{r}(x,(R_1,R_2)) = \underline{r}(x',(R_1,R_2))$. Thus, as $x \in \varphi^M(R_1,R_2)$ and $x' \in \varphi^M(R_1,R'_2)$, we have $\varphi^M(R_1,R_2) = \{x,x'\} = \varphi^M(R_1,R'_2)$, contradicting **(3)**.*

*By **(i)** and **(2)**, there is $y \in \mathcal{X}$ with $r_2(y_2) > r_2(x'_2)$ and $r'_2(y_2) < r'_2(x'_2)$ such that $r_2(x'_2) = r'_2(y_2) = r_2(y_2) - 1$ and $r'_2(x'_2) = r_2(y_2) = r'_2(y_2) + 1$.*

Step 4 (Contradiction of (3)): *for each $u_1 \in \mathcal{U}$ representing R_1, we have*

$\sum_{x\in\varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) > \sum_{x'\in\varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1')$.
The following holds.

- *By Step 3,* $r_2(y_2) > r_2(x_2')$ *and* $r_2'(y_2) < r_2'(x_2')$. *Thus, by* **(1)**, $r_1(y_2) > r_1(x_2')$. *By Lemma 1,* $r_1((y_2)_c) < r_1((x_2'))$. *As* $|N| = 2$, *we have* $(y_2)_c = y_1$ *and* $(x_2')_c = x_1'$. *Thus,* ***(i)*** $r_1(y_1) > r_1(x_1')$.

- *By assumption,* $r_1(x_1') > r_1(x_1)$. *By definition,* $r_1(x_1) \geq \underline{r}(x,(R_1,R_2))$. *As* $x \in \varphi^M(R_1,R_2)$, *we have* $\underline{r}(x,(R_1,R_2)) \geq \underline{r}(y,(R_1,R_2))$. *Thus, by* ***(i)***, $r_1(y_1) > \underline{r}(y,(R_1,R_2))$ *implying* ***(ii)*** $\underline{r}(y,(R_1,R_2)) = r_2(y_2)$.

- *By Step 2,* $\underline{r}(x,(R_1,R_2')) = r_1(x_1)$. *By definition,* $r_1(x_1) \geq \underline{r}(x,(R_1,R_2))$. *As* $x \in \varphi^M(R_1,R_2)$, *we have* $\underline{r}(x,(R_1,R_2)) \geq \underline{r}(y,(R_1,R_2))$. *By* ***(ii)***, $\underline{r}(y,(R_1,R_2)) = r_2(y_2)$. *By Step 3,* $r_2(y_2) = r_2'(x_2')$. *By definition,* $r_2'(x_2') \geq \underline{r}(x',(R_1,R_2'))$. *As* $x' \in \varphi^M(R_1,R_2')$, *we have* $\underline{r}(x',(R_1,R_2')) \geq \underline{r}(x,(R_1,R_2'))$. *Thus,* ***(iii)*** $\underline{r}(x,(R_1,R_2')) = \underline{r}(x,(R_1,R_2)) = \underline{r}(y,(R_1,R_2)) = \underline{r}(x',(R_1,R_2'))$.

As ***(iii)*** *holds, and as* $x \in \varphi^M(R_1,R_2)$ *and* $x' \in \varphi^M(R_1,R_2')$, *we have* $\varphi^M(R_1,R_2) = \{x,y\}$ *and* $\varphi^M(R_1,R_2) = \{x,x'\}$, *contradicting, by* ***(i)***, **(3)**.

Statement 2:

Suppose $|K| = 3$. *By Theorem 2, it can be easily proved that the following lemma holds.*

Lemma 3 *For each* $R \in \mathcal{R}^N$ *and each* $x \in \varphi^M(R)$, *we have* $\underline{r}(x,R) \in \{2^{|K|-1}, 2^{|K|-1}+1\}$.

Then,

Part 1: φ^M ***and*** φ^{Mm} ***satisfy the axioms of Theorem 3.2.***
By Theorem 3.1, φ^M *satisfies each equity axiom. As it never uses objects' names, it also satisfies* neutrality. *As* φ^{Mm} *is a subcorrespondence of* φ^M, *it satisfies* conditional no-envy *and the* identical-preferences lower bound. *As it never uses agents' nor objects' names, it satisfies* anonymity *and* neutrality. *As* $|K| = 3$, *it also satisfies* preference-monotonicity. *Indeed, in the proof of Theorem 3.1, there are two cases, where* φ^{Mm} *may differ from* φ^M.

Case 1: *In Step 3, when* $\underline{r}(x',(R_1,R_2')) = \underline{r}(x,(R_1,R_2')) = \underline{r}(x,(R_1,R_2)) = \underline{r}(x',(R_1,R_2))$ *implies* $\varphi^M(R_1,R_2) = \{x,x'\} = \varphi^M(R_1,R_2')$. *Then,*

- *By Step 1,* $\underline{r}(x',(R_1,R_2)) = r_2(x_2')$. *Thus, as* $\underline{r}(x',(R_1,R_2)) = \underline{r}(x,(R_1,R_2))$ *and* $|N| = 2$, *we have* $\underline{r}(x,(R_1,R_2)) = r_1(x_1)$. *Thus,* $\overline{r}(x',(R_1,R_2)) = r_1(x_1')$ *and* $\overline{r}(x,(R_1,R_2)) = r_2(x_2)$. *Thus, as* $x \in \varphi^{Mm}(R_1,R_2)$ *and* $\underline{r}(x,(R_1,R_2)) = \underline{r}(x',(R_1,R_2))$, ***(i)*** $r_2(x_2) \leq r_1(x_1')$.

- *By Step 2,* $\underline{r}(x,(R_1,R_2')) = r_1(x_1)$. *Thus, as* $\underline{r}(x,(R_1,R_2')) = \underline{r}(x',(R_1,R_2'))$ *and* $|N| = 2$, *we have* $\underline{r}(x',(R_1,R_2')) = r_2'(x_2')$. *Thus,* $\overline{r}(x,(R_1,R_2')) = r_2'(x_2)$ *and* $\overline{r}(x',(R_1,R_2')) = r_1(x_1')$. *Thus, as* $x' \in \varphi^{Mm}(R_1,R_2')$ *and* $\underline{r}(x',(R_1,R_2')) = \underline{r}(x,(R_1,R_2'))$, ***(ii)*** $r_1(x_1') \leq r_2'(x_2)$.

- ***(iii)*** $r_2(x_2) \geq r_2'(x_2)$. *By contradiction, suppose* $r_2(x_2) < r_2'(x_2)$. *By Lemma 2, there is* $z \in X$ *such that* ***(iv)*** $r_2(x_2) < r_2(z_2)$ *and* $r_2'(x_2) > r_2'(z_2)$. *Thus, as* $x \in \varphi^{Mm}(R_1,R_2)$, *we have* $r_1(x_1) > r_1(z_1)$. *By Lemma 1,* $r_1((x_1)_c) < r_1((z_1)_c)$. *As* $|N| = 2$, *we have* $(x_2)_c = x_1$ *and* $(z_2)_c = z_1$. *Thus,* $r_1(x_2) < r_1(z_2)$, *contradicting, by* ***(iv)***, **(1)**.

By ***(i)***, ***(ii)***, *and* ***(iii)***, $r_2'(x_2) = r_2(x_2)$. *Thus,* $\overline{r}(x',(R_1,R_2')) = \overline{r}(x,(R_1,R_2')) = \overline{r}(x,(R_1,R_2)) = \overline{r}(x',(R_1,R_2))$.

Case 2: *In Step 4, when* $\underline{r}(x,(R_1,R_2')) = \underline{r}(x,(R_1,R_2)) = \underline{r}(y,(R_1,R_2)) = \underline{r}(x',(R_1,R_2'))$ *implies* $\varphi^M(R_1,R_2) = \{x,y\}$ *and* $\varphi^M(R_1,R_2) = \{x,x'\}$. *Then,*

- *By* ***(ii)*** *of Step 4 ,* $\underline{r}(y,(R_1,R_2)) = r_2(y_2)$. *By assumption,* $\underline{r}(y,(R_1,R_2)) = \underline{r}(x,(R_1,R_2))$ *and* $|N| = 2$. *Thus,* ***(i)*** $\underline{r}(x,(R_1,R_2)) = r_1(x_1) = r_2(y_2) = \underline{r}(y,(R_1,R_2))$.

- *By assumption,* $\varphi^M(R_1,R_2) = \{x,y\}$. *Thus, by* ***(i)*** *and Lemma 3,* ***(ii)*** $r_1(x_1) = r_2(y_2) \in \{2^{|K|-1}, 2^{|K|-1}+1\}$.

- ***(iii)*** $r_2(y_2) \neq 2^{|K|-1}+1$. *By contradiction, suppose* $r_2(y_2) = 2^{|K|-1}+1$. *First, by Lemma 1,* $r_2((y_2)_c) = 2^{|K|-1}$. *As* $|N| = 2$, *we have* $(y_2)_c = y_1$. *By Step 3,* $r_2(x_2') = r_2(y_2) - 1$. *Thus,* $r_2(y_1) = 2^{|K|-1} = r_2(x_2')$. *Thus, as* R_2 *is strict,* $y_1 = x_2'$. *Second, by assumption,* $r_1(x_1') > r_1(x_1)$. *Thus, by* ***(i)***, $r_1(x_1') > 2^{|K|-1}+1$. *By Lemma 1,* $r_1((x_1')_c) < 2^{|K|-1}$. *As* $|N| = 2$, *we have* $(x_1')_c = x_2'$. *Thus,* $r_1(x_2') < 2^{|K|-1}$. *Thus, as* $r_2(y_2) = 2^{|K|-1}+1$ *and* $y_1 = x_2'$, *we have* $r_1(y_1) < r_2(y_2)$, *contradicting* $\underline{r}(y,(R_1,R_2)) = r_2(y_2)$.

By ***(i)***, ***(ii)***, *and* ***(iii)***, $r_1(x_1) = 2^{|K|-1}$. *By Lemma 1,* $r_2((x_2)_c) = 2^{|K|-1}+1$. *As* $|N| = 2$, *we have* $(x_1)_c = x_2$. *Thus,* $r_1(x_2) = 2^{|K|-1}+1$. *Thus,* $r_2(x_2) < r_1(x_1)$. *Thus, as* φ^M *satisfies* conditional no-envy *and as* $x \in \varphi^M(R_1,R_2)$ *and* $x \in \varphi^M(R_1,R_2')$, *we have* $PF(R_1,R_2) = PF(R_1,R_2') = \emptyset$.

In Cases 1 and 2, by definition of φ^{Mm}, *we have* $\varphi^{Mm}(R_1,R_2) = \varphi^M(R_1,R_2)$ *and* $\varphi^{Mm}(R_1,R_2') = \varphi^M(R_1,R_2')$. *Thus, the proof of Theorem 3.1 also holds for* φ^{Mm}.

Part 2: A rule that satisfies the axioms of Theorem 3.2 is either φ^M ***or*** φ^{Mm}.

Let φ be a rule satisfying these axioms. Let $N = \{1,2\}$, $K = \{a,b,c\}$, and $R = (R_1, R_2) \in \mathcal{R}^N$. In what follows, we prove that $\varphi(R) = \varphi^M(R)$ or $\varphi(R) = \varphi^{Mm}(R)$. Clearly, this holds for each $R \in \mathcal{R}^N$. By assumption, φ satisfies neutrality. *Thus, if there is $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$ such that $\varphi(R) = \varphi^{Mm}(R)$, then for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, we have $\varphi(R) = \varphi^{Mm}(R)$. Thus, either for each $R \in \mathcal{R}^N$, we have $\varphi(R) = \varphi^M(R)$ or for each $R \in \mathcal{R}^N$, we have $\varphi(R) = \varphi^{Mm}(R)$.*

Step 1: *$\varphi(R) \subseteq \varphi^M(R)$.*
Let $x^M \in \varphi^M(R)$ and $x \in \varphi(R)$. By contradiction, suppose $x \notin \varphi^M(R)$. Then,

- *By definition of φ^M, we have $\underline{r}(x,R) < \underline{r}(x^M,R)$. By Lemma 3, $\underline{r}(x^M,R) \in \{2^{|K|-1}, 2^{|K|-1}+1\}$. By assumption, φ satisfies the* identical-preferences lower bound. *Thus, by Theorem 2, $\underline{r}(x,R) \geq 2^{|K|-1}$. Thus, **(i)** $\underline{r}(x,R) = 2^{|K|-1}$ and $\underline{r}(x^M,R) = 2^{|K|-1}+1$.*

- *By definition, for each $i \in N$, we have $r_i(x_i^M) \geq \underline{r}(x^M,R)$. By **(i)**, $\underline{r}(x^M,R) = 2^{|K|-1}+1$. By Lemma 1, for each $i \in N$, we have $r_i((x_i^M)_c) \leq 2^{|K|-1}$. As $|N| = 2$, we have $(x_1^M)_c = x_2^M$ and $(x_2^M)_c = x_1^M$. By Theorem 3.1, $\varphi^M(R) \subseteq P(R)$. Thus, **(ii)** $PF(R) \neq \emptyset$.*

*Without loss of generality, suppose $\underline{r}(x,R) = r_1(x_1)$. By **(i)**, $r_1(x_1) = 2^{|K|-1}$. By Lemma 1, $r_1((x_1)_c) = 2^{|K|-1}+1$. As $|N| = 2$, we have $(x_1)_c = x_2$. Thus, $r_1(x_2) > r_1(x_1)$, contradicting, by **(ii)**,* conditional no-envy.

Step 2: *$\varphi(R) = \varphi^M(R)$ or $\varphi(R) = \varphi^{Mm}(R)$.*
By contradiction, suppose $\varphi(R) \neq \varphi^M(R)$ and $\varphi(R) \neq \varphi^{Mm}(R)$. As $|N| = 2$, we have $|\varphi^M(R)| \in \{1,2\}$. Thus, as $\varphi(R) \subseteq \varphi^M(R)$, we have $|\varphi^M(R)| = 2$ and $\varphi(R) \subsetneq \varphi^M(R)$. Thus, there are $x = (x_1, x_2), x' = (x'_1, x'_2) \in \mathcal{X}$ with $x \neq x'$ such that $\{x, x'\} = \varphi^M(R)$ and $\{x\} = \varphi(R)$. Without loss of generality, suppose $\underline{r}(x,R) = r_1(x_1)$. By Lemma 3, $r_1(x_1) \in \{2^{|K|-1}, 2^{|K|-1}+1\}$. Thus,

Case 1: *$r_1(x_1) = 2^{|K|-1}$. Let $R'_2 \in \mathcal{R}$ be such that $R'_2 = R_1$. Let $x^* = (x_1^*, x_2^*), x^{**} = (x_1^{**}, x_2^{**}) \in \mathcal{X}$ be such that $r_1(x_1^*) = 2^{|K|-1}$, $r'_2(x_2^*) = 2^{|K|-1}+1$, $r_1(x_1^{**}) = 2^{|K|-1}+1$, and $r'_2(x_2^{**}) = 2^{|K|-1}$. Then, $B(R_1, R'_2) = \{x^*, x^{**}\}$. By assumption, φ satisfies* anonymity *and the* identical-preferences lower bound. *Thus, $\varphi(R_1, R'_2) = \{x^*, x^{**}\}$. Thus, as $r_1(x_1^*) = r_1(x_1) = 2^{|K|-1}$ and R_1 is strict, $\varphi(R_1, R'_2) = \{x, x^{**}\}$. By assumption, $\varphi(R) = \{x\}$. Clearly, R'_2 is closer to R_1 than R_2. Also, $r_1(x_1) = 2^{|K|-1}$ and $r_1(x_1^{**}) = 2^{|K|-1}+1$. Thus, for each $u_1 \in \mathcal{U}$ representing R_1, we have $u_1(x_1) < \frac{1}{2}(u_1(x_1)+u_1(x_1^{**}))$, contradicting* preference-monotonicity.

Case 2: *$r_1(x_1) = 2^{|K|-1}+1$. As $|N| = 2$, $|K| = 3$, and $|\varphi^M(R)| = 2$, there is $k, k' \in K$ such that $x = (\{k\}, K/\{k\})$ and $x' = (K/\{k'\}, \{k'\})$. Without* *loss of generality, suppose* $x = (\{a\}, \{b, c\})$ *and* $x' = (\{a, c\}, \{b\})$. *Also, as* $\underline{r}(x, R) = \underline{r}(x', R)$, $\underline{r}(x, R) = r_1(x_1)$, *and* $|N| = 2$, *we have* $\underline{r}(x', R) = r_2(x'_2)$. *Thus, as* $x \neq x'$, *we have* $r_1(x_1) = 2^{|K|-1} + 1$, $r_2(x_2) > 2^{|K|-1} + 1$, $r_1(x'_1) > 2^{|K|-1} + 1$, *and* $r_2(x'_2) = 2^{|K|-1} + 1$. *As* $\varphi(R) \neq \varphi^{Mm}(R)$, *we have* $r_1(x'_1) \leq r_2(x_2)$. *Thus,*

- *Suppose* $r_1(x'_1) = r_2(x_2)$. *Let* $(R'_1, R'_2) = \gamma(R_1, R_2)$ *with* $\gamma(a) = b$, $\gamma(b) = a$, *and* $\gamma(c) = c$. *By assumption,* φ *satisfies* neutrality. *Thus, as* $\varphi(R_1, R_2) = (\{a\}, \{b, c\})$, *we have* $\varphi(R'_1, R'_2) = \{(\{b\}, \{a, c\})\}$. *As* $r_1(x'_1) = r_2(x_2)$ *and* $|K| = 3$, *we have* $(R'_1, R'_2) = (R_2, R_1)$. *Thus,* $\varphi(R_2, R_1) = \{(\{b\}, \{a, c\})\}$, *contradicting* anonymity.

- *Suppose* $r_1(x'_1) < r_2(x_2)$. *Let* $y_2 \in \mathcal{S}$ *be such that* $r_2(y_2) = r_1(x'_1)$. *As* $|K| = 3$, *we have* $r_2(x_2) = r_2(y_2) + 1$. *Let* $R'_2 \in \mathcal{R}$ *be such that* $r'_2(x_2) = r_2(y_2)$, $r'_2(y_2) = r_2(x_2)$ *and for each* $S \in \mathcal{S}/\{y_2, x_2\}$, *we have* $r'_2(S) = r_2(S)$. *Then,* $B(R_1, R'_2) = \{x, x'\}$. *By assumption,* φ *satisfies* anonymity, *the* identical-preferences lower bound, *and* neutrality. *As* $r_1(x'_1) = r'_2(x_2)$ *and* $|K| = 3$, *we have* $r_1(R'_2) = \gamma(R_1, R'_2)$ *with* $\gamma(a) = b$, $\gamma(b) = a$, *and* $\gamma(c) = c$. *Thus,* $\varphi(R_1, R'_2) = \{x, x'\}$. *By assumption,* $\varphi(R) = \{x\}$. *Clearly,* R'_2 *is closer to* R_1 *than* R_2. *Also,* $r_1(x_1) = 2^{|K|-1} + 1$ *and* $r_1(x'_1) > 2^{|K|-1} + 1$. *Thus, for each* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $u_1(x_1) < \frac{1}{2}(u_1(x_1) + u_1(x'_1))$, *contradicting* preference-monotonicity. ■

None of the axioms can be dropped in Theorem 3. (The proofs are in Section 1.7.) The rule such that there is $k \in K$ with for each $R \in \mathcal{R}^N$, if for $x \in \varphi^M(R)$, we have $\underline{r}(x, R) \neq 2^{K-1} + 1$, then the rule selects each $x \in \varphi^M(R)$ and if for $x \in \varphi^M(R)$, we have $\underline{r}(x, R) = 2^{K-1} + 1$, then the rule selects each $x \in \{x \in \varphi^M(R) :$ for each $x' \in \varphi^M(R)$, we have $k \in x\}$, satisfies each axiom, but *neutrality.* The *Leximin rule* satisfies each axiom, but *preference-monotonicity.* The rule that selects for each $R \in \mathcal{R}^N$ with $PF(R) = \emptyset$, each $x \in \{(\emptyset, K), (K, \emptyset)\}$ and for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, each $x \in \varphi^M(R)$, satisfies each axiom, but the *identical-preferences lower bound.* The rule that selects for each $R \in \mathcal{R}^N$, each $x \in PB(R)$, satisfies each axiom, but *conditional no-envy.* Finally, the rule such that there is $i \in N$ with for each $R \in \mathcal{R}^N$, the rule selects each $x \in \{x \in \varphi^M(R) :$ for each $x' \in \varphi^M(R)$, we have $r_i(x'_i) \leq r_i(x_i)\}$, satisfies each axiom, but *anonymity.*

The intuition for why the Maximin rule satisfies *preference-monotonicity,* whereas the Leximin rule does not, is simple. Let $N = \{1, 2\}$, $K = \{a, b, c\}$, and $R = (R_1, R_2) \in \mathcal{R}^N$ be as in Figure 1.5. Let $x = (\{a\}, \{b, c\}) \in \mathcal{X}$ and $x' = (\{a, b\}, \{c\}) \in \mathcal{X}$. The Maximin rule selects $\{x, x'\}$. As x and x'

R_1	R_2	R_1	R'_2
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{b,c\}$	$\{a,b\}$	$\{a,c\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{b,c\}$
$\{b,c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b\}$
$\{a\}$	$\{c\}$	$\{a\}$	$\{c\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\{a\}$
$\{c\}$	$\{a\}$	$\{c\}$	$\{b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.5: *The Leximin rule does not satisfy* preference-monotonicity.

have the same maximal ranks, the Leximin rule also selects $\{x, x'\}$. Suppose that 2's preferences become $R'_2 \in \mathcal{R}$ as in Figure 1.5. Then, agents 1 and 2 now moreover agree on $\{a\}$ and $\{b\}$ and thus on $\{a, c\}$ and $\{b, c\}$, i.e., $\{b, c\}$ is one rank lower in R'_2 than R_2. Thus, in (R_1, R'_2), the rank of $\{b, c\}$ is lower than the one of $\{a, c\}$. Thus, in (R_1, R'_2), as the minimal ranks of x and x' are unchanged, the Maximin rule selects $\{x, x'\}$. However, as the maximal rank of x is lower than the one of x', the Leximin rule only selects $\{x'\}$. Thus, as 1 ranks $\{a, b\}$ higher than $\{a\}$, the Leximin rule violates *preference-monotonicity*. It is exactly that difference between the two rules that makes the former satisfy *preference-monotonicity* and the latter not.

An open question is whether there are rules different from the Maximin rule that satisfy *Pareto-efficiency*, *anonymity*, *conditional no-envy*, the *identical-preferences lower bound*, and *preference-monotonicity* in economies with more than three objects. We can already rule out obvious candidates.

For instance, first, consider the rule that selects for each $R \in \mathcal{R}^N$ with $PF(R) = \emptyset$, each $x \in PB(R)$ and for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, each $x \in PF(R)$, corresponds to the Maximin rule in economies with three objects. As it never uses agents' names, it satisfies *anonymity*. Clearly, it satisfies *Pareto-efficiency* and *conditional no-envy*. As $|N| = 2$, if an allocation is *envy-free*, each agent has at least her $2^{|K|-1} + 1th$ least preferred object. Thus, by Theorem 2, it satisfies the *identical-preferences lower bound*. However, it does not satisfy *preference-monotonicity*. Let $N = \{1, 2\}$, $K = \{a, b, c, d\}$, and $R = (R_1, R_2), R' = (R_1, R'_2) \in \mathcal{R}^N$ be as in Figure 1.6. Let $x = (\{a\}, \{b, c, d\})$, $x' = (\{a, c\}, \{b, d\})$, $x'' = (\{a, b\}, \{c, d\})$, and $x''' = (\{a, d\}, \{b, c\}) \in \mathcal{X}$. Clearly, R'_2 is closer to R_1 than R_2. Also, $\varphi(R) = \{x, x'', x'''\}$ and $\varphi(R') = \{x, x', x'', x'''\}$. Thus, for $u_1 \in \mathcal{U}$ representing R_1 such that $u_1(\{a\}) = 34$, $u_1(\{b\}) = 11$, $u_1(\{c\}) = 10$, and $u_1(\{d\}) = 12$, we have $\frac{1}{3}(u_1(\{a\}) + u_1(\{a, b\}) + u_1(\{a, d\})) < \frac{1}{4}(u_1(\{a\}) +$

R_1	R_2	R_2'	R_1'	R_2''	R_2'''
$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
$\{a,b,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
$\{a,c,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{b,c,d\}$	$\{b,c,d\}$
$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,d\}$	$\{a,b,c\}$	$\{a,b,d\}$
$\{a,d\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,b,c\}$
$\{a,b\}$	$\{c,d\}$	$\{b,d\}$	$\{b,c\}$	$\{c,d\}$	$\{c,d\}$
$\{a,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{b\}$	$\{a,d\}$	$\{a,c\}$
$\{b,c,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,c,d\}$	$\{b,c\}$	$\{b,d\}$
$\{b,d\}$	$\{a,c\}$	$\{a,b\}$	$\{c,d\}$	$\{b,d\}$	$\{b,c\}$
$\{c,d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,b\}$	$\{a,b\}$
$\{b,c\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$
$\{d\}$	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{d\}$	$\{c\}$
$\{b\}$	$\{b\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{a\}$
$\{c\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{b\}$	$\{b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.6: *Rules that are equivalent to the Maximin rule or the Maximin-minimax rule in economies two agents and three objects are not desirable in economies with two agents and more than three objects.*

$u_1(\{a,c\}) + u_1(\{a,b\}) + u_1(\{a,d\}))$.

Second, consider the Maximin-minimax rule. It is a subcorrespondence of the Maximin rule. Thus, by Theorem 3.1, it satisfies *Pareto-efficiency*, *conditional no-envy*, and the *identical-preferences lower bound*. As it never uses agents' names, it satisfies *anonymity*. However, it violates *preference-monotonicity*. Let $N = \{1,2\}$, $K = \{a,b,c,d\}$, and $R'' = (R_1', R_2'')$, $R''' = (R_1', R_2''') \in \mathcal{R}^N$ be as in Figure 1.6. Let $x = (\{a,b\},\{c,d\}) \in \mathcal{X}$ and $x' = (\{b,c\},\{a,d\}) \in \mathcal{X}$. Clearly, R_2''' is closer to R_1' than R_2''. Also, $\varphi^{Mm}(R'') = \{x\}$ and $\varphi^{Mm}(R''') = \{x'\}$. Thus, for each $u_1 \in \mathcal{U}$ representing R_1, we have $u_1(\{a,b\}) < u_1(\{b,c\})$.

1.5 More than two agents economies

We now prove that there is a clear gap between economies with two agents and economies with more than two agents.

First, by Theorem 1, it is in the latter case that there may be no *Pareto-efficient* and *envy-free* allocation. Second, by Theorem 2, in two-agent economies, the *identical-preferences lower bound* guarantees each agent with a minimal welfare level that corresponds to her $2^{|K|-1}th$ least preferred subset, whereas in economies with more than two agents, this level may depend on her preferences. Third and last, in Theorem 4, we prove that our assessment of the Maximin rule does not extend to economies with more than two agents. This rule first and foremost violates *Pareto-efficiency*. Furthermore, this rule and each of its subcorrespondences violate *conditional no-envy*.

Theorem 4

1. *The Maximin rule violates* Pareto-efficiency.
2. *The Maximin rule and each of its subcorrespondences violate* conditional no-envy.

Proof.

Statement 1: *Let* $N = \{1, 2, 3\}$, $K = \{a, b, c, d\}$, *and* $R \in \mathcal{R}^N$ *be as in Figure 1.7. Let* $x = (\{b, c\}, \{d\}, \{a\}) \in \mathcal{X}$ *and* $x' = (\{a\}, \{d\}, \{b, c\}) \in \mathcal{X}$. *Clearly,* $\varphi^M(R) = \{x, x'\}$. *But,* $\varphi^M(R) \nsubseteq P(R)$. *Indeed,* $r_1(x'_1) > r_1(x_1)$, $r_2(x'_2) = r_2(x_2)$, *and* $r_3(x'_3) > r_3(x_3)$. *Thus,* $x \notin P(R)$.

Statement 2: *Let* $N = \{1, 2, 3\}$, $K = \{a, b, c, d\}$, *and* $R' \in \mathcal{R}^N$ *be as in Figure 1.7. Let* $x'' = (\{a, b\}, \{d\}, \{c\}) \in \mathcal{X}$ *and* $x''' = (\{a\}, \{b, d\}, \{c\}) \in \mathcal{X}$. *Then,* $PF(R') \neq \emptyset$. *Indeed,* $x'' \in PF(R')$. *Also,* $\varphi^M(R') = \{x'''\}$.[10] *But,* $r_1(x_1) = 7 < r_1(x_2) = 8$. *Thus,* $x''' \notin F(R')$. ■

Thus, as we have not identified other desirable rules in two-agent economies, another open question is whether this gap implies an incompatibility between *Pareto-efficiency*, *anonymity*, *conditional no-envy*, the *identical-preferences lower bound*, and *preference-monotonicity* in economies with more than two agents.

1.6 Concluding remarks

Our objective was to identify fair allocation rules for problems of indivisible goods without monetary compensation. We assumed strict and additively separable preferences over subsets, and desirability of the indivisible goods.

As any large number of objects will never replace money as a compensating means, the search for *Pareto-efficient* and fair allocations is even harder than when money is available. Thus, we focused on several basic equity properties. *Anonymity* is classical in the sense that it has been much studied in the literature and its adaptation is straightforward. As properties

[10]Proof available to the reader upon request to the author.

R_1	R_2	R_3	R_1	R_2	R_3'
$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
$\{a,b,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$
$\{a,b,c\}$	$\{b,c,d\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{b,c,d\}$	$\{a,c,d\}$
$\{a,b\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
$\{a,c,d\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,b,d\}$	$\{a,b,c\}$
$\{a,d\}$	$\{a,b,c\}$	$\{b,c,d\}$	$\{a,d\}$	$\{a,b,c\}$	$\{b,c\}$
$\{b,c,d\}$	$\{a,d\}$	$\{a,d\}$	$\{b,c,d\}$	$\{a,d\}$	$\{a,c\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{c\}$
$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{a,b,d\}$
$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{a\}$	$\{b,c\}$	$\{b,d\}$
$\{b,c\}$	$\{d\}$	$\{a\}$	$\{b,c\}$	$\{d\}$	$\{a,d\}$
$\{b\}$	$\{c\}$	$\{b\}$	$\{b\}$	$\{c\}$	$\{d\}$
$\{c,d\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$
$\{d\}$	$\{a\}$	$\{d\}$	$\{d\}$	$\{a\}$	$\{b\}$
$\{c\}$	$\{b\}$	$\{c\}$	$\{c\}$	$\{b\}$	$\{a\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.7: *In economies with more than two agents, the Maximin rule violates* Pareto-efficiency *and* conditional no-envy.

avoiding agents to envy one another can only be fulfilled in some economies, we introduced *conditional no-envy.* As properties of welfare lower bounds and monotonicity due to changes in preferences are crucial in order to judge any possible problem on the basis of equity and as their adaptation is not straightforward, we introduced the *identical-preferences lower bound* and *preference-monotonicity.*

We have identified the Maximin rule as a desirable rule for two-agent economies. First, it satisfies each axiom. If there are three objects, it is the only rule, together with one of its subcorrespondences, that satisfies each equity axiom and that does not discriminate between objects. If there are more than three objects, it seems to be among the very few rules that satisfy these properties. Indeed, obvious subcorrespondences and supercorrespondences such as the Maximin-minimax, the Leximin rule, the rule that selects for each $R \in \mathcal{R}^N$, each $x \in PB(R)$, or the rule that selects for each $R \in \mathcal{R}^N$ with $PF(R) = \emptyset$, each $x \in PB(R)$ and for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, each $x \in PF(R)$, fail to satisfy one of these properties. Finally, it can be easily applied: There are procedures that yield allocations it selects and in two-agent economies, any allocations it selects can be obtained as a solution of these procedures (Herreiner and Puppe, 2002).

We have proved that there is a clear gap between economies with two agents and economies with more than two agents. In particular, the Maximin rule does not satisfy *Pareto-efficiency* in economies with more than two agents. By opposition, the Leximin rule and the rule that selects for each $R \in \mathcal{R}^N$, each $x \in P(R) \cap \varphi^M(R)$, clearly satisfy *Pareto-efficiency.* They satisfy *anonymity* and *neutrality*. They may satisfy the *identical-preferences lower bound.* Also, the latter rule may satisfy *preference-monotonicity.* But, as they coincide with the Maximin rule in R' of Theorem 4, they violate *conditional no-envy.* Thus, if indeed there is an incompatibility between these axioms, we should perhaps prefer to choose one of these rules that fairly allocate objects in each problem than a rule that avoids envy in a limited number of problems.

We finish with a remark on the monotonicity property. When adapting it, we had to determine how similar preferences are relative to one another. We believe our definition of close is appropriate. If we use one based on the number of agreements between agents, R_2' is more similar to R_1 than R_2 in the economies of Figure 1.2, where $N = \{1,2\}$, $K = \{a,b,c\}$, and $R = (R_1, R_2), R' = (R_1, R_2') \in \mathcal{R}^N$ respectively. Indeed, agents 1 and 2 in total agree on one more pair of subsets in R' than in R. Formally, for $i, j \in N$ and $R_i, R_j, R_j' \in \mathcal{R}$, we say that R_j' *weakly closer to* R_i *than* R_j if i and j agree on at least one more pair of subsets in (R_i, R_j'), i.e., $|\{(S,S') \in \mathcal{S}^2 : S\ R_i\ S'$ and $S\ R_j'\ S'\}| > |\{(S,S') \in \mathcal{S}^2 : S\ R_i\ S'$ and $S\ R_j\ S'\}|$.[11]

This definition is weaker than ours in the sense that if an agent's preferences are closer to another agent's than other preferences of hers, they are also weakly closer, but the reverse is not true. Thus, it may imply ignoring disagreements that were not there previously.[12] Moreover, it implies a stronger notion of *preference-monotonicity* that the Maximin rule and the Maximin-Minimax rule violate.

First, consider the economies of Figure 1.2, where $N = \{1,2\}$, $K = \{a,b,c\}$, and $R = (R_1, R_2), R' = (R_1, R_2'), R'' = (R_1, R_2'') \in \mathcal{R}^N$ respectively. Let $x = (\{a,c\},\{b\}) \in \mathcal{X}$, $x' = (\{a,b\},\{c\}) \in \mathcal{X}$, and $x'' = (\{a\},\{b,c\}) \in \mathcal{X}$. Clearly, R_2' is weakly closer to R_1 than R_2. Also, $\varphi^M(R) = \{x\}$ and $\varphi^M(R') = \{x', x''\}$. Thus, for $u_1 \in \mathcal{U}$ representing R_1 such that $u(\{a\}) = 35$, $u(\{b\}) = 30$, and $u(\{a\}) = 10$, we have $u_1(\{a,c\}) < \frac{1}{2}(u_1(\{a,b\}) + u_1(\{a\}))$.

Second, consider the economies of Figure 1.8, where $N = \{1,2\}$,

[11] Geometrically, it means that the number hyperplanes crossed from R_j' to R_i is smaller that the one from R_j to R_i, i.e., the Keminy distance between R_j' to R_i is smaller that the one from R_j to R_i.

[12] In our last example, R_2' is weakly closer to R_1 than R_2, but not closer because agents 1 and 2 disagree on pairs of subsets in R' they agree on in R (e.g. $\{b\}$ and $\{c\}$).

R_1	R_2	R_1	R'_2
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,c\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{a,b\}$
$\{b\}$	$\{c\}$	$\{b\}$	$\{a\}$
$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{c\}$
$\{c\}$	$\{b\}$	$\{c\}$	$\{b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Figure 1.8: *A stronger notion of* preference-monotonicity.

$K = \{a,b,c\}$, and $R = (R_1,R_2), R' = (R_1,R'_2) \in \mathcal{R}^N$ respectively. Let $x = (\{a,b\},\{c\}) \in \mathcal{X}$, $x' = (\{b\},\{a,c\}) \in \mathcal{X}$, and $x'' = (\{b,c\},\{a\}) \in \mathcal{X}$. Clearly, R'_2 is weakly closer to R_1 than R_2. Also, $\varphi^M(R) = \{x,x'\}$ and $\varphi^M(R') = \{x''\}$. Thus, for $u_1 \in \mathcal{U}$ representing R_1 such that $u(\{a\}) = 3$, $u(\{b\}) = 6$, and $u(\{a\}) = 2$, we have $\frac{1}{2}(u_1(\{a,b\}) + u_1(\{b\})) < u_1(\{b,c\})$. Thus, no rule satisfies this stronger notion of *preference-monotonicity*, the other equity axioms, and *neutrality*.

1.7 Appendix: independence of axioms

Theorem 5 *Suppose $|N| = 2$. Let φ be a rule.*

1. *If there is $k \in K$ such that for each $R \in \mathcal{R}^N$, if for $x \in \varphi^M(R)$, we have $\underline{r}(x,R) \neq 2^{K-1}+1$, then $\varphi(R) = \varphi^M(R)$ and if for $x \in \varphi^M(R)$, we have $\underline{r}(x,R) = 2^{K-1}+1$, then $\varphi(R) = \{x \in \varphi^M(R) :$ for each $x' \in \varphi^M(R)$, we have $k \in x\}$, then φ satisfies* anonymity, conditional no-envy, *the* identical-preferences lower bound, preference-monotonicity, *but not* neutrality.

2. *If for each $R \in R^N$, we have $\varphi(R) = \varphi^L(R)$, then φ satisfies* anonymity, conditional no-envy, *the* identical-preferences lower bound, *and* neutrality, *but not* preference-monotonicity.

3. *If for each $R \in \mathcal{R}^N$ with $PF(R) = \emptyset$, we have $\varphi(R) = \{(\emptyset,K),(K,\emptyset)\}$ and for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, we have $\varphi(R) = \varphi^M(R)$, then φ satisfies* anonymity, conditional no-envy, preference-monotonicity, *and* neutrality, *but not the* identical-preferences lower bound.

4. *If for each* $R \in \mathcal{R}^N$, *we have* $\varphi(R) = PB(R)$, *then* φ *satisfies* anonymity, *the* identical-preferences lower bound, preference-monotonicity, *and* neutrality, *but not* conditional no-envy.

5. *If there is* $i \in N$ *such that for each* $R \in \mathcal{R}^N$, *we have* $\varphi(R) = \{x \in \varphi^M(R) : \text{for each } x' \in \varphi^M(R), \text{ we have } r_i(x'_i) \leq r_i(x_i)\}$, *then* φ *satisfies* conditional no-envy, *the* identical-preferences lower bound, preference-monotonicity, *and* neutrality, *but not* anonymity.

Proof.

Let $N = \{1, 2\}$.

Statement 1: *Suppose that there is* $k \in K$ *such that for each* $R \in \mathcal{R}^N$, *if for* $x \in \varphi^M(R)$, *we have* $\underline{r}(x, R) \neq 2^{K-1} + 1$, *then* $\varphi(R) = \varphi^M(R)$ *and if for* $x \in \varphi^M(R)$, *we have* $\underline{r}(x, R) = 2^{K-1} + 1$, *then* $\varphi(R) = \{x \in \varphi^M(R) : \text{for each } x' \in \varphi^M(R), \text{ we have } k \in x\}$. *Clearly,* φ *is a subcorrespondence of* φ^M. *Thus, by Theorem 3.1,* φ *satisfies* conditional no-envy *and the* identical-preferences lower bound. *As it never uses agents' names, it satisfies* anonymity. *Also, it satisfies* preference-monotonicity. *Indeed, in the proof of Theorem 3.1, there are two cases, where* φ *may differ from* φ^M.

Case 1: *In Step 3, when* $\underline{r}(x', (R_1, R'_2)) = \underline{r}(x, (R_1, R'_2)) = \underline{r}(x, (R_1, R_2)) = \underline{r}(x', (R_1, R_2))$ *implies* $\varphi^M(R_1, R_2) = \{x, x'\} = \varphi^M(R_1, R'_2)$. *Then,*

- *By Step 1,* $\underline{r}(x', (R_1, R_2)) = r_2(x'_2)$. *Thus, as* $\underline{r}(x', (R_1, R_2)) = \underline{r}(x, (R_1, R_2))$ *and* $|N| = 2$, *we have* $\underline{r}(x, (R_1, R_2)) = r_1(x_1)$. *Thus, as* $x \in \varphi(R_1, R_2)$, ***(i)*** $k \in x_1$.

- *By Step 2,* $\underline{r}(x, (R_1, R'_2)) = r_1(x_1)$. *Thus, as* $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$ *and* $|N| = 2$, *we have* $\underline{r}(x', (R_1, R'_2)) = r'_2(x'_2)$. *Thus, as* $x' \in \varphi(R_1, R'_2)$, ***(ii)*** $k \in x'_2$.

As ***(i)*** *and* ***(ii)*** *hold, and as* $x \in \varphi(R_1, R_2)$ *and* $\underline{r}(x', (R_1, R_2)) = \underline{r}(x, (R_1, R_2))$, *we have* $\varphi(R_1, R_2) = \{x, x'\}$. *Also, as* $x' \in \varphi(R_1, R'_2)$ *and* $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$, $\varphi(R_1, R'_2) = \{x, x'\}$. *Thus,* $\varphi(R_1, R_2) = \varphi^M(R_1, R_2)$ *and* $\varphi(R_1, R'_2) = \varphi^M(R_1, R'_2)$.

Case 2: *In Step 4, when* $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x, (R_1, R_2)) = \underline{r}(y, (R_1, R_2)) = \underline{r}(x', (R_1, R'_2))$ *implies* $\varphi^M(R_1, R_2) = \{x, y\}$ *and* $\varphi^M(R_1, R_2) = \{x, x'\}$. *By* ***(ii)*** *of Step 4 ,* $\underline{r}(y, (R_1, R_2)) = r_2(y_2)$. *Thus, as* $\underline{r}(y, (R_1, R_2)) = \underline{r}(x, (R_1, R_2))$ *and* $|N| = 2$, *we have* $\underline{r}(x, (R_1, R_2)) = r_1(x_1)$. *Thus,* $r_1(x_1) = r_2(y_2) \in \{2^{|K|-1}, 2^{|K|-1} + 1\}$. *Suppose* $r_1(x_1) = r_2(y_2) = 2^{|K|-1} + 1$. *First, by Lemma 1,* $r_2((y_2)_c) = 2^{|K|-1}$. *As* $|N| = 2$, *we have* $(y_2)_c = y_1$. *By Step 3,* $r_2(x'_2) = r_2(y_2) - 1$. *Thus,* $r_2(y_1) = 2^{|K|-1} = r_2(x'_2)$. *Thus, as*

R_2 is strict, $y_1 = x_2'$. Second, by assumption, $r_1(x_1') > r_1(x_1')$. Thus, $r_1(x_1') > 2^{|K|-1} + 1$. By Lemma 1, $r_1((x_1')_c) < 2^{|K|-1}$. As $|N| = 2$, we have $(x_1')_c = x_2'$. Thus, $r_1(x_2') < 2^{|K|-1}$. Thus, as $r_2(y_2) = 2^{|K|-1} + 1$ and $y_1 = x_2'$, we have $r_1(y_1) < r_2(y_2)$, contradicting $\underline{r}(y,(R_1,R_2)) = r_2(y_2)$. Thus, $r_1(x_1) = r_2(y_2) \neq 2^{|K|-1}$. Thus, by definition of φ, we have $\varphi(R_1,R_2) = \varphi^M(R_1,R_2)$ and $\varphi(R_1,R_2') = \varphi^M(R_1,R_2')$.

In Cases 1 and 2, $\varphi(R_1,R_2) = \varphi^M(R_1,R_2)$ and $\varphi(R_1,R_2') = \varphi^M(R_1,R_2')$. Thus, the proof of Theorem 3.1 also holds for φ. However, as φ uses object k's name, it violates neutrality.

Statement 2: *Suppose that for each $R \in R^N$, we have $\varphi(R) = \varphi^L(R)$. Clearly, φ is a subcorrespondence of φ^M. Thus, by Theorem 3.1, φ satisfies* conditional no-envy *and the* identical-preferences lower bound. *As it never uses agents' nor objects' names, it satisfies* anonymity *and* neutrality. *However, it violates* preference-monotonicity. *Indeed, let $K = \{a,b,c\}$, and $R, R' \in \mathcal{R}^N$ be as in Figure 1.5. Let $x = (\{a\},\{b,c\}) \in \mathcal{X}$ and $x' = (\{a,b\},\{c\}) \in \mathcal{X}$. Clearly, R_2' is closer to R_1 than R_2. Also, $\varphi(R) = \{x,x'\}$ and $\varphi(R') = \{x'\}$. Thus, for each $u_1 \in \mathcal{U}$ representing R_1, we have $\frac{1}{2}(u_1(\{a\}) + u_1(\{a,b\})) < u_1(\{a,b\})$.*

Statement 3: *Suppose that for each $R \in \mathcal{R}^N$ with $PF(R) = \emptyset$, we have $\varphi(R) = \{(\emptyset,K),(K,\emptyset)\}$ and for each $R \in \mathcal{R}^N$ with $PF(R) \neq \emptyset$, we have $\varphi(R) = \varphi^M(R)$. As φ never uses agents' nor objects' names, it satisfies* anonymity *and* neutrality. *Clearly, it satisfies* conditional no-envy. *Also, it satisfies* preference-monotonicity. *Indeed, let $(R_1,R_2),(R_1,R_2') \in \mathcal{R}^N$ be such that R_2' is closer to R_1 than R_2. Then,*

- ***(i)*** *$F(R_1,R_2) \supseteq F(R_1,R_2')$. By contradiction, suppose that there is $x \in F(R_1,R_2')$ such that $x \notin F(R_1,R_2)$. Then, $r_1(x_1) \geq 2^{|K|-1}+1$, $r_2(x_2) < 2^{|K|-1}+1$, and $r_2'(x_2) \geq 2^{|K|-1}+1$. By Lemma 1, $r_1((x_1)_c) \leq 2^{|K|-1}$, $r_2((x_2)_c) > 2^{|K|-1}$, and $r_2'((x_2)_c) \leq 2^{|K|-1}$. As $|N| = 2$, we have $(x_1)_c = x_2$ and $(x_2)_c = x_1$. Thus, $r_1(x_1) > r_1(x_2)$, $r_2(x_1) > r_2(x_2)$, and $r_2'(x_1) < r_2'(x_2)$, contradicting that R_2' is closer to R_1 than R_2.*
- *By Theorem 1.2,* ***(ii)*** *for each $R \in \mathcal{R}^N$, we have $PF(R) \neq \emptyset$ if and only if $F(R) \neq \emptyset$.*

By ***(i)*** *and* ***(ii)***, *distinguish three cases.*

Case 1: *$PF(R_1,R_2) \neq \emptyset$ and $PF(R_1,R_2') \neq \emptyset$. Then, $\varphi(R_1,R_2) = \varphi^M(R_1,R_2)$ and $\varphi(R_1,R_2') = \varphi^M(R_1,R_2')$. Thus, by Theorem 3.1, for each $u_1 \in \mathcal{U}$ representing R_1, we have $\sum_{x\in\varphi(R_1,R_2)} \frac{1}{|\varphi(R_1,R_2)|} u_1(x_1) \geq \sum_{x'\in\varphi(R_1,R_2')} \frac{1}{|\varphi(R_1,R_2')|} u_1(x_1')$.*

Case 2: *$PF(R_1, R_2) \neq \emptyset$ and $PF(R_1, R_2') = \emptyset$. Then, $\varphi(R_1, R_2) = \varphi^M(R_1, R_2)$ and $\varphi(R_1, R_2') = \{(\emptyset, K), (K, \emptyset)\}$. Let $x^* \in \varphi(R_1, R_2)$ be such that for each $x \in \varphi(R_1, R_2)$, we have $r_1(x_1) \geq r_1(x_1^*)$. Thus, $\sum_{x\in\varphi(R_1,R_2)} \frac{1}{|\varphi(R_1,R_2)|} u_1(x_1) \geq u_1(x_1^*)$. By Theorem 3.1, φ^M satisfies* conditional no-envy. *By assumption, $PF(R_1, R_2) \neq \emptyset$. Thus, $u_1(x_1^*) > \frac{1}{2}(u_1(x_1^*) + u_1(x_2^*))$. As $|N| = 2$, we have $(x_2^*) = (x_1^*)_c$. By definition, $\frac{1}{2}(u_1(x_1^*) + u_1((x_1^*)_c)) = \frac{1}{2}u_1(K)$. By assumption, $\frac{1}{2}u_1(K) = \frac{1}{2}(u_1(\emptyset) + u_1(K))$. Also, $\frac{1}{2}(u_1(\emptyset) + u_1(K)) = \sum_{x'\in\varphi(R_1,R_2')} \frac{1}{|\varphi(R_1,R_2')|} u_1(x_1')$. Thus, for each $u_1 \in \mathcal{U}$ representing R_1, we have $\sum_{x\in\varphi(R_1,R_2)} \frac{1}{|\varphi(R_1,R_2)|} u_1(x_1) > \sum_{x'\in\varphi(R_1,R_2')} \frac{1}{|\varphi(R_1,R_2')|} u_1(x_1')$.*

Case 3: *$PF(R_1, R_2) = \emptyset$ and $PF(R_1, R_2') = \emptyset$. Then, $\varphi(R_1, R_2) = \varphi(R_1, R_2')$. Thus, for each $u_1 \in \mathcal{U}$ representing R_1, we have $\sum_{x\in\varphi(R_1,R_2)} \frac{1}{|\varphi(R_1,R_2)|} u_1(x_1) = \sum_{x'\in\varphi(R_1,R_2')} \frac{1}{|\varphi(R_1,R_2')|} u_1(x_1')$.*

However, φ violates the identical-preferences lower bound. *Indeed, let $R \in \mathcal{R}^N$ be such that $PF(R) = \emptyset$. Then, $\varphi(R) = \{(\emptyset, K), (K, \emptyset)\}$. By Lemma 1, $r_1(\emptyset) = r_2(\emptyset) = 1$. By Theorem 2, $\underline{r}(x^{PE}(R_{(1)}), R_{(1)}) = \underline{r}(x^{PE}(R_{(2)}), R_{(2)}) = 2^{|K|-1}$. Thus, for each $x \in \varphi(R)$, there is $i \in N$ such that $r_i(x_i) < \underline{r}(x^{PE}(R_{(i)}), R_{(i)})$.*

Statement 4: *Suppose that for each $R \in \mathcal{R}^N$, we have $\varphi(R) = PB(R)$. As φ never uses agents' nor objects' names, it satisfies* anonymity *and* neutrality. *Clearly, by Theorem 3.1, it satisfies the* identical-preferences lower bound. *Also, it satisfies* preference-monotonicity. *Indeed, suppose that 2's preferences become closer to 1's, $R_1 \in \mathcal{R}$. By contradiction, suppose that 1 is better off on average after the change. By Lemma 2, there is a sequence of consecutive switches between adjacent bundles such that after one at least, 1 finds herself better off on average. Formally, there are $R_2, R_2' \in \mathcal{R}$ with*

(1) *R_2' is closer to R_1 than R_2;*
(2) *for each $S, S' \in \mathcal{S}$, we have $r_2(S) > r_2(S')$ and $r_2'(S) < r_2'(S')$ if and only if $r_2(S) = r_2'(S') = r_2(S') + 1$ and $r_2'(S) = r_2(S') = r_2'(S') - 1$;*
(3) *for $u_1 \in \mathcal{U}$ representing R_1, we have*
$$\sum_{x\in\varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) < \sum_{x'\in\varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1').$$

Then,

- ***(i)*** *$P(R_1, R_2) \subseteq P(R_1, R_2')$. By contradiction, suppose that there is $x \in \mathcal{X}$ such that $x \in P(R_1, R_2)$ and $x \notin P(R_1, R_2')$. As preferences are strict and $|N| = 2$, there is $x' \in P(R_1, R_2) \cap P(R_1, R_2')$ such that $r_1(x_1') > r_1(x_1)$, $r_2(x_2') < r_2(x_2)$, and $r_2'(x_2') > r_2'(x_2)$. By Lemma 1, $r_2((x_2')_c) > r_2((x_2)_c)$ and $r_2'((x_2)_c) < r_2'((x_2')_c)$. As $|N| = 2$, we have*

$(x_2')_c = x_1'$ and $(x_2)_c = x_1$. *Thus,* $r_1(x_1') > r_1(x_1)$, $r_2(x_1') > r_2(x_1)$, *and* $r_2'(x_1') < r_2'(x_1)$, *contradicting* **(1)**.

- ***(ii)*** $B(R_1, R_2) = B(R_1, R_2')$. *By contradiction, suppose* $B(R_1, R_2) \neq B(R_1, R_2')$. *Then,*
Case 1: *there is* $x \in \mathcal{X}$ *such that* $x \in B(R_1, R_2)$ *and* $x \notin B(R_1, R_2')$. *Thus,* $r_1(x_1) \geq 2^{K-1}$, $r_2(x_2) \geq 2^{|K|-1}$, *and* $r_2'(x_2) < 2^{|K|-1}$. *Thus, by* **(2)**, *there is* $y \in \mathcal{X}$ *with* $r_2(y_2) < r_2(x_2)$ *and* $r_2'(y_2) > r_2'(x_2)$ *such that* $r_2(x_2') = r_2'(y_2) = r_2(y_2) + 1$ *and* $r_2'(x_2') = r_2(y_2) = r_2'(y_2) - 1$. *Suppose* $y \in B(R_1, R_2')$. *By Lemma 1,* $r_2((y_2)_c) > r_2((x_2)_c)$ *and* $r_2'((y_2)_c) < r_2'((x_2)_c)$. *As* $|N| = 2$, *we have* $(y_2)_c = y_1$ *and* $(x_2)_c = x_1$. *Thus, by* **(1)**, $r_1(y_1) < r_1(x_1)$. *By* **(2)**, $B(R_1, R_2) \backslash B(R_1, R_2') = \{x\}$ *and* $B(R_1, R_2') \backslash B(R_1, R_2) = \{y\}$. *Thus, for each* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $\sum_{x \in \varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) > \sum_{x' \in \varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1')$, *contradicting* **(3)**. *Thus,* $y \notin B(R_1, R_2')$. *By* **(2)**, $B(R_1, R_2) \backslash B(R_1, R_2') = \{x\}$ *and* $B(R_1, R_2') \backslash B(R_1, R_2) = \emptyset$. *Thus, for each* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $\sum_{x \in \varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) > \sum_{x' \in \varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1')$, *contradicting* **(3)**.
Case 2: *there is* $x \in \mathcal{X}$ *such that* $x \notin B(R_1, R_2)$ *and* $x \in B(R_1, R_2')$. *Thus,* $r_1(x_1) \geq 2^{K-1}$, $r_2(x_2) < 2^{|K|-1}$, *and* $r_2'(x_2) \geq 2^{|K|-1}$. *Thus, by* **(2)**, *there is* $y \in \mathcal{X}$ *with* $r_2(y_2) > r_2(x_2)$ *and* $r_2'(y_2) < r_2'(x_2)$ *such that* $r_2(x_2') = r_2'(y_2) = r_2(y_2) - 1$ *and* $r_2'(x_2') = r_2(y_2) = r_2'(y_2) + 1$. *By Lemma 1,* $r_2((y_2)_c) < r_2((x_2)_c)$ *and* $r_2'((y_2)_c) > r_2'((x_2)_c)$. *As* $|N| = 2$, *we have* $(y_2)_c = y_1$ *and* $(x_2)_c = x_1$. *Thus, by* **(1)**, $r_1(y_1) > r_1(x_1)$. *Thus, as* $r_1(x_1) \geq 2^{K-1}$, $r_2'(x_2) \geq 2^{|K|-1}$, *and* $r_2'(x_2') = r_2(y_2)$, *we have* $y \in B(R_1, R_2)$. *By* **(2)**, $B(R_1, R_2) \backslash B(R_1, R_2') = \{y\}$ *and* $B(R_1, R_2') \backslash B(R_1, R_2) = \{x\}$. *Thus, for each* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $\sum_{x \in \varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) > \sum_{x' \in \varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1')$, *contradicting* **(3)**.

As ***(i)*** *and* ***(ii)*** *hold, and as* $PB(R_1, R_2) \neq PB(R_1, R_2')$, *there is* $x \in B(R_1, R_2) = B(R_1, R_2')$ *such that* $x \notin P(R_1, R_2)$ *and* $x \in P(R_1, R_2')$. *Thus, as preferences are strict and* $|N| = 2$, *there is* $x' \in PB(R_1, R_2) \cap PB(R_1, R_2')$ *such that* $r_1(x_1') > r_1(x_1)$, $r_2(x_2') > r_2(x_2)$, *and* $r_2'(x_2') < r_2'(x_2)$. *Thus, for each* $u_1 \in \mathcal{U}$ *representing* R_1, *we have* $\sum_{x \in \varphi^M(R_1,R_2)} \frac{1}{|\varphi^M(R_1,R_2)|} u_1(x_1) > \sum_{x' \in \varphi^M(R_1,R_2')} \frac{1}{|\varphi^M(R_1,R_2')|} u_1(x_1')$, *contradicting* **(3)**.
However, φ *violates* conditional no-envy. *Indeed, let* $K = \{a, b, c\}$ *and* $R = (R_1, R_2) \in \mathcal{R}^N$ *be as in Figure 1.2. Let* $x = (\{a\}, \{b, c\})$ *and* $x' = (\{a, c\}, \{b\}) \in \mathcal{X}$. *Clearly,* $\varphi(R) = \{x, x'\}$. *Also,* $F(R) = PF(R) = \{x'\}$. *Thus,* $\varphi(R) \nsubseteq F(R)$.

Statement 5: *Suppose that there is $i \in N$ such that for each $R \in \mathcal{R}^N$, we have $\varphi(R) = \{x \in \varphi^M(R) : \text{for each } x' \in \varphi^M(R), \text{ we have } r_i(x'_i) \leq r_i(x_i)\}$. Clearly, φ is a subcorrespondence of φ^M. Thus, by Theorem 3.1, φ satisfies* conditional no-envy *and the* identical-preferences lower bound. *As it never uses objects' names, it satisfies* neutrality. *Also, it satisfies* preference-monotonicity. *Indeed, in the proof of Theorem 3.1, there are two cases, where φ may differ from φ^M.*

Case 1: *In Step 3, when $\underline{r}(x', (R_1, R'_2)) = \underline{r}(x, (R_1, R'_2)) = \underline{r}(x, (R_1, R_2)) = \underline{r}(x', (R_1, R_2))$ implies $\varphi^M(R_1, R_2) = \{x, x'\} = \varphi^M(R_1, R'_2)$. First, suppose $i = 1$. By Step 1, $\underline{r}(x', (R_1, R_2)) = r_2(x'_2)$. Thus, as $\underline{r}(x', (R_1, R_2)) = \underline{r}(x, (R_1, R_2))$ and $|N| = 2$, we have $\underline{r}(x, (R_1, R_2)) = r_1(x_1)$. By assumption, $r_1(x'_1) > r_1(x_1)$. Thus, $\varphi(R) = \{x'\}$ and $x \neq x'$, contradicting $x \in \varphi(R)$. Second, suppose $i = 2$. By Step 2, $\underline{r}(x, (R_1, R'_2)) = r_1(x_1)$. Thus, as $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$ and $|N| = 2$, we have $\underline{r}(x', (R_1, R'_2)) = r'_2(x'_2)$. As $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$, $\underline{r}(x, (R_1, R'_2)) = r_1(x_1)$, $x \neq x'$, we have $r'_2(x_2) > r'_2(x'_2)$. Thus, $\varphi(R_1, R'_2) = \{x\}$, contradicting $x' \in \varphi(R_1, R'_2)$.*

Case 2: *In Step 4, when $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x, (R_1, R_2)) = \underline{r}(y, (R_1, R_2)) = \underline{r}(x', (R_1, R'_2))$ implies $\varphi^M(R_1, R_2) = \{x, y\}$ and $\varphi^M(R_1, R_2) = \{x, x'\}$. First, suppose $i = 1$. By **(ii)** of Step 4, $\underline{r}(y, (R_1, R_2)) = r_2(y_2)$. Thus, as $\underline{r}(y, (R_1, R_2)) = \underline{r}(x, (R_1, R_2))$ and $|N| = 2$, we have $\underline{r}(x, (R_1, R_2)) = r_1(x_1)$. By **(i)** of Step 4, $r_1(y_1) > r_1(x_1)$. Thus, $\varphi(R) = \{y\}$ and $x \neq y$, contradicting $x \in \varphi(R)$. Second, suppose $i = 2$. By Step 2, $\underline{r}(x, (R_1, R'_2)) = r_1(x_1)$. Thus, as $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$ and $|N| = 2$, we have $\underline{r}(x', (R_1, R'_2)) = r(x'_2, R'_2)$. As $\underline{r}(x, (R_1, R'_2)) = \underline{r}(x', (R_1, R'_2))$, $\underline{r}(x, (R_1, R'_2)) = r_1(x_1)$, $x \neq x'$, we have $r'_2(x_2) > r'_2(x'_2)$. Thus, $\varphi(R_1, R'_2) = \{x\}$, contradicting $x' \in \varphi(R_1, R'_2)$.*

In Cases 1 and 2, we obtain a contradiction. Thus, the proof of Theorem 3.1 also holds for φ. However, as φ uses i's name, it violates anonymity. ∎

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Chapter 2

Characterizations of Pareto-efficient, fair, and strategy-proof allocation rules in queueing problems

with Çağatay Kayı[1]

Abstract: A set of agents with different waiting costs have to receive a service of equal length of time one after the other. We may set up monetary transfers to compensate the agents who have to wait. We identify the only rule that satisfies *Pareto-efficiency*, a weak equity axiom as *equal treatment of equals in welfare* or *symmetry*, and *strategy-proofness*. We prove that it satisfies *no-envy* and *anonymity*. Furthermore, we prove that the presence of indivisibilities implies a dilemma between rules' *anonymity* and *single-valuedness*.

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2.1 Introduction

A set of agents simultaneously arrive at a service facility that can only serve one agent at a time. Agents require service for the same length of time. The waiting cost may vary from one agent to the other. Efficiency requires to form a queue minimizing the total waiting cost. Equity requires to compensate

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agents having to wait. However, as waiting costs may not be known by the planner, agents may have no incentive to reveal them.

Our objective is to identify solutions to such *queueing problems* that satisfy the following axioms. First is *Pareto-efficiency.* This axiom can be decomposed into two distinct axioms, namely *Pareto-efficiency of queues*, i.e., agents should be served in decreasing order of their waiting costs, and *balancedness*, i.e., transfers should sum up to zero.

Second, agents with equal waiting costs should be treated equally. As agents cannot be served simultaneously, it impossible to give two agents with equal waiting costs equal bundles. However, using monetary transfers, we can give them bundles between which they are indifferent. Selecting more than one queue, we can give them the possibility to be served at the same position. Thus, we require *equal treatment of equals in welfare*, i.e., agents with equal waiting costs should have equal welfares, and *symmetry*, i.e., agents with equal waiting costs should be treated symmetrically.

Both axioms are important that they are necessary conditions for axioms embodying further basic equity properties. The former is implied by *no-envy*, i.e., no agent should prefer another agent's bundle to her own. The latter is implied by *anonymity*, i.e., agents' names should not matter. *Single-valued* rules may satisfy *equal treatment of equals in welfare*, but none satisfies *symmetry.* Thus, because agents cannot be served simultaneously, *anonymity* is possible if and only if we allow multi-valuedness.

Third is *strategy-proofness*, i.e., each agent should find revealing her unit waiting cost at least as desirable as misrepresenting it. As explained, the presence of indivisibilities and equity require to allow rules to select more than one allocation. We assume that an agent finds revealing her unit waiting cost at least as desirable as misrepresenting it if she finds (i) the worst bundle she may receive when revealing her unit waiting cost at least as desirable as the worst bundle she may receive when misrepresenting it and (ii) the best bundle she may receive when revealing her unit waiting cost at least as desirable as the best bundle she may receive when misrepresenting it.

We identify the only allocation *rule* that satisfies these axioms on the domain of linear preferences over positions and transfers. For each problem, it selects all *Pareto-efficient* queues and sets transfers considering each pair of agents in turn, making each agent in the pair pay the cost she imposes on the pair, and distributing the sum of these two payments equally among the others. We refer to it as the *Largest Equally Distributed Two-by-Two Pivotal rule.* Furthermore, we prove that it satisfies *no-envy*, *anonymity*, and stronger incentive compatibility properties. In particular, it is such that each agent always finds any bundle she receives when revealing her unit waiting cost at least as desirable as any bundle she receives when misrepresenting it.

In general social choice problems, weak basic equity axioms as *equal treatment of equals in welfare* or *symmetry* are incompatible with *strategy-proofness* (Gibbard, 1973, Satterthwaite, 1975). Moreover, even if there is hope for rules to satisfy these axioms on more restricted problems, these violate *Pareto-efficiency*. Thus, our results allows us to identify the particular allocation problems we study as being among the few, in which *Pareto-efficiency*, equity, and *strategy-proofness* are compatible.

The other exceptions are as follows. In economies with a public good chosen in an interval over which the agents have continuous and single-peaked preferences, the Generalized Condorcet rules are characterized by *Pareto-efficiency, anonymity*, and *strategy-proofness* (Moulin, 1980).[2] In economies with infinitely divisible private goods over which the agents have continuous and single-peaked preferences, the Uniform rule is characterized by *Pareto-efficiency, equal treatment of equals in welfare*, and *strategy-proofness*, as well as *Pareto-efficiency, symmetry*, and *strategy-proofness* (Sprumont, 1991, Ching, 1994).[3] In economies with infinitely divisible private goods produced by a linear technology, the Equal Income Walrasian rule is characterized by *Pareto-efficiency, equal treatment of equals in welfare*, and *strategy-proofness* (Maniquet and Sprumont, 1999). The allocation problems we study distinguish themselves from the former ones. They involve indivisibilities and hence they call for an efficient, fair, and *strategy-proof* solution that is different from the former ones, in particular that is not *single-valued.*

The literature on queueing problems is divided in two groups of papers. The first group focuses on the identification of rules satisfying equity axioms relative to changes in the set of agents or waiting costs, in addition to the efficiency and equity axioms we impose (Maniquet, 2003, Chun, 2004a, Chun, 2004b, Katta and Sethuraman, 2005). On the domain of linear preferences in positions and transfers, only rules that select *Pareto-efficient* queues and set each agent's transfer equal to the Shapley value of some associated coalitional game, satisfy these axioms (Maniquet, 2003, Chun, 2004a, Katta and Sethuraman, 2005). Also, there are rules that satisfy *Pareto-efficiency* and *no-envy* (Chun, 2004b, Katta and Sethuraman, 2005). However, none satisfies equity axioms relative to changes in waiting costs (Chun, 2004b).

[2]For an a priori fixed list of $n-1$ parameters in the interval, the Generalized Condorcet rule associated with these parameters chooses for each problem, the median of these parameters and the agents' peaks, where n is the number of agents.

[3]The Uniform rule allocates goods among agents as follows. If the sum of the agents' peaks is greater or equal to the amount of goods available, an agent receives her peak if it is smaller than a common bound, otherwise she receives this common bound, which is chosen so the allocation is feasible. If the sum of the agents' peaks is less or equal to the amount of goods available, she receives her peak if it is greater than a common bound, otherwise she receives this common bound, which is chosen so the allocation is feasible.

The second group focuses on the identification of necessary and sufficient conditions for the existence of rules satisfying *Pareto-efficiency* and *strategy-proofness* on the domain of linear preferences in transfers. For such queueing problems, like for any public decision making problem in which the agents have additively separable preferences with respect to transfers, there are rules that satisfy *Pareto-efficiency of queues* and *strategy-proofness* (Groves, 1973). Also, like for any public decision making problem in which preference profiles are convex and hence smoothly connected, only these rules satisfy these properties (Holmström, 1979).[4,5]

These rules however set transfers that need not sum to zero and hence do not satisfy *balancedness* (Green and Laffont, 1977). Thus, unless we further restrict the domain, *Pareto-efficiency* and *strategy-proofness* are incompatible. Combinatory and independence conditions over the structure of waiting costs are necessary and sufficient for these axioms to be compatible (Mitra and Sen, 1998, Mitra, 2001).[6] For instance, if preferences are linear over positions and transfers, there are rules that satisfy *Pareto-efficiency* and *strategy-proofness* (Suijs, 1996, Mitra and Sen, 1998).

To prove our results, we begin with *single-valued* rules. Studying these, then allowing multi-valuedness helps understand the role of each axiom. We identify the class of *single-valued* rules that satisfy *Pareto-efficiency of queues* and *strategy-proofness*. In so doing, we illustrate the rationale behind the characterization that holds in general public decision making problems (Green and Laffont, 1977, Holmström, 1979). We prove that only *single-valued* subcorrespondences of the Largest Equally Distributed Two-by-two Pivotal rule satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. Finally, we prove that these rules satisfy *no-envy*.

We continue by extending these results to non-*single-valued* rules. Furthermore, we prove that only the Largest Equally Distributed Two-by-two Pivotal rule satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Also, we prove this rule satisfies *anonymity*. Thus, as the Largest Equally Distributed Two-by-two Pivotal rule is the union of all the rules that satisfy

[4] Preferences are smoothly connected if for any two profiles in the domain, there is a differentiable deformation of one profile into the other that is also in the domain.

[5] This characterization also holds on the universal domain of preferences (Green and Laffont, 1977).

[6] The combinatory condition requires each agent to always have her $n-1th$ order difference of waiting costs at the first position in the queue equal to zero, where n is the number of agents. The independence condition requires that if an agent precedes another in a *Pareto-efficient* queue and a third agent leaves, then the former should still precedes the other in a *Pareto-efficient* queue for the reduced problem. These conditions are still necessary and sufficient for the existence rules that satisfy *Pareto-efficiency* and *strategy-proofness* in problems with unequal processing times (Mitra, 2002).

Pareto-efficiency, *equal treatment of equals in welfare*, and *strategy-proofness* and as *anonymity* implies *symmetry*, we prove that only this rule satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, *symmetry*, and *strategy-proofness*.

The intuition for our results is simple. Any rule can be obtained by selecting the appropriately chosen queues and setting each agent's transfer equal to the cost she imposes on the others plus an appropriately chosen amount. By *Pareto-efficiency*, a desirable rule should select *Pareto-efficient* queues. Also, as the costs agents impose on the others are always strictly positive (except for the last agent in the queue), it should redistribute the sum of these costs. By equity, it should select all *Pareto-efficient* queues and it should redistribute this sum fairly. By *strategy-proofness*, it should redistribute this sum in such a way that each agent's share only depends on the others' waiting costs.

This is exactly what the Largest Equally Distributed Two-by-two Pivotal rule does. It selects all *Pareto-efficient* queues (so it is efficient and fair). It sets each agent's transfer considering each pair of agents in turn, making each agent in the pair pay the cost she imposes on the pair, and distributing the sum of these two payments (so it is efficient) equally (so it is fair) among the others (so it is *strategy-proof*).

In Section 2.2, we formally introduce the model. In Section 2.3, we define the axioms we impose on rules. In Section 2.4, we identify the only rule that satisfies these axioms. Also, we prove that it satisfies further basic axioms. Finally, we give concluding remarks.

2.2 Model

There is a finite set of agents N. Each agent $i \in N$ may consume a *position* $\sigma_i \in \mathbb{N}$ in a queue and a positive or negative *transfer* $t_i \in \mathbb{R}$. Preferences are linear in position and transfer. Let $\theta_i \in \mathbb{R}_{++}$ be the *unit waiting cost of* $i \in N$. If i is served σ_i-th, her *total waiting cost* is $(\sigma_i - 1)\theta_i$. Her preferences can be represented by the following function: for each $(\sigma_i, t_i) \in \mathbb{N} \times \mathbb{R}$, we have $u(\sigma_i, t_i; \theta_i) = -(\sigma_i - 1)\theta_i + t_i$. For each $(\sigma_i, t_i) \in \mathbb{N} \times \mathbb{R}$, we use the following notational shortcut. For $\theta_i \in \mathbb{R}_{++}$, let $u_i(\sigma_i, t_i) \equiv u(\sigma_i, t_i; \theta_i)$; for $\theta'_i \in \mathbb{R}_{++}$, let $u'_i(\sigma_i, t_i) \equiv u(\sigma_i, t_i; \theta'_i)$; for $\theta''_i \in \mathbb{R}_{++}$, let $u''_i(\sigma_i, t_i) \equiv u(\sigma_i, t_i; \theta''_i)$; and so on. We do not study effects of changes in the set of agents. Thus, for simplicity, a (queueing) *problem* is a list $\theta \equiv (\theta_i)_{i\in N} \in \mathbb{R}^N_{++}$.

An *allocation* is a pair $(\sigma, t) \equiv (\sigma_i, t_i)_{i\in N} \in \mathbb{N}^N \times \mathbb{R}^N$ such that (i) for each pair of agents, their positions in queue σ differ, i.e., for each $\{i, j\} \subseteq N$ with $i \neq j$, we have $\sigma_i \neq \sigma_j$, and (ii) the sum of transfers t is non-positive, i.e., $\sum_{i\in N} t_i \leq 0$. Let Z be the set of all allocations. An (allocation) *rule* φ

is a correspondence that associates with each problem $\theta \in \mathbb{R}^N_{++}$ a non-empty set of allocations $\varphi(\theta) \subseteq Z$.

Let $\theta \in \mathbb{R}^N_{++}$ and $(\sigma, t) \in Z$. The *total waiting cost of* $S \subseteq N$ is $\sum_{i \in S}(\sigma_i - 1)\theta_i$. Let $P_i(\sigma) \equiv \{j \in N | \sigma_j < \sigma_i\}$ be the set of all agents *preceding* $i \in N$ in σ. Let $F_i(\sigma) \equiv \{j \in N | \sigma_j > \sigma_i\}$ be the set of all agents *following* $i \in N$ in σ. Let $B_{ij}(\sigma) \equiv \{l \in N | \min\{\sigma_i, \sigma_j\} < \sigma_l < \max\{\sigma_i, \sigma_j\}\}$ be the set of all agents served *between* $i \in N$ and $j \in N$ in σ.[7] Let σ^{-i} be such that for each $l \in P_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l$ and for each $l \in F_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l - 1$. The *cost* $i \in N$ *imposes on* $S \subseteq N$ is $(\sum_{l \in S}(\sigma_l - 1)\theta_l) - (\sum_{l \in S \setminus \{i\}}(\sigma^{-i}_l - 1)\theta_l) = \sum_{l \in S \cap F_i(\sigma)} \theta_l$. Thus, the cost an agent imposes on the others is always equal to the cost each of her followers incurs by waiting one more unit of time.

2.3 Properties of rules

In this section, we define the axioms we impose on rules. Let φ be a rule.

Efficiency is standard. There should be no allocation that each agent finds at least as desirable as a selected allocation and at least one agent prefers. Formally,

Pareto-efficiency: *For each* $\theta \in \mathbb{R}^N_{++}$ *and each* $(\sigma, t) \in \varphi(\theta)$, *there is no* $(\sigma', t') \in Z$ *such that for each* $i \in N$, *we have* $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ *and for at least one* $j \in N$, *we have* $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$.

Remark that $(\sigma, t) \in Z$ is *Pareto-efficient for* $\theta \in \mathbb{R}^N_{++}$ if and only if *(i)* σ is *Pareto-efficient for* θ, i.e., for each $\sigma' \in \mathbb{N}^N$, we have $\sum_{i \in N}(\sigma'_i - 1)\theta_i \geq \sum_{i \in N}(\sigma_i - 1)\theta_i$, and *(ii)* t is *balanced for* θ, i.e., $\sum_{i \in N} t_i = 0$. Thus, queues minimize the set of agents' total waiting cost independently of transfers. Furthermore, *Pareto-efficient* queues are unique up to any permutation of agents with equal unit waiting costs following one another. Formally, let $\Sigma^*(\theta)$ be the set of all *Pareto-efficient* queues for θ. For each $\theta \in \mathbb{R}^N_{++}$ and each $(\sigma, t) \in Z$, we have $\sigma \in \Sigma^*(\theta)$ if and only if for each $\{i, j\} \subset N$ with $i \neq j$, if $\sigma_i < \sigma_j$, then $\theta_i \geq \theta_j$. Following our first remark, we decompose *Pareto-efficiency* into two axioms.

Pareto-efficiency of queues: *For each* $\theta \in \mathbb{R}^N_{++}$ *and each* $(\sigma, t) \in \varphi(\theta)$, *we have* $\sigma \in \Sigma^*(\theta)$.

Balancedness: *For each* $\theta \in \mathbb{R}^N_{++}$ *and each* $(\sigma, t) \in \varphi(\theta)$, *we have* $\sum_{i \in N} t_i = 0$.

[7] For each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in Z$, and each $\{i, j\} \subseteq N$, we have $B_{ij}(\sigma) = B_{ji}(\sigma)$.

Equity requires to treat agents with equal unit waiting costs equally. As agents cannot be served simultaneously, it is impossible to give two agents with equal unit waiting costs equal bundles. We require agents with equal unit waiting costs to have equal welfares and to be treated symmetrically. Formally,

Equal treatment of equals in welfare: *For each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in \varphi(\theta)$, and each $\{i, j\} \subset N$ with $i \neq j$ and $\theta_i = \theta_j$, we have $u_i(\sigma_i, t_i) = u_j(\sigma_j, t_j)$.*

Symmetry: *For each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in \varphi(\theta)$, and each $\{i, j\} \subset N$ with $i \neq j$ and $\theta_i = \theta_j$, if $(\sigma', t') \in Z$ such that $(\sigma'_i, t'_i) = (\sigma_j, t_j)$, $(\sigma'_j, t'_j) = (\sigma_i, t_i)$, and for each $l \in N \backslash \{i, j\}$, we have $(\sigma'_l, t'_l) = (\sigma_l, t_l)$, then $(\sigma', t') \in \varphi(\theta)$.*

Both axioms are important that they are implied by axioms embodying further basic equity properties. *Equal treatment of equals in welfare* is necessary for no agent to prefer another agent's bundle to her own. *Symmetry* is necessary for agents' names not to matter. This second axiom requires that if we permute agents' preferences, we should permute the selected bundles accordingly. Formally, let Π be the set of all permutations on N. For $\pi \in \Pi$ and $\theta \in \mathbb{R}^N$, let $\pi(\theta) \equiv (\theta_{\pi(i)})_{i \in N}$ and $\pi(\sigma, t) \equiv (\sigma_{\pi(i)}, t_{\pi(i)})_{i \in N}$.

No-envy: *For each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in \varphi(\theta)$, and each $i \in N$, there is no $j \in N \backslash \{i\}$ such that $u_i(\sigma_j, t_j) > u_i(\sigma_i, t_i)$.*

Anonymity: *For each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in \varphi(\theta)$, and each $\pi \in \Pi$, we have $\pi(\sigma, t) \in \varphi(\pi(\theta))$.*

Single-valued and non-*single-valued* rules may satisfy *equal treatment of equals in welfare.* However, no *single-valued* rule satisfies *symmetry.* Thus, due to the presence of indivisibilities, we may require *anonymity* if and only if we allow multi-valuedness.

The last axiom is motivated by strategic considerations. The planner may not know agents' unit waiting costs. Thus, as agents may behave strategically when announcing them, neither efficiency nor equity may be attained. We require that each agent should find revealing her unit waiting cost at least as desirable as misrepresenting it.

As rules may select more than one allocation, we must compare welfare levels derived from non-empty sets of allocations. However, preferences are defined over positions and transfers. Thus, we assume the following. An agent prefers a subset of allocations to another if and only if (i) for each allocation in the latter, there is an allocation in the former that she finds at least as desirable and (ii) for each allocation in the former, there is an allocation in the latter that she does not prefer.[8] Formally, let $\mathcal{X}_i$ be the set of all non-empty sets of positions and transfers in $\mathbb{N} \times \mathbb{R}$ that $i \in N$ may consume. For $\theta_i \in \mathbb{R}_{++}$, let $R_i(\theta_i)$ be such that for each $\{X_i, X_i'\} \subseteq \mathcal{X}_i$, we have $X_i \; R_i(\theta_i) \; X_i'$ if and only if $\min_{(\sigma_i,t_i)\in X_i} u_i(\sigma_i, t_i) \geq \min_{(\sigma_i',t_i')\in X_i'} u_i(\sigma_i', t_i')$ and $\max_{(\sigma_i,t_i)\in X_i} u_i(\sigma_i, t_i) \geq \max_{(\sigma_i',t_i')\in X_i'} u_i(\sigma_i', t_i')$. For $\theta \in \mathbb{R}^N_{++}$, let $\theta_{-i} \equiv (\theta_l)_{l\in N\setminus\{i\}}$ be the list of the unit waiting costs of $N\setminus\{i\}$.

Strategy-proofness: *For each $\theta \in \mathbb{R}^N_{++}$, each $i \in N$, and each $\theta_i' \in \mathbb{R}_{++}$, if $X_i = \bigcup_{(\sigma,t)\in\varphi(\theta)}(\sigma_i, t_i)$ and $X_i' = \bigcup_{(\sigma,t)\in\varphi(\theta_i',\theta_{-i})}(\sigma_i, t_i)$, then $X_i \; R_i(\theta_i) \; X_i'$.*

We end this section with three remarks on this axiom. First, a *single-valued* rule φ satisfies *strategy-proofness* if and only if for each $\theta \in \mathbb{R}^N_{++}$, each $i \in N$, and each $\theta_i' \in \mathbb{R}_{++}$, if $(\sigma, t) = \varphi(\theta)$ and $(\sigma', t') = \varphi(\theta_i', \theta_{-i})$, then $u_i(\sigma_i, t_i) \geq u_i(\sigma_i', t_i')$.

Second, as e.g. in Pattanaik (1973), Dutta (1977), Thomson (1979), *strategy-proofness* requires each agent to find the worst bundle she may receive when she reveals her unit waiting cost at least as desirable as the worst bundle she may receive when she misrepresents it.

Third, it requires each agent to find the best bundle she may receive when she reveals her unit waiting cost at least as desirable as the best bundle she may receive when she misrepresents it. This second requirement is important that it prevents situations as follows. An agent may receive several bundles when misrepresenting her waiting unit cost that she all finds, but the worst, more desirable than the only bundle she receives when revealing it.

Furthermore, it is implied by further basic incentive compatibility properties. It is a necessary condition for rules to be implementable in undominated strategies by bounded mechanisms (Jackson, 1992) and for each agent not to find misrepresenting her unit waiting cost more desirable than revealing it via addition or deletion of allocations (Ching and Zhou, 2000).[9]

2.4 Characterizations

In this section, we identify the only rule that satisfies the axioms we impose on rules. Moreover, we prove that it satisfies *no-envy*, *anonymity*, and

[8] Determining how agents rank non-empty sets given their preferences over singletons has been studied in, e.g., Pattanaik (1973), Barberá (1977), Dutta (1977), Kelly (1977), Feldman (1979, 1980), Gärdenfors (1979), Thomson (1979), Ching and Zhou (2000), Duggan and Schwartz (2000), and Barberá, Dutta, and Sen (2001).

[9] Formally, an agent does not find misrepresenting her unit waiting cost more desirable as revealing it via addition or deletion of allocations if there is no $\theta \in \mathbb{R}^N_{++}$, $i \in N$, and $\theta_i' \in \mathbb{R}_{++}$ such that for $(\sigma', t') \in \varphi(\theta_i', \theta_{-i})\setminus\varphi(\theta)$, we have $u_i(\sigma_i', t_i') > \min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$ or for $(\sigma, t) \in \varphi(\theta)\setminus\varphi(\theta_i', \theta_{-i})$, we have $\max_{(\sigma',t')\in\varphi(\theta_i',\theta_{-i})} u_i(\sigma_i', t_i')$.

stronger incentive compatibility properties. In particular, it is such that each agent always finds any bundle she receives when revealing her unit waiting cost at least as desirable as any bundle she receives when misrepresenting it.

Therefore, we characterize the class of *single-valued* rules that satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* and we prove that these rules satisfy *no-envy* (Theorem 3). Then, we extend these results to rules that may select more than one allocation and we prove that one of these rules is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*, and that it satisfies *anonymity* (Theorem 4).[10]

2.4.1 *Single-valued* rules

We begin by proving that a *single-valued* rule satisfies *Pareto-efficiency of queues* and *strategy-proofness* if and only if for each problem, it is as follows (Theorem 1). It selects a *Pareto-efficient* queue. It sets each agent's transfer as prescribed in Groves (1973), i.e., equal to the total waiting cost of all other agents plus an amount only depending on these agents' unit waiting costs.

As the domain of preference profiles is convex, it is smoothly connected. Thus, this result follows from Holmström's (1979). However, the proof we give illustrates the rationale behind this characterization. Furthermore, it is used in the next subsection to generalize results that hold for *single-valued* rules to rules that may select more than one allocation.

The intuition is as follows. Suppose that was not the case and that if an agent with some waiting cost announces some other waiting cost, the change in her utility to be greater (smaller) than the efficiency loss. Then, there would always be a waiting cost for which she should receive the same position and transfer as with the latter (former) waiting cost and for which announcing the former (latter) waiting cost would increase her utility.

Formally, let $D \equiv \{d|$ for each $\theta \in \mathbb{R}^N_{++}$, we have $d(\theta) \in \Sigma^*(\theta)\}$. Let $H \equiv \{(h_i)_{i\in N}|$ for each $i \in N$, we have $h_i : \mathbb{R}^{N\setminus\{i\}}_{++} \rightarrow \mathbb{R}\}$. A *single-valued* rule φ is a *Groves rule* if and only if there are $d \in D$ and $h \in H$ such that for each $\theta \in \mathbb{R}^N_{++}$, we have $\varphi(\theta) = (\sigma, t) \in Z$ with $\sigma = d(\theta)$ and for each $i \in N$, we have $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$.

Theorem 1 *A* single-valued *rule satisfies* Pareto-efficiency of queues *and* strategy-proofness *if and only if it is a Groves rule.*

Proof.

[10]Extending Theorem 3 to non-*single-valued* rules, we prove that what holds for *single-valued* rules, holds for any rules. Thus, *single-valuedness* and Theorem 4 implies Theorem 3.

Let φ *be a* single-valued *rule.*

Part 1: If φ ***is a Groves rule, then it satisfies*** **Pareto-efficiency of queues** ***and*** **strategy-proofness**

Suppose that φ *is a Groves rule. Then,*

Pareto-efficiency of queues: *Suppose that* φ *is a Groves rule. Let* $\theta \in \mathbb{R}^N_{++}$ *and* $(\sigma, t) = \varphi(\theta)$. *By definition of a Groves rule, there is* $d \in D$ *such that* $\sigma = d(\theta) \in \Sigma^*(\theta)$.

Strategy-proofness: *Suppose that* φ *is a Groves rule. Let* $\theta \in \mathbb{R}^N_{++}$, $i \in N$, $\theta'_i \in \mathbb{R}_{++}$, $(\sigma, t) = \varphi(\theta)$, *and* $(\sigma', t') = \varphi(\theta'_i, \theta_{-i})$. *By definition of a Groves rule, there is* $d \in D$ *such that* $\sigma = d(\theta) \in \Sigma^*(\theta)$. *Also, there is* $h \in H$ *such that for each* $i \in N$, *we have* $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$ *and* $t'_i = -\sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)\theta_l + h_i(\theta_{-i})$. *By contradiction, suppose* $u_i(\sigma'_i, t'_i) > u_i(\sigma_i, t_i)$. *Then,* $-(\sigma'_i - 1)\theta_i - \sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)\theta_l + h_i(\theta_{-i}) > -(\sigma_i - 1)\theta_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$. *Thus,* $-\sum_{l \in N}(\sigma'_l - 1)\theta_l > -\sum_{l \in N}(\sigma_l - 1)\theta_l$, *contradicting* $\sigma \in \Sigma^*(\theta)$.

Part 2: If φ ***satisfies*** **Pareto-efficiency of queues** ***and*** **strategy-proofness*****, then it is a Groves rule.***

Suppose that φ *satisfies* Pareto-efficiency of queues *and* strategy-proofness. *By* Pareto-efficiency of queues, *for each* $\theta \in \mathbb{R}^N_{++}$, *if* $(\sigma, t) = \varphi(\theta)$, *then* $\sigma \in \Sigma^*(\theta)$. *Thus, there is* $d \in D$ *such that for each* $\theta \in \mathbb{R}^N_{++}$, *if* $(\sigma, t) = \varphi(\theta)$, *then* $\sigma = d(\theta)$. *In what follows, we prove that there is* $h \in H$ *such that for each* $\theta \in \mathbb{R}^N_{++}$, *if* $(\sigma, t) = \varphi(\theta)$, *then for each* $i \in N$, *we have* $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$.
Let $(g_i : \mathbb{R}^N_{++} \to \mathbb{R})_{i \in N}$ *be the list of the real-valued functions such that* $\boldsymbol{(i)}$ *for each* $\theta \in \mathbb{R}^N_{++}$, *if* $(\sigma, t) = \varphi(\theta)$, *then for each* $i \in N$, *we have* $t_i = -\sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + g_i(\theta)$. *By contradiction, suppose that for* $\theta \in \mathbb{R}^N_{++}$, $i \in N$, *and* $\theta'_i \in \mathbb{R}_{++}$, *we have* $\boldsymbol{(ii)}$ $g_i(\theta) - g_i(\theta'_i, \theta_{-i}) > 0$. *(The symmetric case is immediate.) Let* $(\sigma, t) = \varphi(\theta)$ *and* $(\sigma', t') = \varphi(\theta'_i, \theta_{-i})$.
By strategy-proofness,

- $u_i(\sigma_i, t_i) - u_i(\sigma'_i, t'_i) \geq 0$. *Thus, by* $\boldsymbol{(i)}$, $[-(\sigma_i - 1)\theta_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + g_i(\theta)] - [-(\sigma'_i - 1)\theta_i - \sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)\theta_l + g_i(\theta'_i, \theta_{-i})] \geq 0$. *Thus,* $g_i(\theta) - g_i(\theta'_i, \theta_{-i}) \geq (\sigma_i - \sigma'_i)\theta_i + \sum_{l \in N\setminus\{i\}}(\sigma_l - \sigma'_l)\theta_l$.

- $u'_i(\sigma'_i, t'_i) - u_i(\sigma'_i, t'_i) \geq 0$. *Thus, by* $\boldsymbol{(i)}$, $[-(\sigma'_i - 1)\theta'_i - \sum_{l \in N\setminus\{i\}}(\sigma'_l - 1)\theta_l + g_i(\theta'_i, \theta_{-i})] - [-(\sigma_i - 1)\theta'_i - \sum_{l \in N\setminus\{i\}}(\sigma_l - 1)\theta_l + g_i(\theta)] \geq 0$. *Thus,* $g_i(\theta'_i, \theta_{-i}) - g_i(\theta) \geq (\sigma'_i - \sigma_i)\theta'_i + \sum_{l \in N\setminus\{i\}}(\sigma'_l - \sigma_l)\theta_l$.

Thus,

$$\textbf{\textit{(iii)}}\ (\sigma_i - \sigma_i')\theta_i' + \textstyle\sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l')\theta_l \geq g_i(\theta) - g_i(\theta_i', \theta_{-i}) \geq \\ (\sigma_i - \sigma_i')\theta_i + \textstyle\sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l')\theta_l.$$

Let us rewrite this expression. By Pareto-efficiency of queues, *for each* $S \subseteq N$, *if for each* $\{k, k'\} \subseteq S$ *with* $k \neq k'$, *we have* $\theta_k = \theta_{k'}$ *and there is no* $k'' \in N\backslash S$ *such that* $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$, *then* $-\sum_{l\in S}(\sigma_l - 1)\theta_l = -\sum_{l\in S}(\sigma_l' - 1)\theta_l$. *Also, there is* $j \in N$ *such that* $\sigma_j = \sigma_i'$. *Thus,* $\sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l')\theta_l = -sign(\sigma_i - \sigma_i')\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l$.[11] *Thus, we may rewrite* ***(iii)*** *as*

$$\textbf{\textit{(iv)}}\ (\sigma_i - \sigma_i')\theta_i' - sign(\sigma_i - \sigma_i')\textstyle\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l \geq g_i(\theta) - g_i(\theta_i', \theta_{-i}) \geq \\ (\sigma_i - \sigma_i')\theta_i - sign(\sigma_i - \sigma_i')\textstyle\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l.$$

Then, distinguish three cases.

Case 1: $(\sigma_i - \sigma_i') = 0$. *Then,* $-sign(\sigma_i - \sigma_i')\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l = 0$. *Thus, by* ***(iv)***, $g_i(\theta) - g_i(\theta_i', \theta_{-i}) = 0$, *contradicting* ***(ii)***.

Case 2: $|\sigma_i - \sigma_i'| = 1$. *Suppose* $\theta_i' > \theta_i$. *(The symmetric case is immediate.) Then,* $(\sigma_i - \sigma_i') = 1$ *and* $-sign(\sigma_i - \sigma_i')\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l = -\theta_j$. *Thus, by* ***(iv)***, $\theta_i' - \theta_j \geq g_i(\theta) - g_i(\theta_i', \theta_{-i}) \geq \theta_i - \theta_j$. *Thus, as* $\theta_i' > \theta_i$, *either* $\theta_i' - \theta_j > g_i(\theta) - g_i(\theta_i', \theta_{-i})$ *or* $g_i(\theta) - g_i(\theta_i', \theta_{-i}) > \theta_i - \theta_j$. *Suppose* $g_i(\theta) - g_i(\theta_i', \theta_{-i}) > \theta_i - \theta_j$. *(The other case is also immediate.) Let* $\theta_i'' \in \mathbb{R}_{++}$ *be such that* ***(v)*** $g_i(\theta) - g_i(\theta_i', \theta_{-i}) > \theta_i'' - \theta_j > 0$. *Let* $(\sigma'', t'') = \varphi(\theta_i'', \theta_{-i})$. *By* ***(iv)*** *and* ***(v)***, $\theta_i' > \theta_i'' > \theta_j > \theta_i$. *Thus, by* Pareto-efficiency of queues, $\sigma_i'' = \sigma_i'$. *Thus,* $(\sigma_i - \sigma_i'') = (\sigma_i - \sigma_i') = 1$ *and* $\sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l'')\theta_l = \sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l')\theta_l = -\theta_j$. *Also, by the logic of Case 1,* $g_i(\theta_i'', \theta_{-i}) = g_i(\theta_i', \theta_{-i})$ *implying* $g_i(\theta) - g_i(\theta_i'', \theta_{-i}) = g_i(\theta) - g_i(\theta_i', \theta_{-i})$. *Thus, by* ***(v)***, $g_i(\theta) - g_i(\theta_i'', \theta_{-i}) > (\sigma_i - \sigma_i'')\theta_i'' + \sum_{l\in N\backslash\{i\}}(\sigma_l - \sigma_l'')\theta_l$. *Thus,* $-(\sigma_i - 1)\theta_i'' - \sum_{l\in N\backslash\{i\}}(\sigma_l - 1)\theta_l + g_i(\theta) > -(\sigma_i'' - 1)\theta_i'' - \sum_{l\in N\backslash\{i\}}(\sigma_l'' - 1)\theta_l + g_i(\theta_i'', \theta_{-i})$. *Thus, by* ***(i)***, $u_i''(\sigma_i, t_i) > u_i''(\sigma_i'', t_i'')$, *contradicting* strategy-proofness.

Case 3: $|\sigma_i - \sigma_i'| > 1$. *By the logic of Case 2, starting from* σ_i' *and the one closer to* σ_i *by one position and ending with the one closer to* σ_i' *by one position and* σ_i, *we obtain each time* $g_i(., \theta_{-i}) = g_i(., \theta_{-i})$. *Thus,* $g_i(\theta) = g_i(\theta_i', \theta_{-i})$, *contradicting* ***(ii)***. ■

The class of *single-valued* rules that are Groves rules, is large. We distinguish subclasses according to their h function. For instance, *Pivotal rules* are Groves rules associated with $h \in H$ such that for each $\theta \in \mathbb{R}^N_{++}$, if (σ, t) is selected, then for each $i \in N$, we have $h_i(\theta_{-i}) = \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)\theta_l$.[12]

[11] Let $sign : \mathbb{R} \to \{-1, 0, 1\}$ be such that for each $a \in \mathbb{R}$, we have $sign(a) = -1$ if and only if $a < 0$, $sign(a) = 0$ if and only if $a = 0$, and $sign(a) = 1$ if and only if $a > 0$.

[12] Pivotal rules are also known as *Clarke's rules* (Clarke, 1971).

By Theorem 1, a *single-valued* rule satisfies *Pareto-efficiency* and *strategy-proofness* if and only if it is a Groves rules and it satisfies *balancedness*. However, in two-agent problems, no *single-valued* rule that is a Groves rule, satisfies *balancedness* (Suijs, 1996). Thus, from now on, we focus on problems with more than two agents.

We now introduce another class of *single-valued* rules. For each problem, such a rule selects a *Pareto-efficient* queue and sets transfers considering each pair of agents in turn, making each agent in the pair pay what a Pivotal rule recommends for the reduced problem consisting in these two agents, and distributing the sum of these two payments equally among the others.

As there may be several *Pareto-efficient* queues for a problem, there are several such rules. Furthermore, for each problem and each selected *Pareto-efficient* queue, each agent's transfer is such that she pays the cost she imposes on each pair of agents she could be in and she receives $(1/n-2)th$ of the cost each other agent imposes on each pair of agents this agent is and she is not. Thus, each agent pays the unit waiting costs of all her followers and receives $(1/n-2)th$ of the unit waiting costs of all each other agent's followers, but her. Formally, let $n = |N|$.

Equally Distributed Two-by-two Pivotal (*EDTP*) rule: For each $\theta \in \mathbb{R}^N_{++}$, if $(\sigma, t) = \varphi(\theta)$, then $\sigma \in \Sigma^*(\theta)$ and for each $i \in N$, we have $t_i = -\sum_{l \in F_i(\sigma)} \theta_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} \theta_l$.

In Theorem 2, we prove that for each problem and each selected *Pareto-efficient* queue, the transfers of an Equally Distributed Two-by-two Pivotal rule can be obtained in three other ways. First, making each agent pay what a Pivotal rule recommends for the whole problem and giving each agent back $(1/n-2)th$ of what the others are paying. Second, giving each agent $(1/n-2)th$ of her predecessors' total waiting cost and making each agent pay $(1/n-2)th$ of her followers' waiting gain from not being last in the queue. Third, giving each agent $1/2$ of her predecessors' unit waiting cost and making each agent pay $1/2$ of her followers' unit waiting cost plus $(1/2(n-2))th$ of the difference in the unit waiting costs of each agent, but her and each of this agent's predecessors's, but her.

Theorem 2 *Let φ be a* single-valued *rule. Then, the following statements are equivalent.*

1. *φ is an Equally Distributed Two-by-two Pivotal rule.*

2. *φ is a Groves rule associated with $h \in H$ such that for each $\theta \in \mathbb{R}^N_{++}$, if $(\sigma, t) = \varphi(\theta)$, then for each $i \in N$, we have $h_i(\theta_{-i}) = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l$.*

3. *φ is such that for each $\theta \in \mathbb{R}^N_{++}$, if $(\sigma, t) = \varphi(\theta)$, then $\sigma \in \Sigma^*(\theta)$ and for each $i \in N$, we have $t_i = \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} \theta_l - \sum_{l \in F_i(\sigma)} \frac{(n-\sigma_l)}{(n-2)} \theta_l$ (Mitra and Sen, 1998, Mitra, 2001).*

4. *φ is such that for each $\theta \in \mathbb{R}^N_{++}$, if $(\sigma, t) = \varphi(\theta)$, then $\sigma \in \Sigma^*(\theta)$ and for each $i \in N$, we have $t_i = \sum_{l \in P_i(\sigma)} \frac{\theta_l}{2} - \sum_{l \in F_i(\sigma)} \frac{\theta_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \frac{\theta_k - \theta_l}{2(n-2)}$ (Suijs, 1996).*

Proof.

Let $\theta \in \mathbb{R}^N_{++}$, $(\sigma, t) = \varphi(\theta)$, and $i \in N$. Let $h \in H$ be as in Statement 2. Then,

$$
\begin{aligned}
t_i &= -\sum_{l \in F_i(\sigma)} \theta_l + \tfrac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} \theta_l \quad \textit{(1.)} \\
&= -\sum_{l \in F_i(\sigma)} \theta_l + \tfrac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l \\
&= -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1)\theta_l + \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l + \tfrac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l \quad \textit{(2.)} \\
&= -\sum_{l \in F_i \sigma} \theta_l + \tfrac{1}{n-2} \sum_{l \in P_i(\sigma)} (\sigma_l - 1)\theta_l + \tfrac{1}{(n-2)} \sum_{l \in F_i(\sigma)} (\sigma_l - 2)\theta_l \\
&= \sum_{l \in P_i(\sigma)} \tfrac{(\sigma_l - 1)}{(n-2)} \theta_l + \sum_{l \in F_i(\sigma)} \tfrac{(\sigma_l - 2) - (n-2)}{(n-2)} \theta_l \\
&= \sum_{l \in P_i(\sigma)} \tfrac{(\sigma_l - 1)}{(n-2)} \theta_l - \sum_{l \in F_i(\sigma)} \tfrac{(n - \sigma_l)}{(n-2)} \theta_l \quad \textit{(3.)} \\
&= \sum_{l \in P_i(\sigma)} \tfrac{\theta_l}{2} - \sum_{l \in F_i(\sigma)} \tfrac{\theta_l}{2} - \sum_{l \in N \setminus \{i\}} \tfrac{(n - 2\sigma_l)\theta_l}{2(n-2)} \\
&= \sum_{l \in P_i(\sigma)} \tfrac{\theta_l}{2} - \sum_{l \in F_i(\sigma)} \tfrac{\theta_l}{2} - \Big[\sum_{l \in N \setminus \{i\}} \tfrac{(n - \sigma_l - 1)\theta_l}{2(n-2)} - \sum_{l \in N \setminus \{i\}} \tfrac{(\sigma_l - 1)\theta_l}{2(n-2)}\Big] \\
&= \sum_{l \in P_i(\sigma)} \tfrac{\theta_l}{2} - \sum_{l \in F_i(\sigma)} \tfrac{\theta_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \tfrac{\theta_k - \theta_l}{2(n-2)}. \quad \textit{(4.)} \ \blacksquare
\end{aligned}
$$

Finally, we prove that only Equally Distributed Two-by-two Pivotal rules satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. Moreover, these rules satisfy *no-envy*. Thus, as *no-envy* implies *equal treatment of equals in welfare*, a *single-valued* rule satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* if and only if it is an Equally Distributed Two-by-two Pivotal rule.

Theorem 3 *Let φ be a* single-valued *rule.*

1. *If φ satisfies* Pareto-efficiency, equal treatment of equals in welfare, *and* strategy-proofness, *then it is an Equally Distributed Two-by-two Pivotal rule.*

2. *If φ is an Equally Distributed Two-by-two Pivotal rule, then it satisfies* Pareto-efficiency, no-envy, *and* strategy-proofness.

Proof.

Statement 1: *Suppose that φ satisfies the axioms of Theorem 3.1. Let*

$\theta \in \mathbb{R}^N_{++}$ and $(\sigma,t) = \varphi(\theta)$. By Pareto-efficiency, *$\sigma \in \Sigma^*(\theta)$. By Theorem 1,* Pareto-efficiency *and* strategy-proofness *imply that φ is a Groves rule, i.e., there is $(h_i)_{i\in N} \in H$ such that for each $i \in N$, we have $t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$. Let $(\gamma_i)_{i\in N} \in H$ be such that for each $i \in N$, we have $t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)\theta_l + \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)\theta_l + \gamma_i(\theta_{-i}) = -\sum_{l\in F_i(\sigma)}\theta_l + \gamma_i(\theta_{-i})$. In what follows, we prove by induction that for each $i \in N$, we have $\gamma_i(\theta_{-i}) = \frac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)\theta_l$. Thus, for each $i \in N$, we have $t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)\theta_l + \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)\theta_l$. Thus, by Theorem 2, for each $i \in N$, we have $t_i = -\sum_{l\in F_i(\sigma)}\theta_l + \frac{1}{(n-2)}\sum_{j\in N\backslash\{i\}}\sum_{l\in F_j(\sigma^{-i})}\theta_l$.*
Without loss of generality, suppose $N = \{1,2,...,n\}$ and $\theta_1 \geq \theta_2 \geq ... \geq \theta_n$. Let $i \in N$. For $S \subseteq N$, let $\theta_S \equiv (\theta_l)_{l\in S}$ be the list of the unit waiting costs of the members of $S \subseteq N$. Then,

***Basic step:** $(\theta_n,...,\theta_n)$.*
By Pareto-efficiency, *$\gamma_1(\theta_n,...,\theta_n) + ... + \gamma_n(\theta_n,...,\theta_n) = \frac{n(n-1)}{2}\theta_n$. By* equal treatment of equals in welfare, *$\gamma_1(\theta_n,...,\theta_n) = ... = \gamma_n(\theta_n,...,\theta_n)$. Thus, for each $i \in N$, we have $\gamma_i(\theta_n,...,\theta_n) = \frac{(n-1)}{2}\theta_n$.*

***Step 1:** $(\theta_1,\theta_n,...,\theta_n)$.*
By Pareto-efficiency, *$\gamma_1(\theta_n,...,\theta_n) + \gamma_2(\theta_1,\theta_n,...,\theta_n) + ... + \gamma_n(\theta_1,\theta_n,...,\theta_n) = \frac{(n-1)n}{2}\theta_n$. By Step 0, $\gamma_1(\theta_n,...,\theta_n) = \frac{(n-1)}{2}\theta_n$. By* equal treatment of equals in welfare, *$\gamma_2(\theta_1,\theta_n,...,\theta_n) = ... = \gamma_n(\theta_1,\theta_n,...,\theta_n)$. Thus, for each $i \in N\backslash\{1\}$, we have $\gamma_i(\theta_1,\theta_n,...,\theta_n) = \frac{(n-1)}{2}\theta_n$. This holds for each $j \in N\backslash\{n\}$. Thus, for each $i \in N$,*

- *if $i = j$, then $\gamma_i(\theta_n,...,\theta_n) = \frac{(n-1)}{2}\theta_n$;*
- *if $i \in N\backslash\{j\}$, then $\gamma_i(\theta_j,\theta_n,...,\theta_n) = \frac{(n-1)}{2}\theta_n$.*

...

***Step s (Induction step):** $(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n)$.*
By Pareto-efficiency, *$\gamma_1(\theta_2,\theta_3,...,\theta_s,\theta_n,...,\theta_n) + \gamma_2(\theta_1,\theta_3,...,\theta_s,\theta_n,...,\theta_n) + ... + \gamma_n(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n) = \sum_{l\in\{1,2,...,s\}}(\sigma_l - 1)\theta_l + \frac{(n-s)(n+s+1)}{2}\theta_n$. By Step $s-1$, for $i \in \{1,2,...,s\}$, we have $\gamma_i(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n) = \sum_{l\in\{1,2,...,s\}\backslash\{i\}}\frac{(\sigma_l^{\{1,2,...,s\}\backslash\{i\}}-1)}{(n-2)}\theta_l + \frac{(n-1-(s-1))(n-2+(s-1))}{2(n-2)}\theta_n$. By* equal treatment of equals in welfare, *$\gamma_{s+1}(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n) = ... = \gamma_n(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n)$. Thus, for each $i \in N\backslash\{1,2,...,s\}$, we have $\gamma_i(\theta_1,\theta_2,...,\theta_s,\theta_n,...,\theta_n) = \sum_{l\in\{1,2,...,s\}}\frac{(\sigma_l^{\{1,2,...,s\}}-1)}{(n-2)}\theta_l + \frac{(n-1-(s))(n-2+(s))}{2(n-2)}\theta_n$. This holds for each $S \subset N\backslash\{n\}$ with $|S| = s$. Thus, for each $i \in N$,*

- *if $i \in S$, then $\gamma_i(\theta_S\backslash\{i\},\theta_n,...,\theta_n) = \sum_{l\in S\backslash\{i\}}\frac{(\sigma_l^{S\backslash\{i\}}-1)}{(n-2)}\theta_l + \frac{(n-1-|S\backslash\{i\}|)(n-2+|S\backslash\{i\}|)}{2(n-2)}\theta_n$;*

- *if* $i \in N\backslash S$, *then* $\gamma_i(\theta_S, \theta_n, ..., \theta_n) = \sum_{l\in S} \frac{(\sigma_l^S-1)}{(n-2)}\theta_l + \frac{(n-1-|S|)(n-2+|S|)}{2(n-2)}\theta_n$.

...

Step n − 1: $(\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n)$.
By Pareto-efficiency, $\gamma_1(\theta_2, \theta_3, ..., \theta_{n-1}, \theta_n) + \gamma_2(\theta_1, \theta_3, ..., \theta_{n-1}, \theta_n) + ... + \gamma_n(\theta_1, \theta_2, ..., \theta_{n-1}) = \sum l \in \{1, 2, ..., n-1\}(\sigma_l - 1)\theta_l$. *By Step* $n-2$, *for* $i \in \{1, 2, ..., n-1\}$, *we have* $\gamma_i(\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n) = \sum_{l\in\{1,2,...,n-1\}\backslash\{i\}} \frac{(\sigma_l^{\{1,2,...,n-1\}\backslash\{i\}}-1)}{(n-2)}\theta_l + \theta_n$. *Thus,* $\gamma_n(\theta_1, \theta_2, ..., \theta_{n-1}) = \sum_{l\in\{1,2,...,n-1\}} \frac{(\sigma_l^{\{1,2,...,n-1\}}-1)}{(n-2)}\theta_l$. *Thus, for each* $i \in N$, *we have* $\gamma_i(\theta_{-i}) = \sum_{l\in N\backslash\{i\}} \frac{(\sigma_l^{N\backslash\{i\}}-1)}{(n-2)}\theta_l = \sum_{l\in N\backslash\{i\}} \frac{(\sigma_l^{-i}-1)}{(n-2)}\theta_l$.

Statement 2:

Pareto-efficiency: *Suppose that* φ *is an* EDTP *rule. Let* $\theta \in \mathbb{R}^N_{++}$ *and* $(\sigma, t) = \varphi(\theta)$. *By definition of an* EDTP *rule,* $\sigma \in \Sigma^*(\theta)$ *and by Theorem 2, for each* $i \in N$, *we have* $t_i = -\sum_{l\in F_i(\sigma)}\theta_l + \frac{1}{(n-2)}\sum_{j\in N\backslash\{i\}}\sum_{l\in F_j(\sigma^{-i})}\theta_l$ *Thus,*

$$\begin{aligned}\sum_{i\in N} t_i &= \sum_{i\in N}[-\sum_{l\in F_i(\sigma)}\theta_l + \tfrac{1}{(n-2)}\sum_{j\in N\backslash\{i\}}\sum_{l\in F_j(\sigma^{-i})}\theta_l] \\ &= \sum_{i\in N}[-\sum_{l\in F_i(\sigma)}\theta_l + \tfrac{1}{(n-2)}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i}-1)\theta_l] \\ &= -\sum_{i\in N}\sum_{l\in F_i(\sigma)}\theta_l + \tfrac{1}{(n-2)}\sum_{i\in N}\sum_{l\in N\backslash\{i\}}(\sigma_l^{-i}-1)\theta_l \\ &= -\sum_{i\in N}(\sigma_i-1)\theta_i + \tfrac{1}{(n-2)}\sum_{i\in N}(n-2)(\sigma_i-1)\theta_i \\ &= 0.\end{aligned}$$

No-envy:[13] *Suppose that* φ *is an* EDTP *rule. Let* $\theta \in \mathbb{R}^N_{++}$, $(\sigma, t) = \varphi(\theta)$, *and* $\{i, j\} \subset N$ *with* $i \neq j$. *By definition of an* EDTP *rule,* $\sigma \in \Sigma^*(\theta)$ *and by Theorem 2,* $t_i = \sum_{l\in P_i(\sigma)}\frac{(\sigma_l-1)}{(n-2)}\theta_l - \sum_{l\in F_i(\sigma)}\frac{(n-\sigma_l)}{(n-2)}\theta_l$ *and* $t_j = \sum_{l\in P_j(\sigma)}\frac{(\sigma_l-1)}{(n-2)}\theta_l - \sum_{l\in F_j(\sigma)}\frac{(n-\sigma_l)}{(n-2)}\theta_l$. *Then, distinguish two cases.*

Case 1: $\sigma_i < \sigma_j$. *Let* $d \in \mathbb{N}$ *be such that* $\sigma_j = \sigma_i + d$. *Then, as, by assumption,* $1 \leq \sigma_i < \sigma_j \leq n$, *we have* $d \leq n - \sigma_i$. *Also, as* $\sigma \in \Sigma^*(\theta)$, *for each* $l \in B_{ij}(\sigma)$, *we have* $\theta_i \geq \theta_l \geq \theta_j$. *Thus,*

$$\begin{aligned}u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= (-(\sigma_i-1)\theta_i - \sum_{l\in B_{ij}(\sigma)}\tfrac{(n-\sigma_l)}{(n-2)}\theta_l - \tfrac{(n-\sigma_j)}{(n-2)}\theta_j) \\ &\quad - (-(\sigma_j-1)\theta_i + \tfrac{(\sigma_i-1)}{(n-2)}\theta_i + \sum_{l\in B_{ij}(\sigma)}\tfrac{(\sigma_l-1)}{(n-2)}\theta_l) \\ &= \tfrac{(n-2)d-(\sigma_i-1)}{(n-2)}\theta_i - \tfrac{(n-1)}{(n-2)}\sum_{l\in B_{ij}(\sigma)}\theta_l - \tfrac{(n-\sigma_i-d)}{(n-2)}\theta_j \\ &\geq (\tfrac{(n-2)d-(\sigma_i-1)-(n-1)(d-1)-(n-\sigma_i-d)}{(n-2)})\theta_i\end{aligned}$$

[13] Chun (2004b) provides a necessary and sufficient condition for a rule φ to satisfy *Pareto-efficiency* and *no-envy*: For each $\theta \in \mathcal{R}^N_{++}$ and each $(\sigma, t) \in \varphi(\theta)$, we have $\sigma \in \Sigma^*(\theta)$, $\sum_{i\in N} t_i = 0$, and for each $\{i, j\} \subset N$, if $\sigma_j = \sigma_i + 1$, then $\theta_i \geq t_j - t_i \geq \theta_j$. An alternative proof thus consists in proving that *EDTP* rules satisfy this condition. In fact, rules in Suijs (1996) satisfy this condition (Katta and Sethuraman, 2005). Thus, by Theorem 2, *EDTP* rules satisfy this condition.

$= 0.$

Case 2: *$\sigma_i > \sigma_j$. Let $d \in \mathbb{N}$ be such that $\sigma_i = \sigma_j + d$. Then, as, by assumption, $n \geq \sigma_i > \sigma_j \geq 1$. Also, as $\sigma \in \Sigma^*(\theta)$, for each $l \in B_{ji}(\sigma)$, we have $\theta_i \leq \theta_l \leq \theta_j$. Thus,*

$$\begin{aligned} u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= \left(-(\sigma_i - 1)\theta_i + \tfrac{(\sigma_j-1)}{(n-2)}\theta_j + \textstyle\sum_{l \in B_{ji}(\sigma)} \tfrac{(\sigma_l-1)}{(n-2)}\theta_l\right) \\ &\quad - \left(-(\sigma_j - 1)\theta_i - \textstyle\sum_{l \in B_{ji}(\sigma)} \tfrac{(n-\sigma_l)}{(n-2)}\theta_l - \tfrac{(n-\sigma_i)}{(n-2)}\theta_i\right) \\ &= \tfrac{-(n-2)d+(n-\sigma_j-d)}{(n-2)}\theta_i + \tfrac{(n-1)}{(n-2)}\textstyle\sum_{l \in B_{ji}(\sigma)} \theta_l + \tfrac{(\sigma_j-1)}{(n-2)}\theta_j \\ &\geq \left(\tfrac{-(n-2)d+(n-\sigma_j-d)+(n-1)(d-1)+(\sigma_j-1)}{(n-2)}\right)\theta_i \\ &= 0. \end{aligned}$$

Strategy-proofness: *Suppose that φ is an* EDTP *rule. By Theorem 2, φ is a Groves rule. Thus, by Theorem 1, φ satisfies* strategy-proofness. ■

Remark that *Pareto-efficiency*, *no-envy*, and *strategy-proofness* are independent of one another. First, any rule such that for each $\theta \in \mathbb{R}^N_{++}$, if (σ, t) is selected, then $\sigma \in \Sigma^*(\theta)$ and for $\alpha \in \mathcal{R}^N_{++}$ such that if $i \in N$ with $\sigma_i \neq 1$ and $\{j, k\} \subset N$ are such that $\sigma_j = \sigma_i - 1$ and $\sigma_k = \sigma_i + 1$, then $\alpha_i \in [\theta_j, \theta_k]$ and $t_i = \sum_{l \in P_i(\sigma) \cup \{i\}} \alpha_l$ and if $i \in N$ with $\sigma_i = 1$, then $t_i = \alpha_i$ and $\sum_{l \in N} t_l = 0$, satisfies all axioms, but *strategy-proofness* (Chun, 2004b). Second, any Groves rule associated with $h \in H$ such that for each $\theta \in \mathbb{R}^N_{++}$ and for $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, if (σ, t) is selected, $h_1 = \sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1)\theta_l + \frac{1}{n-2}\sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1)\theta_l + \lambda$ and for each $i \in N \setminus \{1\}$, we have $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_i + \frac{1}{n-2}\sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l - \frac{\lambda}{(n-1)}$ satisfies all axioms, but *no-envy*. Third, any Groves rule associated with $h \in H$ such that for each $\theta \in \mathbb{R}^N_{++}$ and for $\lambda \in \mathbb{R}_{++}$, if (σ, t) is selected, then for each $i \in N$, we have $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_i + \frac{1}{n-2}\sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l - \lambda$ satisfies all axioms, but *Pareto-efficiency*.

2.4.2 *Single-valued* and non-*single-valued* rules

We now come to our central result. We identify the only rule that satisfies the axioms we impose on rules. Moreover, we prove that it satisfies *no-envy*, *anonymity*, and stronger incentive compatibility properties.

For each problem, this rule selects all *Pareto-efficient* queues and sets each agent's transfer such that each agent pays the unit waiting costs of all her followers and receives $(1/n-2)th$ of the unit waiting costs of all each other agent's followers, but her. Formally,

The Largest Equally Distributed Two-by-two Pivotal rule, φ^{LEDTP}: For each $\theta \in \mathbb{R}^N_{++}$, we have $(\sigma, t) \in \varphi^{LEDTP}(\theta)$ if and only if $\sigma \in \Sigma^*(\theta)$ and for each $i \in N$, we have $t_i = -\sum_{l \in F_i(\sigma)} \theta_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} \theta_l$.

For each problem and each *Pareto-efficient* queue, the transfers of the Largest Equally Distributed Two-by-two Pivotal rule can be obtained as the transfers of any rule described in Theorem 2. Thus, for each $\theta \in \mathbb{R}^N_{++}$, each $(\sigma, t) \in \varphi^{LEDTP}(\theta)$, and each $i \in N$, we have $t_i = -\sum_{l \in F_i(\sigma)} \theta_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} \theta_l = \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} \theta_l - \sum_{l \in F_i(\sigma)} \frac{(n-\sigma_l)}{(n-2)} \theta_l = \sum_{l \in P_i(\sigma)} \frac{\theta_l}{2} - \sum_{l \in F_i(\sigma)} \frac{\theta_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \frac{\theta_k - \theta_l}{2(n-2)}$.

In Theorem 4, we prove that the Largest Equally Distributed Two-by-two Pivotal is the only rule, together with any of its subcorrespondences, that satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. As any of it subcorrespondences, it satisfies *no-envy*. It is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Also, it satisfies *anonymity*. Thus, as *no-envy* implies *equal treatment of equals in welfare* an as *anonymity* implies *symmetry*, we prove that this rule is the only rule that satisfies the axioms we impose on rules.

Furthermore, proving Statement 2, we obtain that for each $\theta \in \mathbb{R}^N_{++}$, each $i \in N$, each $\theta'_i \in \mathbb{R}_{++}$, each $(\sigma, t) \in \varphi(\theta)$, and each $(\sigma', t') \in \varphi(\theta'_i, \theta_{-i})$, we have $u_i(\sigma_i, t_i) \geq u_i(\sigma'_i, t'_i)$. Thus, the Largest Equally Distributed Two-by-two Pivotal satisfies stronger incentives compatibility properties. In particular, it is such that each agent always finds any bundle she may receive when revealing her unit waiting cost at least as desirable as any bundle she may receive when misrepresenting it.

Theorem 4 *Let φ be a rule.*

1. *If φ satisfies* Pareto-efficiency, equal treatment of equals in welfare, *and* strategy-proofness, *then it is a subcorrespondence of the Largest Equally Distributed Two-by-two Pivotal rule.*

2. *If φ is a subcorrespondence of the Largest Equally Distributed Two-by-two Pivotal rule, then it satisfies* Pareto-efficiency, no-envy, *and* strategy-proofness.

3. *If φ satisfies* Pareto-efficiency, symmetry, *and* strategy-proofness, *then φ is the Largest Equally Distributed Two-by-two Pivotal rule.*

4. *If φ is the Largest Equally Distributed Two-by-two Pivotal rule, then φ satisfies* Pareto-efficiency, anonymity, *and* strategy-proofness.

Proof.

Statement 1: *Suppose that φ satisfies the axioms of Theorem 4.1. Let $\theta \in \mathbb{R}^N_{++}$ and $(\sigma, t) \in \varphi(\theta)$. By* Pareto-efficiency, *$\sigma \in \Sigma^*(\theta)$. In Claims 1 and 2, we prove that* Pareto-efficiency *and* strategy-proofness *imply that* *there is* $\{\underline{h}, \overline{h}\} \subseteq H$ *such that for each* $i \in N$,

- *if* $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{h}_i(\theta_{-i})$;
- *if* $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)\theta_l + \overline{h}_i(\theta_{-i})$.

Thus, repeating the proof by induction of Theorem 3, for each $i \in N$,

- $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)\theta_l$;
- $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)\theta_l + \sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)\theta_l$.

Thus, by Pareto-efficiency, *for each* $i \in N$, *we have* $u_i(\underline{\sigma}, \underline{t}) = u_i(\overline{\sigma}, \overline{t})$. *Thus, for each* $i \in N$, *we have* $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)\theta_l + \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)\theta_l$. *Thus, by Theorem 2, for each* $i \in N$, *we have* $t_i = -\sum_{l\in F_i(\sigma)}\theta_l + \frac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{l\in F_j(\sigma^{-i})}\theta_l$.

Claim 1: *There is* $\overline{h} \in H$ *such that for each* $\theta \in \mathbb{R}^N_{++}$ *and each* $i \in N$, *if* $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)\theta_l + \overline{h}_i(\theta_{-i})$.
Let $(\overline{g}_i : \mathbb{R}^N_{++} \to \mathbb{R})_{i\in N}$ *be the list of real-valued functions such that* ***(i)*** *for each* $\theta \in \mathbb{R}^N_{++}$ *and each* $i \in N$, *if* $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)\theta_l + \overline{g}_i(\theta)$. *By contradiction, suppose that for* $\theta \in \mathbb{R}^N_{++}$, $i \in N$, *and* $\theta'_i \in \mathbb{R}$, *we have* ***(ii)*** $\overline{g}_i(\theta) - \overline{g}_i(\theta'_i, \theta_{-i}) \neq 0$. *Let* $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$ *and* $(\overline{\overline{\sigma}}, \overline{\overline{t}}) \in \arg\max_{(\sigma,t)\in\varphi(\theta'_i,\theta_{-i})} u'_i(\sigma_i, t_i)$. *Then,*

- *By* strategy-proofness, $u_i(\overline{\sigma}_i, \overline{t}_i) \geq \max_{(\sigma',t')\in\varphi(\theta'_i,\theta_{-i})} u_i(\sigma'_i, t'_i)$ *and* $u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) \geq \max_{(\sigma,t)\in\varphi(\theta)} u'_i(\sigma_i, t_i)$.
- *By definition,* $\max_{(\sigma',t')\in\varphi(\theta'_i,\theta_{-i})} u_i(\sigma'_i, t'_i) \geq u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i)$ *and* $\max_{(\sigma,t)\in\varphi(\theta)} u'_i(\sigma_i, t_i) \geq u'_i(\overline{\sigma}_i, \overline{t}_i)$.

Thus, ***(iii)*** $u_i(\overline{\sigma}_i, \overline{t}_i) - u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) \geq 0$ *and* $u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) - u'_i(\overline{\sigma}_i, \overline{t}_i) \geq 0$. *By the logic of Theorem 1,* ***(i)***, ***(ii)***, *and* ***(iii)*** *imply a contradiction.*

Claim 2: *There is* $\underline{h} \in H$ *such that for each* $\theta \in \mathbb{R}^N_{++}$ *and each* $i \in N$, *if* $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{h}_i(\theta_{-i})$.
Let $(\underline{g}_i : \mathbb{R}^N_{++} \to \mathbb{R})_{i\in N}$ *be the list of real-valued functions such that for each* $\theta \in \mathbb{R}^N_{++}$ *and each* $i \in N$, *if* $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *then* $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{g}_i(\theta)$. *In what follows, we prove that there is* $(\underline{h}_i)_{i\in N} \in H$ *such that for each* $\theta \in \mathbb{R}^N$ *and each* $i \in N$, *we have* $\underline{h}_i(\theta_{-i}) = \underline{g}_i(\theta)$. *For simplicity, let* $\theta \in \mathbb{R}^N_{++}$, $i \in N$, $\theta'_i \in \mathbb{R}_{++}$, $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, *and* $(\underline{\underline{\sigma}}, \underline{\underline{t}}) \in \arg\min_{(\sigma,t)\in\varphi(\theta'_i,\theta_{-i})} u'_i(\sigma_i, t_i)$.

Step 1: *There is* $\underline{h}_i : \mathbb{R}^{N\setminus\{i\}}_{++} \to \mathbb{R}$ *such that if* $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u'_i(\sigma_i, t_i)$ *and* $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(\theta'_i,\theta_{-i})} u_i(\sigma_i, t_i)$, *then*

*$t_i^* = -\sum_{l\in N\setminus\{i\}}(\sigma_l^* - 1)\theta_l + \underline{h}_i(\theta_{-i})$ and $t_i^{**} = -\sum_{l\in N\setminus\{i\}}(\sigma_l^{**} - 1)\theta_l + \underline{h}_i(\theta_{-i})$. Let $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i'(\sigma_i, t_i)$ and $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(\theta_i',\theta_{-i})} u_i(\sigma_i, t_i)$. Let $g_i^* : \mathbb{R}_{++}^N \to \mathbb{R}$ be a real-valued function such that **(i)** $t_i^* = -\sum_{l\in N\setminus\{i\}}(\sigma_l^* - 1)\theta_l + g_i^*(\theta)$ and $t_i^{**} = -\sum_{l\in N\setminus\{i\}}(\sigma_l^{**} - 1)\theta_l + g_i^*(\theta_i', \theta_{-i})$. By contradiction, suppose **(ii)** $g_i^*(\theta) - g_i^*(\theta_i', \theta_{-i}) \neq 0$. Then,*

- *By* strategy-proofness, *$u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma_i^{**}, t_i^{**})$ and $u_i'(\sigma_i^*, t_i^*) \leq u_i'(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i)$.*
- *By assumption, $u_i(\sigma_i^*, t_i^*) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$ and $u_i'(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i) \leq u_i'(\sigma_i^{**}, t_i^{**})$.*

*Thus, **(iii)** $u_i(\sigma_i^*, t_i^*) - u_i(\sigma_i^{**}, t_i^{**}) \geq 0$ and $u_i'(\sigma_i^{**}, t_i^{**}) - u_i'(\sigma_i^*, t_i^*) \geq 0$. By the logic of Theorem 1, **(i)**, **(ii)**, and **(iii)** imply a contradiction. This holds for each $\theta_i' \in \mathbb{R}_{++}$.*

Step 2: *$\underline{g}_i(\theta) = \underline{h}_i(\theta_{-i})$ and $\underline{g}_i(\theta_i', \theta_{-i}) = \underline{h}_i(\theta_{-i})$.*
By contradiction, suppose $\underline{g}_i(\theta) - \underline{h}_i(\theta_{-i}) \neq 0$. (The other case is immediate.) Then,

- *By assumption, $u_i(\sigma_i^*, t_i^*) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$. Thus, $-(\sigma_i^* - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\sigma_l^* - 1)\theta_l + \underline{h}_i(\theta_{-i}) \geq -(\underline{\sigma}_i - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{g}_i(\theta)$. Thus, $-\sum_{l\in N}(\sigma_l^* - 1)\theta_l + \underline{h}_i(\theta_{-i}) \geq -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{g}_i(\theta)$. Thus, by* Pareto-efficiency, *$\underline{h}_i(\theta_{-i}) \geq \underline{g}_i(\theta)$.*
- *By* strategy-proofness, *$u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma_i^{**}, t_i^{**})$. Thus, $-(\underline{\sigma}_i - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{g}_i(\theta) \geq -(\sigma_i^{**} - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\sigma_l^{**} - 1)\theta_l + \underline{h}_i(\theta_{-i})$. Thus, $\underline{g}_i(\theta) \geq (\underline{\sigma}_i - \sigma_i^{**})\theta_i + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{**})\theta_l + \underline{h}_i(\theta_{-i})$.*

Thus,

$$\textbf{(iv)}\ \underline{h}_i(\theta_{-i}) \geq \underline{g}_i(\theta) \geq (\underline{\sigma}_i - \sigma_i^{**})\theta_i + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{**})\theta_l + \underline{h}_i(\theta_{-i}).$$

Let us rewrite this expression. By Pareto-efficiency, *for each $S \subseteq N$, if for each $\{k, k'\} \subseteq S$ with $k \neq k'$, we have $\theta_k = \theta_{k'}$ and there is no $k'' \in N\setminus S$ such that $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$, then $\sum_{l\in S} -(\sigma_l - 1)\theta_l = \sum_{l\in S} -(\sigma_l' - 1)\theta_l$. Also, there is $j \in N$ such that $\underline{\sigma}_j = \sigma_i^{**}$. Thus, $\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{**})\theta_l = -sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l$. Thus, we may rewrite **(iv)** as*

$$\textbf{(v)}\ \underline{h}_i(\theta_{-i}) \geq \underline{g}_i(\theta) \geq (\underline{\sigma}_i - \sigma_i^{**})\theta_i - sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l + \underline{h}_i(\theta_{-i}).$$

Then, distinguish three cases.

Case 1: *$(\underline{\sigma}_i - \sigma_i^{**}) = 0$. Then, $-sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l = 0$. Thus, by **(v)**, $\underline{g}_i(\theta) = \underline{h}_i(\theta_{-i})$, contradicting $\underline{g}_i(\theta) - \underline{h}_i(\theta_{-i}) \neq 0$.*

Case 2: $|\underline{\sigma}_i - \sigma_i^{**}| = 1$. *Suppose $\theta_i' > \theta_i$. (The symmetric case is immediate.) Then, $(\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $-sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}}\theta_l = -\theta_j$. Thus, by* **(v)**, *$\underline{h}_i(\theta_{-i}) \geq \underline{g}_i(\theta) \geq (\theta_i - \theta_j) + \underline{h}_i(\theta_{-i})$. Let $\theta_i'' \in \mathbb{R}_{++}$ be such that* **(vi)** *$\underline{g}_i(\theta) > (\theta_i'' - \theta_j) + \underline{h}_i(\theta_i)$ and $\theta_i' > \theta_i'' > \theta_i$. Let $(\sigma^{***}, t^{***}) \in \arg\min_{(\sigma,t)\in\varphi(\theta_i'',\theta_{-i})} u_i(\sigma_i, t_i)$. By* Pareto-efficiency of queues, *$\sigma_i^{***} = \sigma_i^{**}$. Thus, $(\underline{\sigma}_i - \sigma_i^{***}) = (\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})\theta_l = \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l')\theta_l = -\theta_j$. Thus, by* **(vi)**, *$\underline{g}_i(\theta) > (\underline{\sigma}_i - \sigma_i^{***})\theta_i'' + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})\theta_l + \underline{h}_i(\theta_i)$. Thus, $-(\underline{\sigma}_i - 1)\theta_i'' - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{g}_i(\theta) > -(\sigma_i^{***} - 1)\theta_i'' - \sum_{l\in N\setminus\{i\}}(\sigma_l^{***} - 1)\theta_l + \underline{h}_i(\theta_{-i})$. Thus, $u_i''(\underline{\sigma}_i, \underline{t}_i) > u_i''(\sigma_i^{***}, t_i^{***})$. Also, $u_i''(\sigma_i^{***}, t_i^{***}) \geq \min_{(\sigma,t)\in\varphi(\theta_i'',\theta_{-i})} u_i''(\sigma_i, t_i)$. Thus, $u_i''(\underline{\sigma}_i, \underline{t}_i) > \min_{(\sigma,t)\in\varphi(\theta_i'',\theta_{-i})} u_i''(\sigma_i, t_i)$, contradicting* strategy-proofness.

Case 3: $|\underline{\sigma}_i - \sigma_i^{**}| > 1$. *By the logic of Case 2, starting from σ_i^{**} and the one closer to $\underline{\sigma}_i$ by one position and ending with the one closer to σ_i^{**} by one position and $\underline{\sigma}_i$, we obtain each time $\underline{g}_i(.,\theta_{-i}) = \underline{h}_i(\theta_{-i})$. Thus, $\underline{g}_i(\theta) = \underline{h}_i(\theta_{-i})$, contradicting $\underline{g}_i(\theta) - \underline{h}_i(\theta_{-i}) \neq 0$.*

Statement 2:

Pareto-efficiency: *Straightforward from Theorem 3.*

No-envy: *Straightforward from Theorem 3.*

Strategy-proofness: *Suppose that φ is an* EDTP *rule. Let $\theta \in \mathbb{R}^N_{++}$, $i \in N$, $\theta_i' \in \mathbb{R}_{++}$, $(\sigma, t) \in \varphi(\theta)$, and $(\sigma', t') \in \varphi(\theta_i', \theta_{-i})$. By definition of an* EDTP *rule, $\sigma \in \Sigma^*(\theta)$ and by Theorem 2, there is $h \in H$ such that for each $i \in N$, we have $h_i(\theta_{-i}) = \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)\theta_l = \sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i} - 1)\theta_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i} - 1)\theta_l$ and $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$ and $t_i' = -\sum_{l\in N\setminus\{i\}}(\sigma_l' - 1)\theta_l + h_i(\theta_{-i})$. Suppose $u_i(\sigma_i', t_i') > u_i(\sigma_i, t_i)$. Then, $-(\sigma_i' - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\sigma_l' - 1)\theta_l + h_i(\theta_{-i}) > -(\sigma_i - 1)\theta_i - \sum_{l\in N\setminus\{i\}}(\sigma_l - 1)\theta_l + h_i(\theta_{-i})$. Thus, $-\sum_{l\in N}(\sigma_l' - 1)\theta_l > -\sum_{l\in N}(\sigma_l - 1)\theta_l$, contradicting $\sigma \in \Sigma^*(\theta)$. Thus, $u_i(\sigma_i', t_i') \leq u_i(\sigma_i, t_i)$. This holds for each $(\sigma, t) \in \varphi(\theta)$ and each $(\sigma', t') \in \varphi(\theta_i', \theta_{-i})$. Thus, if $X_i = \bigcup_{(\sigma,t)\in\varphi(\theta)}(\sigma_i, t_i)$ and $X_i' = \bigcup_{(\sigma,t)\in\varphi(\theta_i',\theta_{-i})}(\sigma_i, t_i)$, then $X_i\ R_i(\theta_i)\ X_i'$.*

Statement 3: *Suppose that φ satisfies the axioms of Theorem 4.3. Let $\theta \in \mathbb{R}^N_{++}$ and $(\sigma) \in \varphi(\theta)$. By* Pareto-efficiency, *$\sigma \in \Sigma^*(\theta)$. By Statement 1,* Pareto-efficiency *and* strategy-proofness *imply that there is $\{\underline{h}, \overline{h}\} \subseteq H$ such that for each $i \in N$, if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)\theta_l + \underline{h}_i(\theta_{-i})$ and if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(\theta)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)\theta_l + \overline{h}_i(\theta_{-i})$. By* symmetry, *for each $\{i, j\} \subset N$, if $\theta_{-i} = \theta_{-j}$, then $\underline{h}_i(\theta_{-i}) = \underline{h}_j(\theta_{-j})$ and $\overline{h}_i(\theta_{-i}) = \overline{h}_j(\theta_{-j})$. Thus, for each $\{i, j\} \subset N$, if $\theta_i = \theta_j$, then $\underline{h}_i(\theta_{-i}) = \underline{h}_j(\theta_{-j})$ and $\overline{h}_i(\theta_{-i}) = \overline{h}_j(\theta_{-j})$. This is* *true for each* $\theta \in \mathcal{R}_{++}$. *Thus, repeating the proof by induction of Theorem 3, for each* $i \in N$, *we have* $\underline{t}_i = -\sum_{l \in F_i(\underline{\sigma})} \theta_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\underline{\sigma}_l^{-i} - 1)\theta_l$ *and* $\bar{t}_i = -\sum_{l \in F_i(\bar{\sigma})} \theta_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\bar{\sigma}_l^{-i} - 1)\theta_l$. *Thus, by* Pareto-efficiency, *for each* $i \in N$, *we have* $u_i(\underline{\sigma}, \underline{t}) = u_i(\bar{\sigma}, \bar{t})$. *Thus, for each* $i \in N$, *we have* $t_i = -\sum_{l \in F_i(\sigma)} \theta_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)\theta_l$. *Thus, by Theorem 2,* $t_i = -\sum_{l \in F_i(\sigma)} \theta_l + \frac{1}{(n-2)} \sum_{j \in N \setminus \{i\}} \sum_{l \in F_j(\sigma^{-i})} \theta_l$. *Thus, for each* $\theta \in \mathbb{R}^N_{++}$, *we have* $\varphi(\theta) \subseteq \varphi^{LEDTP}(\theta)$. *Thus, by* symmetry, $\varphi(\theta) = \varphi^{LEDTP}(\theta)$.

Statement 4: *Suppose that for each* $(\theta) \in \mathbb{R}^N_{++}$, *we have* $\varphi(\theta) = \varphi^{LEDTP}(\theta)$. *By Theorem 4.2,* φ *satisfies* Pareto-efficiency *and* strategy-proofness. *Also,* φ *does not depend on agents' names. In particular,* t_i *has the same structure for each* $i \in N$. *Thus,* φ *satisfies* anonymity. ■

Remark that *Pareto-efficiency, anonymity,* and *strategy-proofness* are independent of one another. First, any rule that selects all *Pareto-efficient* queues and sets each agent's transfer equal to the Shapley value of the associated coalitional game, where the worth of a coalition is the minimum possible sum of its members waiting costs (Maniquet, 2003), satisfies all axioms, but *strategy-proofness.* Second, any proper subcorrespondence of a rule that is the union of all the *single-valued* rules that are Groves rules associated with $h \in H$ and that satisfy *balancedness,* satisfies all axioms, but *anonymity.* Third, any rule such that for each $\theta \in \mathbb{R}^N_{++}$ and for $\lambda \in \mathbb{R}_+$, we have that (σ, t) is selected if and only if $\sigma \in \{1, 2, ..., n\}^N$ and for each $i \in N$, we have $t_i = -\lambda$, satisfies all axioms, but *Pareto-efficiency.*

2.5 Concluding comments

Our objective was to identify allocation rules for queueing problems that satisfy efficiency, equity, and incentive compatibility properties simultaneously on the domain of linear preferences in positions and transfers. We proved that the Largest Equally Distributed Two-by-two Pivotal rule is the only such rule. Indeed, only this rule satisfies *Pareto-efficiency, equal treatment of equals in welfare, symmetry,* and *strategy-proofness.* Moreover, it satisfies *no-envy, anonymity,* and stronger incentive compatibility properties.

Situations in which waiting costs are linear in time are frequent, especially in short term problems. Think of a firm with several units having each its chain out of service. These units simultaneously need help from the repair and maintenance center. Each unit remaining unattended, still pays each worker on its chain her wage per hour. Thus, each such unit incurs a cost that linearly increases in the time it is down. In longer term problems, as agents discount future, waiting costs tend to exponentially increase in time.

We draw three lessons from our results. First, the queueing problems we studied are among the few allocation problems in which *Pareto-efficiency*, a weak equity axiom as *equal treatment of equals in welfare* or *symmetry*, and *strategy-proofness* are compatible. The natural step now is to determine if this compatibility extends to other queueing problems, in particular to those in which agents have different processing times. However, in queueing problems in which agents have waiting costs varying non-linearly across positions, no rule satisfies *Pareto-efficiency* and *strategy-proofness* (Mitra, 2002).

Second, while *Pareto-efficiency* and *strategy-proofness* leave us with a large class of *single-valued* rules, adding a weak equity axiom as *equal treatment of equals in welfare* imposes a unique way of setting transfers. The open question is to determine what is the class of multi-valued rules that *Pareto-efficiency* and *strategy-proofness* recommend. If as we conjecture, it also large, the contrast induced by treating equal agents equally would then generalize to rules that may select more than one allocation.

Finally, in the queueing problems we studied, simply requiring to treat equal agents equally in addition to *Pareto-efficiency* and *strategy-proofness*, guarantees further basic properties. First, it prevents agents from envying one another. In allocation problems of private goods, *equal treatment of equals in welfare* and *coalition strategy-proofness*, i.e., no coalition should gain by simultaneously misrepresenting their preferences, imply *no-envy* (Moulin, 1993). In general public decision making problems in which the agents have strictly monotonically closed preferences, *equal treatment of equals in welfare*, *strategy-proofness*, and *non-bossiness*, i.e., if a change in an agent's waiting cost does not change her bundle, then it should not change other agents' bundles either, imply *no-envy* (Fleurbaey and Maniquet, 1997).

These results do not apply to the queueing problems we studied. Indeed, no rule satisfies *Pareto-efficiency* and *coalition strategy-proofness* (Kayı and Ramaekers, 2006). Also, as preferences are linear in positions and transfers, they are non-monotonically closed. In fact, no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness* (Kayı and Ramaekers, 2006).

Furthermore, it prevents agents' names to matter. Finally, it guarantees each agent with a minimal welfare level. Indeed, in allocation problems of at most one indivisible private good per agent, *no-envy* implies the *identical-preferences lower bound*, i.e., each agent should find her bundle at least as desirable as any bundle *Pareto-efficiency* and *equal treatment of equals in welfare* recommend when the others have her preferences (Beviá, 1996).

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Chapter 3

Stability, consistency, and monotonicity in matching markets with contracts when agents negotiate

Abstract: We study *two-sided many-to-one matching markets with contracts*, in which two finite and disjoint sets of agents must be matched under terms of contracts and given that one side of the market may sign at most one contract. We prove that to predict the outcomes of such markets, as agents may not agree on terms of the contract under which they should be matched and hence may have to negotiate them, we should not restrict our attention to *stable* allocations. Moreover, independently of the fact that agents may negotiate, the only solution for such markets that satisfies efficiency, equity, and strategic compatibility properties simultaneously is to select all *stable* allocations. Finally, by opposition to when there is only one way to match agents, it is not efficiency and equity properties that impose a conflict between the common interests of each of the two sides of market, but efficiency, equity, and incentive compatibility properties.

JEL Classification: C78, D71, D78, J41.
Keywords: Matching, matching with contracts, population-monotonicity, consistency, stability, Nash-implementability.

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3.1 Introduction

We study *two-sided many-to-one matching markets with contracts*, in which two finite and disjoint sets of agents must be matched under terms of contracts and given that one side of the market may sign at most one contract. Such problems are frequent. Think of academic job markets, in which e.g.

newly graduated doctors look for post-doctoral grants or tenure track assistant professorships, tenured professors look for tenured professorships with limited administrative duties and teaching, and university departments offer research grands and positions in different research domains. Think of medical job markets, in which e.g. specific doctors look for full-time positions, public hospitals offer positions in few specific domains under benchmark wages, and private hospitals offer various positions under attractive wages. Think of school and college admission problems, in which each student looks for such an institution and each such institution looks for as many students as its quota allows it. Think of dating problems, in which each single person looks for a single person of opposite sex.

We study any such matching problem, i.e., in which agents may be matched together under different terms or not, and in which terms may be cover different topics (job profile, wage, job localization, ...) or not. We assume that there is more than two agents. Each agent has strict preferences. Each agent that may be matched with more than one agent has *substitutable* preferences, i.e., if each such agent chooses a contract from a set of contracts, she should still choose it from any subset of this set that includes it.

Following an axiomatic approach, our aim is to determine which properties solutions for such problems should satisfy and based on these properties, to justify solutions. Our primary concern is efficiency and equity. However, as the planner may not force agents into some matching, contracts are based on voluntary participation. Also, agents may negotiate contracts with one another. Besides, the planner may not know agents' preferences. Thus, agents may behave strategically rejecting their allocated bundle or when announcing their preferences. Our secondary concern is motivated by such strategic considerations.

We identify the following efficiency, equity, and incentive compatibility properties as adapted to the problems we study. Efficiency is expressed as *Pareto-efficiency*. It implies *unanimity*, i.e., if an allocation is the choice of each agent over the set of available contracts, it should be the only selected allocation.

Equity is focused on consequences of exogenous changes in a problem's parameters. First, we require *own-side population-monotonicity*, i.e., if one side's population decreases, each agent on this side should find the worst bundle she could be allocated in the new situation at least as desirable as the worst bundle she could be allocated in the previous situation. Second, we require *consistency*, i.e., for each problem and each allocation selected for this problem, if agents leave with whom they are matched to in this allocation, the reduced allocation of this allocation relative to the agents still there should be selected for this reduced problem.

Incentive compatibility properties are expressed as follows. First, *stability*, i.e., no agent should find rejecting her allocated bundle (*i*) *to sign none or only some of the contracts it contains, or* (*ii*) *to sign with other agents contracts it does not contain*, more desirable than accepting it. It requires to select only *stable* allocations, i.e., allocations such that no agent finds rejecting her allocated bundle (*i*) *to sign none or only some of the contracts it contains, or* (*ii*) *to sign with other agents contracts it does not contain*, more desirable than accepting it. Second, *Nash-implementability*, i.e., the set of all selected allocations should correspond to set of all Nash-equilibria of some game. Maskin (1977, 1999) proves that this axiom implies *Maskin-monotonicity*, i.e., following a preferences change, if there is a previously selected allocation such that each agent's bundle has improved, this allocation should still be selected.

First, we prove that only the rule that selects for each problem, all *stable* allocations satisfies *Pareto-efficiency*, *own-side population-monotonicity*, *consistency*, and *Nash-implementability*. We refer to it as the *Stable rule*. This characterization follows results of Kayı, Ramaekers, and Yengin (2006), and Haake and Klaus (2005). Indeed, Kayı, Ramaekers, and Yengin (2006) prove that on the domain of problems, in which each agent has strict preferences and each agent that may be matched with more than one agent has *substitutable* preferences, the Stable rule satisfies both equity axioms.[1] Also, they prove that only this rule satisfies *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity*. Haake and Klaus (2005) prove that on the subdomain of problems, in which there is more than two agents, it satisfies *Nash-implementability*. Clearly, it also satisfies *Pareto-efficiency*. Thus, as *Pareto-efficiency* implies *unanimity* and *Nash-implementability* implies *Maskin-monotonicity*, the Stable rule is characterized by only a few of the axioms we impose on rules. We state and prove results of Kayı, Ramaekers, and Yengin (2006) to provide the main intuitions. In so doing, we bring out the role of the multiplicity of contracts and hence of negotiations in two-sided many-to-one matching markets with contracts.

Second, we prove that if we take into account the fact that agents may be matched together under different terms of contracts and we accordingly weaken *stability*, the Stable rule is not the only rule that satisfies *Pareto-efficiency* and *consistency*. Indeed, the strategic considerations behind *stability* assume that if agents may improve upon an allocation signing particular contracts with one another, they coordinate on being matched together under the terms of these contracts.

However, agents may be matched together under different terms of contracts. As they may not agree on these terms, they may have to negotiate

[1] Kayı, Ramaekers, and Yengin (2006) refer to *consistency* as *weak consistency*.

them. The outcome of such negotiations depends on the agents' negotiation powers. The agents do not a priori know these powers. Thus, before entering into particular negotiations, each agent compares the following alternatives. If she does not enter into these negotiations, she may sign none, some, or all of the contracts her allocated bundle contains, or sign with others contracts it does not contain. If she enters into these negotiations, she may end up with a contract only advantageous for the other. Thus, if an agent finds the former alternative more desirable than the latter, she does not enter into these negotiations.

We introduce *stability with negotiations*, i.e., no agent should find rejecting her allocated bundle (*i*) *to sign none or only some of the contracts it contains, or* (*ii*) *to negotiate with others contracts it does not contain*, more desirable than accepting it. It requires to select only *stable allocations with negotiations*, i.e., allocations such that no agent finds rejecting her allocated bundle (*i*) *to sign none or only some of the contracts it contains, or* (*ii*) *to negotiate with others contracts it does not contain*, more desirable than accepting it. We prove that the rule that selects for each problem, all *Pareto-efficient* and *stable allocations with negotiations*, satisfies *consistency*. We refer to this rule as the *Stable rule with negotiations*. Moreover, we prove that the set of allocations the *Stable rule with negotiations* selects need not be a lattice. However, this rule violates *Maskin-monotonicity*.

Our results hold for each two-sided many-to-one matching problem with contracts. Thus, they hold for each general problem. For instance, the *medical job market problem*, introduced by Roth (1984) and studied by Hatfield and Milgrom (2005), in which each doctor looks to be matched with a hospital under the terms of a contract and each hospital looks to be matched with one or more doctors under the terms of contracts. They also hold for each more restricted problem, in particular in which there is only one way to match agents. For instance, the *college admission problem*, introduced by Roth (1985), in which each student looks for a college and each college looks for as many students as its quota allows it. The *job matching problem*, introduced by Kelso and Crawford (1982), in which each worker looks for a firm and each firm looks for workers. The *marriage problem*, introduced by Gale and Shapley (1962), in which each man looks for a woman and each woman looks for a man.[2]

The literature on two-sided many-to-one matching markets with contracts predicts the outcomes of such markets as *stable* allocations.[3] Hatfield and Milgrom (2005) prove that in each such market, if each agent has strict

[2]For formal proofs that these problems are restricted two-sided many-to-one matching problems with contracts, see Haake and Klaus (2005).

[3]For a survey on matching markets, see Roth and Sotomayor (1990).

preferences and each agent that may be matched with more than one agent has *substitutable* preferences, then such allocations exist.[4] Also, the set of *stable* allocations is a lattice.[5] Thus, each outcome of such markets leads to a conflict between the common interests of each of the two sides of the market.

Furthermore, in particular such markets, in which there is only one way to match agents, efficiency and equity properties always impose such a conflict. Indeed, in each college admission problem, in which each agent has strict preferences and each college has responsive preferences, Toda (2006) proves that only the Stable rule satisfies *unanimity*, *own-side population-monotonicity*, and a stronger axiom of *consistency*, i.e., for each problem and each selected allocation for this problem, if (*i*) either colleges leave with all students they are matched to in this allocation or (*ii*) students leave with the college they are matched to in this allocation only if all students this college is matched to in this allocation leave, then the reduced allocation of this allocation relative to the agents still there should be selected for this new problem.

In each marriage problem, in which each agent has strict preferences, Toda (2006) proves that only the Stable rule satisfies *unanimity*, *own-side population-monotonicity*, and *consistency*. Also, Sasaki and Toda (1992) prove that only the Stable rule satisfies *Pareto-efficiency*, *converse consistency*, i.e., for each problem and each allocation, if for each reduced problem including exactly two men and two women who are matched together in this allocation, these agents are matched together for this reduced problem, then this allocation should be selected for the original problem, *anonymity*, i.e., agents' names should not matter, and *consistency*.

We draw three lessons from our analysis. First, to predict the outcomes of two-sided many-to-one matching markets with contracts, as agents may not agree on the terms of the contract under which they should be matched and hence may have to negotiate them, we should not restrict our attention to

[4]Thus, this result holds on each more restricted market. In fact, it then holds under some weaker assumptions. Roth (1985) proves it on the domain of all college admission problems, in which each agent has strict preferences over singletons and each hospital has *responsive* preferences, i.e., each such agent finds adding a contract to a subset of contracts at least as desirable as adding another contract to this subset of contracts if and only if she finds the former contract at least as desirable as the latter contract. Gale and Shapley (1962) prove it on the domain of all marriage problems.

[5]Thus, this result holds on each more restricted market. In fact, it then holds under some weaker assumptions. Roth and Sotomayor (1990) prove it on the domain of all college admission problems, in which each agent has strict preferences over singletons and each hospital has *responsive* preferences. Knuth (1976) prove it on the domain of all marriage problems, in which each agent has strict preferences. He attributes his result to Conway.

stable allocations, but also focus on *stable allocations with negotiations*. Second, independently of the fact that agents may negotiate, the only solution for such markets that satisfies efficiency, equity, and strategic compatibility properties simultaneously is to select all *stable* allocations. Third, by opposition to when there is only one way to match agents, it is not efficiency and equity properties that impose a conflict between the common interests of each of the two sides of market, but efficiency, equity, and strategic compatibility properties.

In Section 3.2, we formally introduce the model. Without loss of generality, we use the medical job markets. In Section 3.3, we define the axioms we impose on rules. In Section 3.4, we state and prove our results. Finally, we give concluding remarks.

3.2 Model

We consider the medical job markets, in which *doctors* and *hospitals* must be matched under terms of *contracts*. We follow Hatfield and Milgrom (2005) for the model.[6] Let $\mathbb{D}$, $\mathbb{H}$, and $\mathbb{X}$ be the infinite sets of potential doctors, hospitals, and contracts, respectively. Let $\mu : \mathbb{X} \rightarrow \mathbb{D} \times \mathbb{H}$ be the function that specifies the bilateral structure of each contract, i.e., for each $x \in \mathbb{X}$, we have $\mu(x) = (d, h) \in \mathbb{D} \times \mathbb{H}$ if and only if x is a contract between doctor d and hospital h. For each $\{x, x'\} \subseteq \mathbb{X}$, if $\mu(x) = \mu(x')$ and $x \neq x'$, then x and x' are contracts between the same doctor and hospital, but under different terms. Let $\mathcal{D}$ and $\mathcal{H}$ be the sets of all non-empty and finite subsets of $\mathbb{D}$ and $\mathbb{H}$, respectively. Let $\mathcal{X}$ be the set of all finite subsets of $\mathbb{X}$.

Each doctor may be matched to at most one hospital, whereas each hospital may be matched to several doctors. Each agent may stay unmatched, i.e., each doctor may stay unemployed and each hospital may employ no doctor. We refer to this situation as the *null contract*. We denote it by $\emptyset$. By abuse of language, we say that the null contract matches the agent to herself. Formally, for each $d \in \mathbb{D}$, let $\mathcal{X}_d \equiv \{x \in \mathbb{X} :$ there is $h \in \mathbb{H}$ such that $\mu(x) = (d, h)\} \cup \emptyset$ be the set of all contracts in which d may be matched, including to herself. For each $h \in \mathbb{H}$, let $\mathcal{X}_h \equiv \{X \in \mathcal{X}$: for each $x \in X$, there is $d \in \mathbb{D}$ such that $\mu(x) = (d, h)\}$ be the set of all sets of contracts in which h may be matched, including to itself.

Each $d \in \mathbb{D}$ has a complete and transitive preference relation $\mathcal{R}_d$ over $\mathcal{X}_d$.[7] Let $\mathfrak{R}_d$ be the set of all preferences of $d \in \mathbb{D}$. Let $\mathcal{X}_d|_X \equiv \{x \in \mathcal{X}_d : x \in X\}$ be the *reduced set of all contracts in which $d \in \mathbb{D}$ may be matched (including*

[6] Haake and Klaus (2005), Kayı, Ramaekers, and Yengin (2006) also follow this model.

[7] Let $\mathcal{P}_d$ be the strict preference relation associated with $\mathcal{R}_d$.

to herself) of $\mathcal{X}_d$ relative to $X \in \mathcal{X}$. Let $C(.,\mathcal{R}_d)$ be the *choice function of $d \in \mathbb{D}$ with $\mathcal{R}_d \in \mathfrak{R}_d$* that assigns to each set of contracts $X \in \mathcal{X}$ the most preferred contract $C(X,\mathcal{R}_d) \in \mathcal{X}_d|_X$. Formally, for each $X \in \mathcal{X}$, we have $C(X,\mathcal{R}_d) \equiv max_{\mathcal{R}_d}\{x \in \mathcal{X}_d|_X\}$. For each $d \in \mathbb{D}$, let $\mathcal{R}_d|_X$ be the *restricted preferences of $\mathcal{R}_d \in \mathfrak{R}_d$ relative to $X \in \mathcal{X}$*. Formally, $\mathcal{R}_d|_X$ is the complete and transitive binary relation over $\mathcal{X}_d|_X$ such that for each $x, x' \in \mathcal{X}_d|_X$, we have $x \; \mathcal{R}_d|_X \; x'$ if and only if $x \; \mathcal{R}_d \; x'$.

Each $h \in \mathbb{H}$ has a complete and transitive preference relation $\mathcal{R}_h$ over $\mathcal{X}_h$.[8] Let $\mathfrak{R}_h$ be the set of all preferences of $h \in \mathbb{H}$. Let $\mathcal{X}_h|_X \equiv \{X' \in \mathcal{X}_h : X' \subseteq X\}$ be the *reduced set of all sets of contracts in which $h \in \mathbb{H}$ may be matched (including to itself) of $\mathcal{X}_h$ relative to $X \in \mathcal{X}$*. Let $C(.,\mathcal{R}_h)$ be the *choice correspondence of $h \in \mathbb{H}$ with $\mathcal{R}_h \in \mathfrak{R}_h$* that assigns to each set of contracts $X \in \mathcal{X}$ the most preferred subset of contracts $C(X,\mathcal{R}_h) \in \mathcal{X}_h|_X$. Formally, for each $X \in \mathcal{X}$, we have $C(X,\mathcal{R}_h) \equiv max_{\mathcal{R}_h}\{X' \in \mathcal{X}_h|_X\}$. For each $h \in \mathbb{H}$, let $\mathcal{R}_h|_X$ be the *restricted preferences of $\mathcal{R}_h \in \mathfrak{R}_h$ relative to $X \in \mathcal{X}$*. Formally, $\mathcal{R}_h|_X$ is the complete and transitive binary relation over $\mathcal{X}_h|_X$ such that for each $X', X'' \in \mathcal{X}_h|_X$, we have $X' \; \mathcal{R}_h|_X \; X''$ if and only if $X' \; \mathcal{R}_h \; X''$.

A two-sided many-to-one matching market with contracts or simply *market M* is a quadruple (D,H,X,R) such that: (i) $D \in \mathcal{D}$, (ii) $H \in \mathcal{H}$, (iii) $X \in \mathcal{X}$ with $\{i \in \mathbb{D} \cup \mathbb{H} : \mathcal{X}_i|_X \neq \emptyset\} = D \cup H$, and (iv) $R = (R_i)_{i \in D \cup H}$ with for each $i \in D \cup H$, there is $\mathcal{R}_i \in \mathfrak{R}_i$ such that $R_i = \mathcal{R}_i|_X$. Let $\mathcal{M}$ be the set of all markets.

We assume the following. First, each market contains more than two agents. Formally, for each $M = (D,H,X,R) \in \mathcal{M}$, we have $|D| + |H| > 2$. Second, each agent has *strict* preferences. Formally, for each $d \in \mathbb{D}$, each $\mathcal{R}_d \in \mathfrak{R}_d$, and each $x, x' \in \mathcal{X}_d$ with $x \neq x'$, either $x\mathcal{P}_d x'$ or $x'\mathcal{P}_d x$. Also, for each $h \in \mathbb{H}$, each $\mathcal{R}_h \in \mathfrak{R}_h$, and each $X, X' \in \mathcal{X}_h$ with $X \neq X'$, either $X\mathcal{P}_h X'$ or $X'\mathcal{P}_h X$. Finally, each hospital has *substitutable* preferences, i.e., if a hospital chooses a contract from a set of contracts, it chooses it from any subset of this set that includes it. Formally, for each $h \in \mathbb{H}$, each $\mathcal{R}_h \in \mathfrak{R}_h$, and each $X, X' \in \mathcal{X}_h$ with $X' \subsetneq X$, we have $X' \cap C(X,\mathcal{R}_h) \subseteq C(X',\mathcal{R}_h)$.

An *allocation A for $(D,H,X,R) \in \mathcal{M}$* is a list of bundles $(A_i)_{i \in D \cup H} \in \prod_{i \in D \cup H} \mathcal{X}_i|_X$ such that for each $d \in D$ and each $h \in H$, if there is $x \in A_d \cup A_h$, then $\{x\} = A_d \cap A_h$.[9] Let $\mathcal{A}(M)$ be the set of all allocations for $M \in \mathcal{M}$. An (allocation) *rule* φ is a correspondence that assigns to each market $M \in \mathcal{M}$ a non-empty set of allocations $\varphi(M) \subseteq \mathcal{A}(M)$.

Let us illustrate the model with an example. Three doctors, a paediatrician, an ophthalmologist, and a dermatologist, look for positions under

[8] Let $\mathcal{P}_h$ be the strict preference relation associated with $\mathcal{R}_h$.

[9] We write A_d, instead of $\{A_d\}$. We use this notational shortcut throughout the paper.

R_{d_p}	R_{d_o}	R_{d_d}	$R_{h_{pu}}$	$R_{h_{pr}}$
b	c	d	$\{e\}$	$\{a,c\}$
a	$\emptyset$	e	$\{f\}$	$\{c,d\}$
	f	$\emptyset$	$\emptyset$	$\{c\}$
			$\{f,e\}$	$\{d\}$
				$\{a\}$
				$\emptyset$
				$\{a,b\}$
				$\{a,c,d\}$
				$\{a,b,c,d\}$
				$\{a,b,c\}$
				$\{b\}$
				$\{b,c,d\}$
				$\{a,d\}$
				$\{b,c\}$
				$\{b,d\}$

Figure 3.1: *Typical two-sided many-to-one matching market with contracts.*

good wage and retirement plan. A public hospital offers positions in ophthalmology and dermatology under non-bargainable benchmark wages. A private hospital offers positions in paediatrics, ophthalmology, and dermatology under attractive wages, job profiles, and retirement plans. We refer to this market as $M = (D, H, X, R) \in \mathcal{M}$, where $D = \{d_p, d_o, d_d\}$, $H = \{h_{pu}, h_{pr}\}$, $X = \{a, b, c, d, e, f, g\}$ with $\mu(a) = (d_p, h_{pr})$, $\mu(b) = (d_p, h_{pr})$, $\mu(c) = (d_o, h_{pr})$, $\mu(d) = (d_d, h_{pr})$, $\mu(e) = (d_d, h_{pu})$, $\mu(f) = (d_o, h_{pu})$, and R is as in Figure 3.1. Clearly, $A = (a, c, e, \{a, c\}, \{e\}) \in \mathcal{A}(M)$.

The paediatrician and the public hospital have no contract available between them. The paediatrician and the private hospital have two contracts available between them, but under different terms. One guarantees a high wage, an attractive retirement plan, but a low job profile. The other guarantees an average wage, an average retirement plan, but a high job profile. The paediatrician and the private hospital do not agree on the terms of contract under which they should be matched.

The public hospital does not display complementarities in its preferences. Indeed, for each $\mathcal{R}_{h_{pu}} \in \mathfrak{R}_{h_{pu}}$ with $\mathcal{R}_{h_{pu}}|_X = R_{h_{pu}}$ and each $X', X'' \subseteq \mathcal{X}_{h_{pu}}|_X$ with $X'' \subsetneq X'$, we have $X'' \cap C(X', \mathcal{R}_{h_{pu}}) \subseteq C(X'', \mathcal{R}_{h_{pu}}\})$.[10] The pri-

[10]We do not restrict our attention to preferences that rule both complementarities and substitutabilities. Formally, for $h \in \mathbb{H}$ and $\mathcal{R}_h \in \mathfrak{R}_h$, we have that $\mathcal{R}_h$ is *separable* if for each $X \in \mathcal{X}_h$ and each $\{x\} \in \mathcal{X}_h \backslash X$, we have $X \cup \{x\}\ \mathcal{R}_h\ X$ if and only if $\{x\}\ \mathcal{R}_h\ \emptyset$. For

vate hospital displays complementarities in its preferences. For instance, $\{a, d\} \cap C(\{a, b, c, d\}, R_{h_{pr}}) = \{a\}$ and $C(\{a, d\}, R_{h_{pr}}\}) = \{d\}$. Thus, for each $\mathcal{R}_{h_{pr}} \in \mathfrak{R}_{h_{pr}}$ with $\mathcal{R}_{h_{pr}}|_X = R_{h_{pr}}$, we have that $\mathcal{R}_{pr}$ is not substitutable.

We end this section with notations. Let $M = (D, H, X, R) \in \mathcal{M}$. First, for $D' \subseteq D$ with $D' \neq \emptyset$ and $H' \subseteq H$ with $H' \neq \emptyset$, let $X|_{D' \cup H'} \equiv \{x \in X :$ there are $d \in D'$ and $h \in H'$ such that $\mu(x) = (d, h)\}$ be the *reduced set of contracts of X relative to $D' \cup H'$*. Also, let $A|_{D' \cup H'} = (A_i|_{D' \cup H'})_{i \in D' \cup H'}$ be the *reduced allocation of $A = (A_i)_{i \in D \cup H} \in \mathcal{A}(M)$ relative to $D' \cup H'$*. Formally, for each $i \in D' \cup H'$, we have $A_i|_{D' \cup H'} \equiv \{x \in A_i :$ there is $j \in D' \cup H'$ such that $\mu(x) = (i, j)$ or $\mu(x) = (j, i)\}$.

Second, for $X' \subseteq X$ with $X' \neq \emptyset$ and $i \in D \cup H$, let $R_i|_{X'}$ be the *restricted preferences of R_i relative to X'*. Formally, if $i \in D$, then $R_i|_{X'}$ is the complete and transitive binary relation over $\mathcal{X}_i|_{X'}$ such that for each $x, x' \in \mathcal{X}_i|_{X'}$, we have $x \; R_i|_{X'} \; x'$ if and only if $x \; R_i \; x'$. If $i \in H$, then $R_i|_{X'}$ is the complete and transitive binary relation over $\mathcal{X}_i|_{X'}$ such that for each $X', X'' \in \mathcal{X}_i|_{X'}$, we have $X' \; R_i|_{X'} \; X''$ if and only if $X' \; R_i \; X''$.

Finally, $A \in \mathcal{A}$ is a *doctor-optimal allocation in $\mathcal{A} \subseteq \mathcal{A}(M)$* if for each $A' \in \mathcal{A}$ and each $d \in D$, we have $A_d \; R_d \; A'_d$. Also, $A \in \mathcal{A}$ is a *hospital-optimal allocation in $\mathcal{A} \subseteq \mathcal{A}(M)$* if for each $A' \in \mathcal{A}$ and each $h \in H$, we have $A_h \; R_h \; A'_h$.

3.3 Properties of rules

In this section, we define the axioms we impose on rules. Let φ be a rule.

Efficiency is standard. There should be no allocation that each agent finds at least as desirable as a selected allocation and at least one agent prefers. Formally, $A \in \mathcal{A}(M)$ is *Pareto-efficient for $M = (D, H, X, R) \in \mathcal{M}$* if there is no $A' \in \mathcal{A}(M)$ such that for each $i \in D \cup H$, we have $A'_i \; R_i \; A_i$ and for at least one $j \in D \cup H$, we have $A'_i \; P_i \; A_i$. Let $P(M)$ be the set of *Pareto-efficient* allocations for $M \in \mathcal{M}$.

Pareto-efficiency: *For each $M \in \mathcal{M}$, we have $\varphi(M) \subseteq P(M)$.*

This axiom implies that if an allocation is the choice of each agent over the set of available contracts, it should be the only selected allocation. Formally,

Unanimity: *For each $M = (D, H, X, R) \in \mathcal{M}$, if there is $A \in \mathcal{A}(M)$ such that for each $i \in D \cup H$, we have $A_i = C(X, R_i)$, then $\varphi(M) = \{A\}$.*

---

each $h \in \mathbb{H}$ and each $\mathcal{R}_h \in \mathfrak{R}_h$, if $\mathcal{R}_h$ is separable, $\mathcal{R}_h$ is substitutable. The converse is not true. Indeed, in Figure 3.1, $\{e\} \; R_{h_{pu}} \; \{e, f\}$ and $\{f\} \; R_{h_{pu}} \; \emptyset$. Thus, for each $\mathcal{R}_{h_{pu}} \in \mathfrak{R}_{h_{pu}}$ with $\mathcal{R}_{h_{pu}}|_X = R_{h_{pu}}$, we have that $\mathcal{R}_{h_{pu}}$ is not separable.

Equity is formulated as follows. First, following a decrease in doctors' (hospitals') population, less doctors (hospitals) fight over the same hospitals (doctors). Efficiency requires to take advantage of this induced welfare surplus. If none of the doctors (hospitals) left is responsible for the decrease, equity requires to solidarily take advantage of it. Thus, if doctors' (hospitals') population decreases, each doctor (hospital) left should find her (its) new situation at least as desirable as her (its) previous situation.

However, as rules are correspondences, a doctor (hospital) could find some of the bundles she (it) may receive in the new situation at least as desirable as some of the bundles she (it) may receive in the previous situation and the other bundles she (it) may receive in the new situation less desirable than some of the bundles she (it) may receive in the previous situation. Without any refinements, we may not determine the overall "sign"of welfare variations.

We require that if doctors' (hospitals') population decreases, each doctor (hospital) left should find the worst bundle she (it) could be allocated in the new situation at least as desirable as the worst bundle she (it) could be allocated in the previous situation. Formally, $(D', H', X', R') \in \mathcal{M}$ is a *doctor-reduction of* $(D, H, X, R) \in \mathcal{M}$ if $D' \subsetneq D$, $H' = H$, $X' = X|_{D' \cup H'}$, and $R' = (R_i|_{X'})_{i \in D' \cup H'}$. Also, $(D', H', X', R') \in \mathcal{M}$ is a *hospital-reduction of* $(D, H, X, R) \in \mathcal{M}$ if $D' = D$, $H' \subsetneq H$, $X' = X|_{D' \cup H'}$, and $R' = (R_i|_{X'})_{i \in D' \cup H'}$.

Own-side population-monotonicity: *For each* $M = (D, H, X, R) \in \mathcal{M}$, *each* $M' = (D', H', X', R') \in \mathcal{M}$, *and each* $A' \in \varphi(M')$, *(i) if* M' *is a doctor-reduction of* M, *then there is* $A \in \varphi(M)$ *such that for each* $d \in D'$, *we have* $A'_d \; R_d \; A_d$, *and (ii) if* M' *is a hospital-reduction of* M, *then there is* $A \in \varphi(M)$ *such that for each* $h \in H'$, *we have* $A'_h \; R_h \; A_h$.

Toda (2006) introduces this axiom in the marriage problems, then generalizes it to the college admission problems.[11] Kayı, Ramaekers, and Yengin (2006) generalize it to the medical job market problems.

Second, suppose that for a market, the planner selects an allocation. Then, some agents leave with what they are allocated. To be consistent with this allocation, the planner should select for this reduced market, the reduced allocation of this allocation relative to the agents still there.[12] This should in particular be true when some agents leave with the agents they are matched to in this allocation.

We require that for each market and each allocation selected for this market, if agents leave with whom they are matched to in this allocation,

[11] He refers to it as *population-monotonicity*.

[12] For a survey on axioms embodying this property in game theory, public economics, and fair allocation, see Thomson (2005).

the reduced allocation of this allocation relative to the agents still there be selected for this reduced market. Formally, for each $M = (D, H, X, R) \in \mathcal{M}$, each $D' \subseteq D$ with $D' \neq \emptyset$, each $H' \subseteq H$ with $H' \neq \emptyset$, and each $A \in \varphi(M)$, let $M|^A_{D' \cup H'}$ be the *reduced market of* M *relative to* $D' \cup H'$ *at* A if: (i) for each $i \in D' \cup H'$, either $A_i = \emptyset$ or for each $x \in A_i$, there is no $j \in (D \cup H) \backslash (D' \cup H')$ with $\{x\} = A_i \cap A_j$, (ii) $X' = X|_{D' \cup H'}$, and (iii) $R' = (R_i|_{X'})_{i \in D' \cup H'}$.

Consistency: *For each* $M = (D, H, X, R) \in \mathcal{M}$, *each* $M' = (D', H', X', R') \in \mathcal{M}$, *and each* $A \in \varphi(M)$, *if* $M' = M|^A_{D' \cup H'}$, *then* $A|_{D' \cup H'} \in \varphi(M')$.

Let us add two remarks on this axiom. First, for each $M = (D, H, X, R) \in \mathcal{M}$, each $M' = (D', H', X', R') \in \mathcal{M}$, and each $A \in \varphi(M)$, if $M' = M|^A_{D' \cup H'}$ and $A' = A|_{D' \cup H'}$, then for each $i \in D' \cup H'$, we have $A'_i = A_i$. Thus, for each market, each allocation selected for this market, and each subset of agents that are matched together in this allocation, an allocation that matches these agents together should still be select after eliminating all "outside" agents.

Second, for each medical job market problem and each allocation selected for this problem, another problem is a reduced problem of this problem at this allocation only if (i) the doctors who leave, leave with their contracts and the hospitals they are matched to in this allocation and (ii) the hospitals that leave, leave with their contracts and all doctors they are matched to in this allocation. For each college admission problem and each allocation selected for this problem, another problem is a reduced problem of this problem at this allocation only if the students who have left, have left with the college they were matched to in this allocation and the colleges that have left, have left with all students they were matched to in this allocation. For each marriage problem and each allocation selected for this problem, another problem is a reduced problem of this problem at this allocation only if (i) the men who leave, leave with the women they were matched to in this allocation and (ii) the women who leave, leave with the men they were matched to in this allocation.

These remarks allow us to link *consistency* with formerly introduced axioms. It is the weakest property we may impose on rules to be consistent with what they select. Sasaki and Toda (1992) introduce it in the marriage problems. Kayı, Ramaekers, and Yengin (2006) generalize it to the medical job market problems.[13] Toda (2006) studies it in the marriage problems, then introduces a stronger axiom in the college admission problems.[14]

[13] They refer to it as *weak consistency*.

[14] Indeed, *consistency* requires that for each college admission problem and each selected

The next axiom is motivated by strategic considerations. As the planner may not force agents into matchings, contracts are based on voluntary participation. Also, agents may negotiate contracts with one another. Thus, as agents may behave strategically forming coalitions, neither efficiency nor equity may be attained. We require that no agent should find rejecting her allocated bundle (i) *to sign none or only some of the contracts it contains, or* (ii) *to sign with other agents contracts it does not contain*, more desirable than accepting it. Formally, $A \in \mathcal{A}(M)$ is *stable for* $M = (D, H, X, R) \in \mathcal{M}$ if:

(i) no agent $i \in D \cup H$ *blocks* A, i.e., there is no $i \in D \cup H$ such that $C(A_i, R_i) \neq A_i$;

(ii) no pair of subsets $(D', H') \subseteq D \times H$ *blocks* A, i.e., there is no pair of subsets $(D', H') \subseteq D \times H$ such that there is $X' \subseteq X$ with: (a) $\{i \in \mathbb{D} \cup \mathbb{H} : \mathcal{X}_i|_{X'} \neq \emptyset\} = D' \cup H'$ and for each $i \in D' \cup H'$, $(b.1)$ $\mathcal{X}_i|_{X'} \nsubseteq A_i$ and $(b.2)$ $C(A_i \cup \mathcal{X}_i|_{X'}, R_i) = \mathcal{X}_i|_{X'}$.

Let $S(M)$ be the set of *stable* allocations for $M \in \mathcal{M}$.

Stability: *For each* $M \in \mathcal{M}$*, we have* $\varphi(M) \subseteq S(M)$.

However, agents may be matched together under different terms of contracts. As they may not agree on these terms, they may have to negotiate them. The outcome of such negotiations depends on the agents' negotiation powers. The agents do not a priori know these powers. Thus, before entering into particular negotiations, each agent compares the following alternatives. If she does not enter into these negotiations, she may sign none, some, or all of the contracts her allocated bundle contains, or sign with others contracts it does not contain. If she enters into these negotiations, she may end up with a contract only advantageous for the other. Thus, if an agent finds the former alternative more desirable than the latter, she does not enter into these negotiations.

For instance, let $M = (D, H, X, R), M' = (D, H, X, R'), M'' = (D, H, X, R'') \in \mathcal{M}$ be such that $D = \{d_1\}$, $H = \{h_1, h_2\}$, $X =$

allocation for this problem, if (i) either colleges leave with all students they are matched to in this allocation or (ii) students leave with the college they are matched to in this allocation, then the reduced allocation of this allocation relative to the agents still there should be selected for this reduced problem. Toda (2006) requires that for each such problem and each selected allocation for this problem, if (i) either colleges leave with all students they are matched to in this allocation or (ii) students leave with the college they are matched to in this allocation *only if all students this college is matched to in this allocation leave*, then the reduced allocation of this allocation relative to the agents still there should be selected for this new problem.

R_{d_1}	R_{h_1}	R_{h_2}	R_{d_1}	R'_{h_1}	R_{h_2}	R'_{d_1}	R_{h_1}	R_{h_2}
a	$\{b\}$	$\{c\}$	a	$\{b\}$	$\{c\}$	b	$\{b\}$	$\{c\}$
b	$\emptyset$	$\emptyset$	b	$\{a\}$	$\emptyset$	a	$\emptyset$	$\emptyset$
c			c	$\emptyset$		c		
$\emptyset$			$\emptyset$			$\emptyset$		

Figure 3.2: *Stability with negotiation powers.*

$\{a,b,c\}$ with $\mu(a) = (d_1,h_1)$, $\mu(b) = (d_1,h_1)$, $\mu(c) = (d_1,h_2)$, and $R = (R_{d_1}, R_{h_1}, R_{h_2}), R' = (R_{d_1}, R'_{h_1}, R_{h_2}), R'' = (R'_{d_1}, R_{h_1}, R_{h_2})$ as Figure 3.2.[15] Let $A = (c, \emptyset, \{c\})$. Clearly, $A \in \mathcal{A}(M) = \mathcal{A}(M') = \mathcal{A}(M'')$. The question is whether in these markets, agents find rejecting their bundle in A to stay on their own or to negotiate with others, more desirable than accepting it.

We begin with M. First, d_1, h_1, nor h_2 finds rejecting her allocated contract to stay on her own more, desirable than keeping it, i.e., $C(\{c\}, R_{d_1}) = \{c\}$, $C(\emptyset, R_{h_1}) = \emptyset$, and $C(\{c\}, R_{h_2}) = \{c\}$. Second, d_1 and h_1 find rejecting their allocated contract c to be matched together under the terms of the available contract b, more desirable than keeping c, i.e., $C(\{c\} \cup \{b\}, R_{d_1}) = \{b\}$ and $C(\{c\} \cup \{b\}, R_{h_1}) = \{b\}$. Also, d_1 finds being matched with h_1 under the terms of the available contract a, more desirable than being matched with it under the terms of any of other available contract, i.e. there is no $x \in X\backslash\{a\}$ such that $\mu(x) = \mu(a)$ and $x \; R_{d_1} \; a$, and h_1 finds being matched with d_1 under the terms of b, more desirable than being matched with her under the terms of any of other available contract, i.e., there is no $x \in X\backslash\{b\}$ such that $\mu(x) = \mu(b)$ and $x \; R_{d_1} \; b$. Thus, if d_1 and h_1 enter into negotiations with one another, depending on their negotiation powers, they will agree on being matched together under the terms of a or b. However, as h_1 finds rejecting c to be matched with d_1 under the terms of a less desirable than keeping c and as it does not a priori know these powers, h_1 will not enter into negotiations with d_1. Thus, A will survive these strategic considerations in M.

Now, consider M'. First, d_1, h_1, nor h_2 finds rejecting her allocated contract to stay on her own more, desirable than keeping it, i.e., $C(\{c\}, R_{d_1}) = \{c\}$, $C(\emptyset, R'_{h_1}) = \emptyset$, and $C(\{c\}, R_{h_2}) = \{c\}$. Second, d_1 and h_1 find rejecting their allocated contract c to be matched together under the terms of the available contract a or b, more desirable than keep-

[15]As we are interested in allocations such that no agent finds rejecting her allocated bundle to sign none or only some of the contracts it contains, more desirable than accepting it, we only represent each agent's preferences relation over the bundles that she finds at least as desirable as the null contract.

ing c, i.e., $C(\{c\} \cup \{a\}, R_{d_1}) = \{a\}$ and $C(\{c\} \cup \{a\}, R'_{h_1}) = \{a\}$, and $C(\{c\} \cup \{b\}, R_{d_1}) = \{b\}$, and $C(\{c\} \cup \{b\}, R'_{h_1}) = \{b\}$. Also, d_1 finds being matched with h_1 under the terms of a, more desirable than being matched with it under the terms of any of other available contract, i.e., there is no $x \in X \backslash \{a\}$ such that $\mu(x) = \mu(a)$ and $x \; R_{d_1} \; a$, and h_1 finds being matched with d_1 under the terms of b, more desirable than being matched with her under the terms of any of other available contract, i.e., there is no $x \in X \backslash \{b\}$ such that $\mu(x) = \mu(b)$ and $x \; R'_{h_1} \; b$. Thus, if d_1 and h_1 enter into negotiations with one another, depending on their negotiation powers, they will agree on being matched together under the terms of a or b. However, this time, as both find rejecting c to be matched together under the terms of a or b, more desirable than keeping it, they will enter into negotiations with one another. Thus, A will not survive these strategic considerations.

Finally, consider M''. First, d_1, h_1, nor h_2 finds rejecting her allocated contract to stay on her own, more desirable than keeping it, i.e., $C(\{c\}, R'_{d_1}) = \{c\}$, $C(\emptyset, R_{h_1}) = \emptyset$, and $C(\{c\}, R_{h_2}) = \{c\}$. Second, d_1 and h_1 find rejecting their allocated contract c to be matched together under the terms of the available contract b, more desirable than keeping c, i.e., $C(\{c\} \cup \{b\}, R'_{d_1}) = \{b\}$ and $C(\{c\} \cup \{b\}, R_{h_1}) = \{b\}$. Also, d_1 and h_1 find being matched together under the terms of b, more desirable than being matched together under the terms of any of other available contract, i.e., there is no $x \in X \backslash \{b\}$ such that $\mu(x) = \mu(b)$, $x \; R'_{d_1} \; b$, and $\{x\} \; R_{h_1} \; \{b\}$. Thus, this time, if d_1 and h_1 enter into negotiations with one another, whatever their negotiation powers, they will agree on being matched together under the terms of b. Thus, as both find rejecting c to be matched together under the terms of b, more desirable than keeping it, they will enter into negotiations with one another. Thus, A will not survive these strategic considerations.

The reasons why A will not survive strategic considerations in M' and M'' by opposition to M are different. In M', it is because d_1 and h_1 agree that being matched together under any available terms of contract is more desirable than not being matched together at all. However, they still disagree on these terms. In M, it is because they agree that being matched together under the terms of b is more desirable than not being matched together at all. However, they still disagree that being matched together under any available terms of contract is more desirable than not being matched together at all.

Thus, we introduce a weaker notion of *stability*. No agent should find rejecting her allocated bundle (i) *to sign none or only some of the contracts it contains, or* (ii) *to negotiate with others contracts it does not contain*, more desirable than accepting it. Formally, $A \in \mathcal{A}(M)$ is *stable with negotiations for* $M = (D, H, X, R) \in \mathcal{M}$ if:

(i) no agent $i \in D \cup H$ *blocks* A;

(ii) no pair of subsets $(D', H') \subseteq D \times H$ *blocks with negotiations* A, i.e., there is no pair of subsets $(D', H') \subseteq D \times H$ such that there is $X' \subseteq X$ with: (a) $\{i \in \mathbb{D} \cup \mathbb{H} : \mathcal{X}_i|_{X'} \neq \emptyset\} = D' \cup H'$ and for each $i \in D' \cup H'$, $(b.1)$ $\mathcal{X}_i|_{X'} \not\subseteq A_i$, $(b.2)$ $C(A_i \cup \mathcal{X}_i|_{X'}, R_i) = \mathcal{X}_i|_{X'}$, and $(b.3)$ for each $x \in \mathcal{X}_i|_{X'}$, there is no $x' \in X \backslash \{x\}$ with $\mu(x') = \mu(x)$ such that $C(A_i \cup \mathcal{X}_i|_{X'} \backslash \{x\}, R_i)$ R_i $(\mathcal{X}_i|_{X'} \backslash \{x\} \cup \{x'\})$ and if $j \in D' \cup H'$ with $\mu(x') = (i, j)$ or $\mu(x') = (j, i)$, then $(\mathcal{X}_j|_{X'} \backslash \{x\} \cup \{x'\})$ R_j $\mathcal{X}_i|_{X'}$.

Let $SN(M)$ be the set of *stable allocations with negotiations* for $M \in \mathcal{M}$.

Stability with negotiations: *For each $M \in \mathcal{M}$, we have $\varphi(M) \subseteq SN(M)$.*

The last axiom is also motivated by strategic considerations. The planner may not know agents' preferences. Thus, as agents may behave strategically when announcing them, neither efficiency nor equity may be attained. However, Roth (1982) proves that there is an incompatibility between *stability* and *strategy-proofness*, i.e., no agent finds misrepresenting her preferences more desirable than revealing them.

We require that the set of all selected allocations correspond to the set of all Nash-equilibria of some game. Formally, let $D \in \mathcal{D}$, $H \in \mathcal{H}$, and $X \in \mathcal{X}$. A *mechanism* $\Gamma = (\Sigma, g)$ is a *strategy space* $\Sigma = \prod_{i \in D \cup H} \Sigma_i$ and *an outcome function* g that assigns to each strategy profile $\sigma \in \Sigma$, a list of subsets of contracts $g(\sigma) = (g_i(\sigma))_{i \in D \cup H} \in \prod_{i \in D \cup H} \mathcal{X}_i|_X$ such that for each $d \in D$ and each $h \in H$, if there is $x \in g_d(\sigma) \cup g_h(\sigma)$, then $\{x\} = g_d(\sigma) \cap_h g(\sigma)$. Let $R = (R_i)_{i \in D \cup H}$ be such that $(D, H, X, R) \in \mathcal{M}$. A strategy $\sigma \in \Sigma$ is a *Nash-equilibrium for R in* $\Gamma = (\Sigma, g)$ if for each $i \in D \cup H$ and each $\sigma_i' \in \Sigma_i$, we have $g_i(\sigma_i', \sigma_{-i})$ R_i $g_i(\sigma_i', \sigma_{-i})$. Let $NE^{\Gamma}(D, H, X, R)$ be the set of all *Nash-equilibria for R in* Γ.

Nash-implementability: *For each $D \in \mathcal{D}$, each $H \in \mathcal{H}$, and each $X \in \mathcal{X}$, there is a mechanism Γ such that for each $R = (R_i)_{i \in D \cup H}$ with $(D, H, X, R) \in \mathcal{M}$, we have $\varphi(D, H, X, R) = g(NE^{\Gamma}(D, H, X, R))$.*

Maskin (1977, 1999) proves that this axiom implies that following a preferences change, if there is a previously selected allocation such that each agent's bundle has improved, this allocation should still be selected. Formally, let $M = (D, H, X, R) \in \mathcal{M}$ and $A \in \mathcal{A}(M)$. For each $i \in D \cup H$ and each R_i' such that there is $\mathcal{R}_i \in \mathfrak{R}_i$ with $R_i' = \mathcal{R}_i|_X$, if for each $X' \subseteq X$ with A_i P_i X', we have A_i P_i' X', then R_i' is a *Maskin-monotonic transformation of R_i at A_i*. For each $R' = (R_i')_{i \in D \cup H}$ such for each $i \in D \cup H$, there is $\mathcal{R}_i \in \mathfrak{R}_i$ with $R_i' = \mathcal{R}_i|_X$, if for each $i \in D \cup H$, we have that R_i' is a Maskin-monotonic transformation of R_i at A_i, then R' is a Maskin-monotonic transformation of R at A.

Maskin-monotonicity: *For each* $M = (D, H, X, R) \in \mathcal{M}$, *each* $M' = (D', H', X', R')$, *and each* $A \in \varphi(M)$, *if* $D' = D$, $H' = H$, $X' = X$, *and* R' *is a Maskin-monotonic transformation of* R *at* A, *then* $A \in \varphi(M')$.

3.4 Results

In this section, we come to our results. We prove that only the rule that selects for each problem, all *stable* allocations satisfies *Pareto-efficiency*, *own-side population-monotonicity*, *consistency*, and *Nash-implementability*. Then, we prove that if we weaken *stability* to *stability with negotiations*, there is another rule that satisfies *Pareto-efficiency* and *consistency*.

To come to these results, first, we prove that there is a rule that satisfies *Pareto-efficiency*, *own-side population-monotonicity*, and *consistency*. We prove that it is the only rule that satisfies *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity*. Haake and Klaus (2005) prove that this rule also satisfies *Nash-implementability*. Thus, as *Pareto-efficiency* implies *unanimity* and as *Nash-implementability* implies *Maskin-monotonicity*, we prove that this rule is characterized by only a few of the axioms we impose on rules. For each market, this rule selects all *stable* allocations. Formally,

The Stable rule, φ^S: *For each* $M \in \mathcal{M}$, *we have* $\varphi^S(M) = S(M)$.

This first series of results that are in Kayı, Ramaekers, and Yengin (2006), allow us to conclude that the only solution for two-sided many-to-one matching markets with contracts that satisfies efficiency, equity, and strategic compatibility properties simultaneously is to select all *stable* allocations.

The intuition for this conclusion is three-fold. First, a weak efficiency property as *unanimity* and *own-side population-monotonicity* imply that if some agents' choices over the set of available contracts is to be matched together under the terms of particular contracts, these agents should be matched together under terms of these contracts (Statement 2, Step 1). Indeed, suppose that a subset of agents' choices over the set of available contracts is to be matched together under the terms of particular contracts. Then, for each agent not in the formerly cited subset, pick a potential agent and a potential contract such that this pair of agents' choices over the set of now available contracts is to be matched together under the terms of this particular contract.

Then, add these agents and contracts in addition to the agents and the contracts present in the original market. By *unanimity*, in this bigger market, each agent is allocated her choice over the set of available contracts, in particular each agent in the formerly cited subset. Thus, by *own-side population-monotonicity*, in each doctor-reduction of this bigger market, each doctor in the formerly cited subset is allocated her choice over the set of available contracts and hence each hospital in the formerly cited subset is allocated her choice over the set of available contracts. Thus, by *own-side population-monotonicity*, in each hospital-reduction of each doctor-reduction of this bigger market, in particular in the original market, each hospital in the formerly cited subset is allocated her choice over the set of available contracts and hence each doctor in the formerly cited subset is allocated her choice over the set of available contracts.

By the same logic, if some agents' choices over their allocated bundle is to be matched with herself, these agents should not be allocated this bundle. Thus, *unanimity* and *own-side population-monotonicity* further imply that no agent should find rejecting her allocated bundle to sign none the contracts it contains, more desirable than accepting it (Statement 2, Step 2).

Second, *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity* imply that no agent should find rejecting her allocated bundle to sign only some of the contracts it contains, or to sign with other agents contracts it does not contain, more desirable than accepting it (Statement 2, Step 3). By contradiction, suppose that agents find rejecting their allocated bundles to sign with other agents contracts it contains or it does not contain, more desirable than accepting it. Then, suppose that there is a Maskin-monotonic transformation of these agents' preferences such that their choices over the set of available contracts is to be matched according to what they find more desirable than accepting their allocated bundle. By the previous argument, these agents then have to be matched according to what they find more desirable than accepting their allocated bundle. Thus, the allocation selected in the original market is not be selected in the new market, contradicting *Maskin-monotonicity*.

Third, by Haake and Klaus (2005), no subcorrespondence of the Stable rule satisfies *Maskin-monotonicity* (Statement 2, Step 4). We use these arguments in the second part of this section, to bring out the role of the multiplicity of contracts in two-sided many-to-one matching markets with contracts.

Theorem 1

1. *The Stable rule satisfies* Pareto-efficiency, own-side population-monotonicity, consistency, *and* Nash-implementability.

2. *Only the Stable rule satisfies* unanimity, own-side population-monotonicity, *and* Maskin-monotonicity.

Proof.

Before proving Statements 1 and 2, note that for each $M = (D, H, X, R) \in \mathcal{M}$ *and each* $A \in \mathcal{A}(M)$*, we have* $A \in S(M)$ *if and only if:*

- *there is no* $d \in D$ *with* $A_d \neq \emptyset$ *such that* $C(A_d, R_d) = \emptyset$*;*
- *there is no* $h \in H$ *such that* $C(A_h, R_h) \subsetneq A_h$*;*
- *there is no pair of subsets* $(D', \{h\}) \subseteq D \times H$ *such that there is* $X' \subseteq \mathcal{X}_h|_X$ *with:*
    - $\{d \in \mathbb{D} : \mathcal{X}_d|_{X'} \neq \emptyset\} = D'$*;*
    - $X' \not\subseteq A_h$*;*
    - $C(A_h \cup X', R_h) = X'$*;*
    - *for each* $d \in D'$*, there is* $x \in X'$ *such that* $\mathcal{X}_d|_{X'} = \{x\}$*,* $A_d \neq x$*, and* $C(A_d \cup \{x\}, R_d) = \{x\}$*.*

Statement 1:

Pareto-efficiency: *Straightforward.*

Own-side population-monotonicity: *Let* $M = (D, H, X, R) \in \mathcal{M}$ *and* $\tilde{M} = (\tilde{D}, \tilde{H}, \tilde{X}, \tilde{R}) \in \mathcal{M}$ *be such that* $\tilde{M}$ *is a doctor-reduction of* M*. (The symmetric result holds when* $\tilde{M}$ *is a hospital-reduction of* M*.) Let* $A^H \in \mathcal{A}(M)$ *and* $\tilde{A}^H \in \mathcal{A}(\tilde{M})$ *be the hospital-optimal allocation in* $\varphi^S(M)$ *and* $\varphi^S(\tilde{M})$*, respectively. By Theorem 4.3 of Ostrovsky (2005), for each* $d \in D$*, we have* $\tilde{A}^H_d \; R_d \; A^H_d$*. By Theorem 4 of Hatfield and Milgrom (2005), for each* $\tilde{A} \in \varphi^S(\tilde{M})$ *and each* $d \in D$*, we have* $\tilde{A}_d \; R_d \; \tilde{A}^H_d$*. Thus, for each* $\tilde{A} \in \varphi^S(\tilde{M})$ *and each* $d \in D$*, there is* $A^H \in \varphi^S(M)$ *such that* $\tilde{A}_d \; R_d \; A^H_d$*.*

Consistency:. *Let* $M = (D, H, X, R) \in \mathcal{M}$ *and* $A \in \varphi^S(M)$*. Let* $M' = (D', H', X', R') \in \mathcal{M}$ *be the reduced market of* M *relative to* $D' \cup H'$ *at* A*, i.e.,* $M' = M|^A_{D' \cup H'}$*. Let* $A' \in \mathcal{A}(M')$ *be the reduced allocation of* A *relative to* $D' \cup H'$*, i.e.,* $A' = A|_{D' \cup H'}$*. By contradiction, suppose* $A' \notin \varphi^S(M')$*. Then, distinguish three cases.*

Case 1: *Suppose that there is* $d^* \in D'$ *with* $A'_{d^*} \neq \emptyset$ *such that* $C(A'_{d^*}, R'_{d^*}) = \emptyset$*. As* $A' = A|_{D' \cup H'}$*, we have* $A'_{d^*} = A_{d^*}$ *implying* $C(A'_{d^*}, R'_{d^*}) = C(A_{d^*}, R'_{d^*})$*. As* $R'_{d^*} = R_{d^*}|_{X'}$*, we have* $C(A_{d^*}, R'_{d^*}) = C(A_{d^*}, R_{d^*})$*. Thus,* $C(A_{d^*}, R_{d^*}) = \emptyset$*. As* $A'_{d^*} \neq \emptyset$ *and* $A'_{d^*} = A_{d^*}$*, we have* $A_{d^*} \neq \emptyset$*. Altogether, there is* $d^* \in D$ *with* $A_{d^*} \neq \emptyset$ *such that* $C(A_{d^*}, R_{d^*}) = \emptyset$*, contradicting* $A \in \varphi^S(M)$*.*

Case 2: *Suppose that there is* $h^* \in H'$ *such that* $C(A'_{h^*}, R'_{h^*}) \subsetneq A'_{h^*}$*. Let*

$X^* \subsetneq A'_{h^*}$ be such that $X^* = C(A'_{h^*}, R'_{h^*})$. As $A' = A|_{D' \cup H'}$, we have $A'_{h^*} = A_{h^*}$ implying $C(A'_{h^*}, R'_{h^*}) = C(A_{h^*}, R'_{h^*})$. As $R'_{h^*} = R_{h^*}|_{X'}$, we have $C(A_{h^*}, R'_{h^*}) = C(A_{h^*}, R_{h^*})$. Thus, $C(A_{h^*}, R_{h^*}) = X^*$. As $X^* \subsetneq A'_{h^*}$ and $A'_{h^*} = A_{h^*}$, we have $X^* \subsetneq A_{h^*}$. Altogether, there is $h^* \in H$ such that $C(A_{h^*}, R_{h^*}) \subsetneq A_{h^*}$, contradicting $A \in \varphi^S(M)$.

Case 3: *Suppose that there is $(D^*, \{h*\}) \subseteq D' \times H'$ such that there is $X^* \subseteq \mathcal{X}_{h^*}|_{X'}$ with $\{d \in \mathbb{D} : \mathcal{X}_d|_{X^*} \neq \emptyset\} = D^*$, $X^* \nsubseteq A'_{h^*}$, $C(A'_{h^*} \cup X^*, R'_h) = X^*$, and for each $d \in D^*$, there is $x^d \in X^*$ such that $\mathcal{X}_d|_{X^*} = \{x^d\}$, $A'_d \neq x^d$, and $C(A'_d \cup \{x^d\}, R'_d) = \{x^d\}$. As $A' = A|_{D' \cup H'}$, we have $A'_{h^*} = A_{h^*}$ and for each $d \in D^*$, we have $A'_d = A_d$ implying $C(A'_{h^*} \cup X^*, R'_{h^*}) = C(A_{h^*} \cup X^*, R'_{h^*})$ and for each $d \in D^*$, we have $C(A'_d \cup \{x^d\}, R'_d) = C(A_d \cup \{x^d\}, R'_d)$. As $R' = R|_{X'}$, we have $C(A_{h^*} \cup X^*, R'_{h^*}) = C(A_{h^*} \cup X^*, R_{h^*})$ and for each $d \in D^*$, we have $C(A_d \cup \{x^d\}, R'_d) = C(A_d \cup \{x^d\}, R_d)$. Thus, $C(A_{h^*} \cup X^*, R_{h^*}) = X^*$ and for each $d \in D^*$, we have $C(A'_d \cup \{x^d\}, R'_d) = \{x^d\}$. As $X^* \subseteq \mathcal{X}_{h^*}|_{X'}$ and $\mathcal{X}_{h^*}|_{X'} = (\mathcal{X}_{h^*}|_X)|_{D' \cup H'}$, we have $X^* \subseteq \mathcal{X}_{h^*}|_X$. As $X^* \nsubseteq A'_{h^*}$ and $A'_{h^*} = A_{h^*}$, we have $X^* \nsubseteq A_{h^*}$. For each $d \in D^*$, as $\mathcal{X}_d|_{X^*} \neq A'_d$ and $A'_d = A_d$, we have $\mathcal{X}_d|_{X^*} \neq A_d$. Altogether, there is $(D^*, \{h*\}) \subseteq D \times H$ such that there is $X^* \subseteq \mathcal{X}_{h^*}|_X$ with $\{d \in \mathbb{D} : \mathcal{X}_d|_{X^*} \neq \emptyset\} = D^*$, $X^* \nsubseteq A_{h^*}$, $C(A'_{h^*} \cup X^*, R_h) = X^*$, and for each $d \in D^*$, there is $x^d \in X^*$ such that $A_d \neq \{x^d\}$, and $C(A_d \cup \{x^d\}, R_d) = \{x^d\}$, contradicting $A \in \varphi^S(M)$.*

Nash-implementability: *Theorem 1 of Haake and Klaus (2005).*

Statement 2: *Suppose that φ satisfies the axioms of Statement 2. In the following step process, we prove that for each $M \in \mathcal{M}$, we have $\varphi(M) = \varphi^S(M)$.*

Step 1: *For each $M = (D, H, X, R) \in \mathcal{M}$, if there is $h \in H$ such that for each $x \in C(X, R_h)$, there is $d \in D$ with $x = C(X, R_d)$, then for each $A \in \varphi(M)$, we have $A_h = C(X, R_h)$.*
Let $M = (D, H, X, R) \in \mathcal{M}$ and $h^* \in H$ be such that for each $x \in C(X, R_{h^*})$, there is $d \in D$ with $x = C(X, R_d)$. Let $D^* \equiv \{d \in D :$ there is $x \in X$ such that $x = C(X, R_d)$ and $x \in C(X, R_{h^*})\}$. For each $d \in D^*$, let $x^{dh^*} \in X$ be such that $x^{dh^*} = C(X, R_d)$ and $x^{dh^*} \in C(X, R_{h^*})$. For each $h \in H \backslash \{h^*\}$, let $d^h \in \mathbb{D}$ and $x^{d^h h} \in \mathbb{X}$ be such that $\mu(x^{d^h h}) = (d^h, h)$. For each $d \in D \backslash D^*$, let $h^d \in \mathbb{H}$ and $x^{dh^d} \in \mathbb{X}$ be such that $\mu(x^{dh^d}) = (d, h^d)$. Let $\hat{D} \equiv \bigcup_{h \in H \backslash \{h^*\}} \{d^h\}$, $\hat{H} \equiv \bigcup_{d \in D \backslash D^*} \{h^d\}$, and $\hat{X} \equiv \bigcup_{d \in D \backslash D^*} \{x^{dh^d}\} \cup \bigcup_{h \in H \backslash \{h^*\}} \{x^{d^h h}\}$. Then,

- *Let $M' = (D', H', X', R') \in \mathcal{M}$ be such that $D' = D \cup \hat{D}$, $H' = H \cup \hat{H}$, $X' = X \cup \hat{X}$, and R' is as follows. First, for each $i \in D^* \cup \{h^*\}$, we have $R'_i|_X = R_i$ and $C(X', R'_i) = C(X, R_i)$. Second, for each $d \in D \backslash D^*$,* *we have $R'_d|_X = R_d$ and $C(X', R'_d) = x^{dh^d}$, and for $h^d \in \hat{H}$, we have $C(X', R'_{h^d}) = \{x^{dh^d}\}$. Third, for each $h \in H\backslash\{h^*\}$, we have $R'_h|_X = R_h$ and $C(X', R'_h) = \{x^{d^h h}\}$, and for $d^h \in \hat{D}$, we have $C(X', R'_{d^h}) = x^{d^h h}$. Let $A' = (A'_i)_{i\in D'\cup H'}$ be such that for each $d \in D^*$, we have $A'_d = x^{dh^*}$, $A'_{h^*} = \bigcup_{d\in D^*}\{x^{dh^*}\}$, for each $d \in D\backslash D^*$, we have $A'_d = x^{dh^d}$ and for $h^d \in \hat{H}$, we have $A'_{h^d} = \{x^{dh^d}\}$, and for each $h \in H\backslash\{h^*\}$, we have $A'_h = \{x^{d^h h}\}$ and for $d^h \in \hat{D}$, we have $A'_{d^h} = x^{d^h h}$. Clearly, $A' \in \mathcal{A}(M')$. Also, **(i)** for each $i \in D' \cup H'$, we have $A'_i = C(X', R'_i)$. Thus, by* unanimity, ***(ii)*** *$\varphi(M') = \{A'\}$.*

- *Let $M'' = (D'', H'', X'', R'') \in \mathcal{M}$ be such that $D'' = D$, $H'' = H'$, $X'' = X'|_{D''\cup H''}$, and $R'' = (R'_i|_{X''})_{i\in D''\cup H''}$. Clearly, M'' is a doctor-reduction of M'. Thus, by **(ii)** and by* own-side population-monotonicity, *for each $A \in \varphi(M'')$ and each $d \in D^*$, we have $A_d\ R'_d\ A'_d$. By **(i)**, for each $d \in D^*$, we have $A'_d = C(X', R'_d)$. Thus, for each $A \in \varphi(M'')$ and each $d \in D^*$, we have $A_d = A'_d$. Thus, for each $A \in \varphi(M'')$, we have $A_{h^*} = A'_{h^*}$. As $R''_{h^*} = R'_{h^*}|_{X''}$, $C(X', R'_{h^*}) = A'_{h^*}$, and $A'_{h^*} \subseteq X''$, we have $C(X'', R''_{h^*}) = A'_{h^*}$. Thus, **(iii)** for each $A \in \varphi(M'')$, we have $A_{h^*} = C(X'', R''_{h^*})$.*

Clearly, M is a hospital-reduction of M''. Let $A \in \varphi(M)$. Thus, by own-side population-monotonicity, *there is $A'' \in \varphi(M'')$ such that $A_{h^*}\ R''_{h^*}\ A''_{h^*}$. By **(iii)**, $A''_{h^*} = C(X'', R''_{h^*})$. Thus, $A_{h^*} = C(X'', R''_{h^*})$. As $R_{h^*} = R''_{h^*}|_X$, $C(X'', R''_{h^*}) = A''_{h^*}$, and $A''_{h^*} \subseteq X$, we have $C(X, R_{h^*}) = C(X'', R''_{h^*})$. Thus, $A_{h^*} = C(X, R_{h^*})$.*

Step 2: *For each $M = (D, H, X, R) \in \mathcal{M}$, each $A \in \varphi(M)$, and each $i \in D \cup H$ with $A_i \neq \emptyset$, we have $C(A_i, R_i) \neq \emptyset$.*
Let $M = (D, H, X, R) \in \mathcal{M}$, $A \in \varphi(M)$, and $i^* \in D \cup H$ be such that $A_{i^*} \neq \emptyset$. By contradiction, suppose $C(A_{i^*}, R_{i^*}) = \emptyset$. Then, distinguish two cases.

Case 1: *$i^* \in D$. Let $d^* \in D$ be such that $d^* = i^*$. Let $\tilde{D} = \{d \in D : A_d \neq \emptyset$ and $C(A_d, R_d) = \emptyset\}$. Let $\tilde{H} = \{h \in H : A_h \neq \emptyset$ and $C(A_h, R_h) = \emptyset\}$. For each $h \in H$, let $d^h \in \mathbb{D}$ and $x^{d^h h} \in \mathbb{X}$ be such that $\mu(x^{d^h h}) = (d^h, h)$. Let $\hat{D} \equiv \bigcup_{h\in H}\{d^h\}$. Let $\hat{X} = \bigcup_{h\in H} x^{\hat{d^h h}}$. Then,*

- *Let $M' = (D', H', X', R') \in \mathcal{M}$ be such that $D' = D$, $H' = H$, $X' = X$, and R' is as follows. First, for each $i \in (D \cup H)\backslash(\tilde{D} \cup \tilde{H})$, we have $C(X', R'_i) = A_i$. Second, for each $i \in \tilde{D} \cup \tilde{H}$, we have $C(X', R'_i) = \emptyset$ and for each $X'' \subseteq X'$ with $X'' \neq \emptyset$, we have $C(A_i \cup X'', R'_i) = A_i$. Clearly, R' is a Maskin-monotonic transformation of R at A. Let $A' \in \mathcal{A}(M')$ be such that for each $i \in D \cup H$, we have $A'_i = A_i$. By* Maskin-monotonicity, *$A' \in \varphi(M')$. Clearly, $A'_{d^*} \neq \emptyset$.*

- *Let $M'' = (D'', H'', X'', R'') \in \mathcal{M}$ be such that $D'' = D' \cup \hat{D}$, $H'' = H'$, $X'' = X' \cup \hat{X}$, and R'' is as follows. First, for each $h \in H''$, we have $R''_h|'_X = R'_h$ and $C(X'', R''_h) = \{x^{d^h h}\}$, and for each $d^h \in \hat{D}$, we have $C(X'', R''_{d^h}) = x^{d^h h}$. Second, for each $d \in D''\backslash\hat{D}$, we have $R''_d = R'_d$. By Step 1,* unanimity *and* own-side population-monotonicity *imply that for each $A'' \in \varphi(M')$ and each $h \in H''$, we have $A''_h = C(X'', R''_h)$. Thus, for each $A'' \in \varphi(M'')$ and each $h \in H''$, we have $A''_h = \{x^{d^h h}\}$. Thus, for each $A'' \in \varphi(M'')$ and each $d \in D''\backslash\hat{D}$, we have $A''_d = \emptyset$. Thus, for each $A'' \in \varphi(M')$, we have $A''_{d^*} = \emptyset$.*

Clearly, M' is a doctor-reduction of M''. For each $A'' \in \varphi(M')$, we have $A''_{d^*} = \emptyset$. Also, $A' \in \varphi(M')$ such that $A'_{d^*} \neq \emptyset$. Thus, as $C(A_{d^*}, R_{d^*}) = \emptyset$, we have $A''_{d^*}\ P_{d^*}\ A'_{d^*}$, contradicting own-side population-monotonicity.

Case 2: *$i^* \in H$. Let $h^* \in H$ be such that $h^* = i^*$. Let $\tilde{D} = \{d \in D : A_d \neq \emptyset \text{ and } C(A_d, R_d) = \emptyset\}$. Let $\tilde{H} = \{h \in H : A_h \neq \emptyset \text{ and } C(A_h, R_h) = \emptyset\}$. Let $\hat{h} \in \mathbb{H}\backslash\{h^*\}$ be such that for each $d \in D$, there is $x^{d\hat{h}} \in \mathbb{X}$ with $\mu(x^{d\hat{h}}) = (d, \hat{h})$. Let $\hat{X} = \bigcup_{d\in D} x^{d\hat{h}}$. Then,*

- *Let $M' = (D', H', X', R') \in \mathcal{M}$ be such that $D' = D$, $H' = H$, $X' = X$, and R' is as follows. First, for each $i \in (D \cup H)\backslash(\tilde{D} \cup \tilde{H})$, we have $C(X', R'_i) = A_i$. Second, for each $i \in \tilde{D} \cup \tilde{H}$, we have $C(X', R'_i) = \emptyset$ and for each $X'' \subseteq X'$ with $X'' \neq \emptyset$, we have $C(A_i \cup X'', R'_i) = A_i$. Clearly, R' is a Maskin-monotonic transformation of R at A. Let $A' \in \mathcal{A}(M')$ be such that for each $i \in D \cup H$, we have $A'_i = A_i$. By* Maskin-monotonicity, *$A' \in \varphi(M')$. Clearly, $A'_{h^*} \neq \emptyset$.*

- *Let $M'' = (D'', H'', X'', R'') \in \mathcal{M}$ be such that $D'' = D'$, $H'' = H' \cup \{\hat{h}\}$, $X'' = X' \cup \hat{X}$, and R'' is as follows. First, for each $d \in D''$, we have $R''_d|_{X'} = R'_d$ and $C(X'', R''_d) = x^{d\hat{h}}$, and for $\hat{h} \in H''$, we have $C(X'', R''_{\hat{h}}) = \hat{X}$. Second, for each $h \in H''\backslash\{\hat{h}\}$, we have $R''_h = R'_h$. By Step 1,* unanimity *and* own-side population-monotonicity *imply that for each $A'' \in \varphi(M')$, we have $A''_{\hat{h}} = C(X'', R''_{\hat{h}})$. Thus, for each $A'' \in \varphi(M'')$, we have $A''_{\hat{h}} = \hat{X}$. Thus, for each $A'' \in \varphi(M'')$ and each $d \in D''$, we have $A''_d = x^{d\hat{h}}$. Thus, for each $A'' \in \varphi(M')$, we have $A''_{h^*} = \emptyset$.*

Clearly, M' is a hospital-reduction of M''. For each $A'' \in \varphi(M')$, we have $A''_{h^*} = \emptyset$. Also, $A' \in \varphi(M')$ such that $A'_{h^*} \neq \emptyset$. Thus, as $C(A_{h^*}, R_{h^*}) = \emptyset$, we have $A''_{h^*}\ P_{h^*}\ A'_{h^*}$, contradicting own-side population-monotonicity.

Step 3: *For each $M \in \mathcal{M}$, we have $\varphi(M) \subseteq \varphi^S(M)$.*
Let $M = (D, H, X, R) \in \mathcal{M}$ and $A \in \varphi(M)$. By contradiction, suppose

$A \not\in \varphi^S(M)$. *By Step 2,* unanimity, own-side population-monotonicity, *and* Maskin-monotonicity *imply that for each $d \in D$ with $A_d \neq \emptyset$, we have $C(A_d, R_d) \neq \emptyset$. Thus, there is a blocking pair $(D^*, \{h^*\}) \subseteq D \times H$ such that there is $X^* \in \mathcal{X}_{h^*}|_X$ with $\{d \in \mathbb{D} : \mathcal{X}_d|_{X^*} \neq \emptyset\} = D^*$, $X^* \neq A_{h^*}$, $C(A_{h^*} \cup X^*, R_{h^*}) = X^*$ and for each $d^* \in D^*$, we have $|\mathcal{X}_{d^*}|_{X^*}| = 1$, $\mathcal{X}_{d^*}|_{X^*} \neq A^*_{d^*}$, and $C(A_{d^*} \cup \mathcal{X}_{d^*}|_{X^*}, R_{d^*}) = \mathcal{X}_{d^*}|_{X^*}$. For each $d^* \in D^*$, let $x^{d^*} \in X^*$ be such that $\{x^{d^*}\} = \mathcal{X}_{d^*}|_{X^*}$. Then,*

- *Let $M' = (D', H', X', R') \in \mathcal{M}$ be such that $D' = D$, $H' = H$, $X' = X$, and R' is as follows. First, for each $X'' \subseteq \mathcal{X}_{h^*}|_{X'}$, we have $X''\ R'_{h^*}\ \emptyset$ if and only if $X'' \subseteq A_{h^*} \cup X^*$. Second, for each $d^* \in D^*$, we have $x^{d^*}\ R_{d^*}\ A_{d^*}$ and for each $x \in X'_{d^*}$, we have $x\ R'_{d^*}\ \emptyset$ if and only if $x \in A_{d^*} \cup X^*$. Third, for each $i \in [D \backslash D^*] \cup [H \backslash \{h^*\}]$, we have $R'_i = R_i$. By Step 2,* unanimity, own-side population-monotonicity, *and* Maskin-monotonicity *imply that for each $i \in D^* \cup \{h^*\}$, we have $A_i\ R_i\ \emptyset$. Thus, **(i)** R' is a Maskin-monotonic transformation of R at A. Also, $C(X', R'_{h^*}) = X^*$ and for each $d^* \in D^*$, we have $C(X', R'_{d^*}) = x^{d^*}$. Thus, by Step 1,* unanimity *and* own-side population-monotonicity *imply that for each $A' \in \varphi(M')$, we have $A'_{h^*} = C(X', R'_{h^*})$. Thus, for each $A' \in \varphi(M')$, we have $A'_{h^*} = X^*$. By assumption, $X^* \neq A_{h^*}$. Thus, $A \not\in \varphi(M')$, contradicting, by **(i)**,* Maskin-monotonicity.

Step 4: *For each $M \in \mathcal{M}$, we have $\varphi(M) = \varphi^S(M)$.*
By Corollary 1 of Haake and Klaus (2005), if a rule is a subcorrespondence of the Stable rule and it satisfies Maskin-monotonicity, *then it is the Stale rule. Thus, by Step 3 and* Maskin-monotonicity, *for each $M \in \mathcal{M}$, we have $\varphi(M) = \varphi^S(M)$.* ■

In Theorem 2, we prove that the Stable rule is not the only rule that satisfies *Pareto-efficiency* and *consistency.* Moreover, we prove that the sets of allocations the other rule selects need not be a lattice. However, this rule violates *Maskin-monotonicity.* For each market, it selects all *Pareto-efficient* and *stable allocations with negotiations.* Formally,

The Stable rule with negotiations, φ^{SN}**:** *For each $M \in \mathcal{M}$, we have $\varphi^{SN}(M) = P(M) \cap SN(M)$.*

This second series of results allow us to conclude that by opposition to when there is only one way to match agents, it is not efficiency and equity properties that impose a conflict between the common interests of each of the two sides of market, but efficiency, equity, and strategic compatibility properties.

The intuition for this conclusion is simple. The Stable rule with negotiations and the Stable rule satisfy efficiency and equity properties. When there is only one way to match agents, as agents do not have to negotiate, the Stable rule with negotiations corresponds to the Stable rule. When there is more than one way to match agents, as agents may have to negotiate, the Stable rule with negotiations includes the Stable rule.

However, if the agents agree on the contracts under which they may be matched, the set of *Pareto-efficient* and *stable allocations with negotiations* corresponds to the set of *stable* allocations. Also, incentive compatibility properties imply that what is selected when the agents agree on the contracts under which they may be matched, must be selected when the agents do not agree on these contracts. Thus, as the *stable allocations with negotiations* that are not *stable* should then not be selected, only the Stable rule satisfies efficiency, equity, and incentive compatibility properties.

Theorem 2

1. *The Stable rule with negotiations satisfies* Pareto-efficiency *and* consistency.
2. *The sets of allocations the Stable rule with negotiations selects need not be a lattice.*
3. *The Stable rule with negotiations violates* Maskin-monotonicity.

Proof.

Before proving Statements 1, 2, and 3, note that for each $M = (D, H, X, R) \in \mathcal{M}$ *and each* $A \in \mathcal{A}(M)$*, we have* $A \in SN(M)$ *if and only if:*

- *there is no* $d \in D$ *with* $A_d \neq \emptyset$ *such that* $C(A_d, R_d) = \emptyset$*;*
- *there is no* $h \in H$ *such that* $C(A_h, R_h) \subsetneq A_h$*;*
- *there is no pair of subsets* $(D', \{h\}) \subseteq D \times H$ *such that there is* $X' \subseteq \mathcal{X}_h|_X$ *with:*
  - $\{d \in \mathbb{D} : \mathcal{X}_d|_{X'} \neq \emptyset\} = D'$*;*
  - $X' \not\subseteq A_h$*;*
  - $C(A_h \cup X', R_h) = X'$*;*
  - *for each* $d \in D'$*, there is* $x \in X'$ *such that* $\mathcal{X}_d|_{X'} = \{x\}$*,* $A_d \neq x$*, and* $C(A_d \cup \{x\}, R_d) = \{x\}$ *and there is no* $x' \in (\mathcal{X}_d|_X \cap X_h|_X)\backslash\{x\}$ *such that either* $A_d\ R_d\ \{x'\}$ *and* $X'\backslash\{x\}\cup\{x'\}\ R_h\ X'$ *or* $C(A_h \cup X'\backslash\{x\}, R_h)\ R_h\ X'\backslash\{x\}\cup\{x'\}$

and $x' \; R_d \; x$.

Statement 1:

Pareto-efficiency: *Straightforward.*

Consistency:. *Let $M = (D, H, X, R) \in \mathcal{M}$ and $A \in \varphi^{SN}(M)$. Let $M' = (D', H', X', R') \in \mathcal{M}$ be the reduced market of M relative to $D' \cup H'$ at A, i.e., $M' = M|^{A}_{D' \cup H'}$. Let $A' \in \mathcal{A}(M')$ be the reduced allocation of A relative to $D' \cup H'$, i.e., $A' = A|_{D' \cup H'}$. By contradiction, suppose $A' \notin \varphi^{SN}(M')$. Then, either $A' \notin P(M')$ or $A' \notin SN(M')$. If $A' \notin P(M')$, then, by Theorem 1, $A \notin P(M)$, contradicting $A \in \varphi^{SN}(M)$. If $A' \notin SN(M')$. Then, distinguish three cases.*

Case 1: *Suppose that there is $d^* \in D'$ with $A'_{d^*} \neq \emptyset$ such that $C(A'_{d^*}, R'_{d^*}) = \emptyset$. As $A' = A|_{D' \cup H'}$, we have $A'_{d^*} = A_{d^*}$ implying $C(A'_{d^*}, R'_{d^*}) = C(A_{d^*}, R'_{d^*})$. As $R'_{d^*} = R_{d^*}|_{X'}$, we have $C(A_{d^*}, R'_{d^*}) = C(A_{d^*}, R_{d^*})$. Thus, $C(A_{d^*}, R_{d^*}) = \emptyset$. As $A'_{d^*} \neq \emptyset$ and $A'_{d^*} = A_{d^*}$, we have $A_{d^*} \neq \emptyset$. Altogether, there is $d^* \in D$ with $A_{d^*} \neq \emptyset$ such that $C(A_{d^*}, R_{d^*}) = \emptyset$, contradicting $A \in \varphi^{SN}(M)$.*

Case 2: *Suppose that there is $h^* \in H'$ such that $C(A'_{h^*}, R'_{h^*}) \subsetneq A'_{h^*}$. Let $X^* \subsetneq A'_{h^*}$ be such that $X^* = C(A'_{h^*}, R'_{h^*})$. As $A' = A|_{D' \cup H'}$, we have $A'_{h^*} = A_{h^*}$ implying $C(A'_{h^*}, R'_{h^*}) = C(A_{h^*}, R'_{h^*})$. As $R'_{h^*} = R_{h^*}|_{X'}$, we have $C(A_{h^*}, R'_{h^*}) = C(A_{h^*}, R_{h^*})$. Thus, $C(A_{h^*}, R_{h^*}) = X^*$. As $X^* \subsetneq A'_{h^*}$ and $A'_{h^*} = A_{h^*}$, we have $X^* \subsetneq A_{h^*}$. Altogether, there is $h^* \in H$ such that $C(A_{h^*}, R_{h^*}) \subsetneq A_{h^*}$, contradicting $A \in \varphi^{SN}(M)$.*

Case 3: *Suppose that there is $(D^*, \{h*\}) \subseteq D' \times H'$ such that there is $X^* \subseteq \mathcal{X}_{h^*}|_{X'}$ with $\{d \in \mathbb{D} : \mathcal{X}_d|_{X^*} \neq \emptyset\} = D^*$, $X^* \not\subseteq A'_{h^*}$, $C(A'_{h^*} \cup X^*, R'_h) = X^*$, and for each $d \in D^*$, there is $x^d \in X^*$ such that $\mathcal{X}_d|_{X^*} = \{x^d\}$, $A'_d \neq x^d$, and $C(A'_d \cup \{x^d\}, R'_d) = \{x^d\}$ and there is no $\hat{x}^d \in (\mathcal{X}_d|_{X'} \cap \mathcal{X}_{h^*}|_{X'}) \backslash \{x^d\}$ such that either $A'_d \; R'_d \; \{\hat{x}^d\}$ and $(X^* \backslash \{x^d\} \cup \{\hat{x}^d\}) \; R'_{h^*} \; X^*$ or $\hat{x}^d \; R'_d \; x^d$ and $C(A'_{h^*} \cup X^* \backslash \{x^d\}, R'_{h^*}) \; R'_{h^*} \; (X^* \backslash \{x^d\} \cup \{\hat{x}^d\})$. As $A' = A|_{D' \cup H'}$, we have $A'_{h^*} = A_{h^*}$ and for each $d \in D^*$, we have $A'_d = A_d$ implying $C(A'_{h^*} \cup X^*, R'_{h^*}) = C(A_{h^*} \cup X^*, R'_{h^*})$ and for each $d \in D^*$, we have $C(A'_d \cup \{x^d\}, R'_d) = C(A_d \cup \{x^d\}, R'_d)$. As $R' = R|_{X'}$, we have $C(A_{h^*} \cup X^*, R'_{h^*}) = C(A_{h^*} \cup X^*, R_{h^*})$ and for each $d \in D^*$, we have $C(A_d \cup \{x^d\}, R'_d) = C(A_d \cup \{x^d\}, R_d)$. Thus, $C(A_{h^*} \cup X^*, R_{h^*}) = X^*$ and for each $d \in D^*$, we have $C(A'_d \cup \{x^d\}, R'_d) = \{x^d\}$. As $X^* \subseteq \mathcal{X}_{h^*}|_{X'}$ and $\mathcal{X}_{h^*}|_{X'} = (\mathcal{X}_{h^*}|_X)|_{D' \cup H'}$, we have $X^* \subseteq \mathcal{X}_{h^*}|_X$. As $X^* \not\subseteq A'_{h^*}$ and $A'_{h^*} = A_{h^*}$, we have $X^* \not\subseteq A_{h^*}$. For each $d \in D^*$, as $\mathcal{X}_d|_{X^*} \neq A'_d$ and $A'_d = A_d$, we have $\mathcal{X}_d|_{X^*} \neq A_d$. As $X' = X|_{D' \cup H'}$, as $A' = A|_{D' \cup H'}$, and as $R' = R|_{X'}$, for each $d \in D^*$, there is no $\hat{x}^d \in (\mathcal{X}_d|_X \cap \mathcal{X}_{h^*}|_X) \backslash \{x^d\}$ such that either $A_d \; R_d \; \{\hat{x}^d\}$ and $(X^* \backslash \{x^d\} \cup \{\hat{x}^d\}) \; R_{h^*} \; X^*$ or $\hat{x}^d \; R_d \; x^d$ and $C(A_{h^*} \cup X^* \backslash \{x^d\}, R_{h^*}) \; R_{h^*}$ $(X^* \backslash \{x^d\} \cup \{\hat{x}^d\})$. Altogether, there is $(D^*, \{h*\}) \subseteq D \times H$ such that there is*

$X^* \subseteq \mathcal{X}_{h^*}|_X$ *with* $\{d \in \mathbb{D} : \mathcal{X}_d|_{X^*} \neq \emptyset\} = D^*$, $X^* \not\subseteq A_{h^*}$, $C(A_{h^*} \cup X^*, R_h) = X^*$, *and for each* $d \in D^*$, *there is* $x^d \in X^*$ *such that* $\mathcal{X}_d|_{X^*} = \{x^d\}$, $A_d \neq x^d$, *and* $C(A_d \cup \{x^d\}, R_d) = \{x^d\}$ *and there is no* $\hat{x}^d \in (\mathcal{X}_d|_X \cap \mathcal{X}_{h^*}|_X)\backslash\{x^d\}$ *such that either* $A_d\ R_d\ \{\hat{x}^d\}$ *and* $(X^*\backslash\{x^d\} \cup \{\hat{x}^d\})\ R_{h^*}\ X^*$ *or* $\hat{x}^d\ R_d\ x^d$ *and* $C(A_{h^*} \cup X^*\backslash\{x^d\}, R_{h^*})\ R_{h^*}\ (X^*\backslash\{x^d\} \cup \{\hat{x}^d\})$, *contradicting* $A \in \varphi^{SN}(M)$.

Statement 2: *Suppose that for each* $M \in \mathcal{M}$, *we have* $\varphi(M) = \varphi^{SN}(M)$. *In what follows, we prove that the sets of allocations* φ^{SN} *selects need not be lattice. Let* $M = (D, H, X, R) \in \mathcal{M}$ *be such that* $D = \{d_1\}$, $H = \{h_1, h_2\}$, $X = \{a, b, c\}$ *with* $\mu(a) = (d_1, h_1)$, $\mu(b) = (d_1, h_1)$, $\mu(c) = (d_1, h_2)$, *and* $R = (R_{d_1}, R_{h_1}, R_{h_2})$ *as Figure 3.2. Let* $A = (c, \emptyset, \{c\})$ *and* $A' = (b, \{b\}, \emptyset)$. *Clearly,* $P(M) \cap SN(M) = \{A, A'\}$. *As* $A_{h_1}\ R_{h_1} A'_{h_1}$ *and* $A'_{h_2}\ R_{h_2}\ A_{h_2}$, *there is no hospital-optimal allocation in* $\{A, A'\}$. *Thus,* $P(M) \cap SN(M)$ *is not a lattice. Thus, the set of allocations* φ^{SN} *selects in* M *is not a lattice.*

Statement 3: *Suppose that for each* $M \in \mathcal{M}$, *we have* $\varphi(M) = \varphi^{SN}(M)$. *In what follows, we prove that* φ^{SN} *violates* Nash-implementability. *This result directly follows from Theorem 1. Let* $M = (D, H, X, R) \in \mathcal{M}$ *be such that* $D = \{d_1\}$, $H = \{h_1, h_2\}$, $X = \{a, b, c\}$ *with* $\mu(a) = (d_1, h_1)$, $\mu(b) = (d_1, h_1)$, $\mu(c) = (d_1, h_2)$, *and* $R = (R_{d_1}, R_{h_1}, R_{h_2})$ *as Figure 3.2. Let* $A = (c, \emptyset, \{c\})$ *and* $A' = (b, \{b\}, \emptyset)$. *Clearly,* $\varphi^{SN}(M) = \{A, A'\}$. *Let* $M' = (D', H', X', R')$ *be such that* $D' = D$, $H' = H$, $X' = X$, *and* $R' = (R'_{d_1}, R'_{h_1}, R'_{h_2})$ *as follows. First, for each* $x \in \mathcal{X}_{d_1}|_X$, *we have* $x\ R'_{d_1}\ a$ *and for each* $x, x' \in \mathcal{X}_{d_1}|_X$, *we have* $x\ R'_{d_1}\ x'$ *if and only if* $x\ R_{d_1}\ x'$. *Second,* $R'_{h_1} = R_{h_1}$. *Third,* $R'_{h_2} = R_{h_2}$. *Clearly,* R' *is a Maskin-monotonic transformation of* R *at* A. *But,* $\varphi^{SN}(M') = \{A'\}$. *Thus,* $A \notin \varphi^{SN}(M')$. *Thus,* φ^{SN} *violates* Maskin-monotonicity. ■

3.5 Concluding remarks

Following a normative approach, our objective was to identify which properties solutions for two-sided many-to-one matching problems with contracts should satisfy and based on these properties, to justify solutions.

First, we proved that to predict the outcomes of such markets we should not restrict our attention to *stable* allocations, but also focus on *stable allocations with negotiations.* Indeed, agents may not agree on the terms of contract under which they should be matched. Thus, they may have to negotiate them.

Based on the strategic considerations behind *stability with negotiations*, we introduced a new rule. We proved that it satisfies *consistency.* Also, we proved that the set of allocations it selects need not be a lattice. Questions still open are to determine under which assumptions the set of all *stable allocations with negotiations* is non-empty, if there are assumptions under which it is always a lattice, or if the Stable rule with negotiations satisfies other equity properties.

We conjecture that under weak assumptions on preferences, the Stable rule with negotiations also satisfies *own-side population-monotonicity* and *other-side population-monotonicity*, i.e., if one side's population decreases, each agent on the other side should find the worst bundle she could be allocated in the previous situation at least as desirable as the worst bundle she could be allocated in the new situation. Kayı, Ramaekers, and Yengin (2006) introduce this latter axiom in medical job market problems. They prove that on the domain of all such problems, in which each agent has strict preferences and each doctor has *substitutable* preferences, the Stable rule satisfies it.

Furthermore, we proved that independently of the fact that agents may negotiate, the only solution for two-sided many-to-one matching markets with contracts that satisfies efficiency, equity, and incentive compatibility properties simultaneously is to select all *stable* allocations. This conclusion strengthens the conclusion that holds for particular restricted problems, in which there is only one way to match agents. Indeed, on the domain of all college admission problems, in which each agent has strict preferences and each college has responsive preferences, Toda (2006) proves that only the Stable rule satisfies *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity*.[16]

We explained how Kayı, Ramaekers, and Yengin (2006) generalize this result on the domain of all medical job market problems, in which each agent has strict preferences and each hospital has substitutable preferences. They use generalizations of arguments introduced by Toda (2006). They prove that *unanimity*, *own-side population-monotonicity*, and *Maskin-monotonicity* are independent of one another. Indeed, the rule that selects all allocations satisfies these axioms, but *unanimity*. The rule that selects all *Pareto-efficient* allocations satisfies these axioms, but *weak own-side population-monotonicity*. The rule that selects the hospital-optimal and doctor-optimal allocations in the set of all allocations satisfies these axioms, but not *Maskin-monotonicity*. Questions still open are to determine if on the domain of all medical job mar-

[16]Toda (2006) first proves this characterization on the domain of all marriage problems, in which each agent has strict preferences. He proves both results with an axiom weaker than *unanimity*, referred to as *weak unanimity*. It requires that if an allocation is the choice of each agent over the set of available contracts and is such that each agent is matched to another agent, then it should be the only selected allocation. He refers to *own-side population-monotonicity* as *population-monotonicity*.

ket problems, in which each agent has strict preferences and each hospital has substitutable preferences, the Stable rule satisfies further equity properties as *converse consistency* or *anonymity*.

Finally, we proved that by opposition to when there is only one way to match agents, it is not efficiency and equity properties that impose a conflict between the common interests of each of the two sides of market, but efficiency, equity, and incentive compatibility properties. Efficiency and incentive compatibility properties may also impose such a conflict. Indeed, on the domain of all medical job market problems, in which each agent has strict preferences and each hospital has substitutable preferences, Haake and Klaus (2005) prove that each rule that satisfies *Pareto-efficiency*, *individual rationality*, i.e., each agent should find keeping all their allocated contracts at least as desirable as rejecting some or all of them, and *Maskin-monotonicity* is a supercorrespondence of the Stable rule.

The natural step now is to see if these conclusions still hold when both sides of the market may sign more than one contract.

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Female-Headed Households, Poverty and Inequality in Colombia

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José Daniel Salinas Rincón

1 Introduction

Poverty and inequality in developing countries have been widely studied. Among other, household composition and female headship are considered important determinants of poverty (Haughton and Khandker 2009). The effects of vulnerability and gender inequality have received special attention because of their negative welfare implications for women. In fact, the higher prevalence of poverty among women has been called "feminization of poverty". This concept is related to three main characteristics: the great proportion of women in poverty; the increasing incidence of this phenomenon; and women's increasing share of poverty linked to the rising incidence of female household headship, (Chant 2006, Moghadam 2005).

Since the 1970's, there has been an increasing interest in female poverty with special attention to Female-Headed Households (FHH). There are a number of studies analysing several aspects of poverty of FHH. The academic literature has identified three sets of factors as potential determinants of the positive relation between poverty and female headship. First, characteristics of household composition. FHHs tend to contain a higher ratio of non-workers to workers. Second, the gender of the main earner. Households where women are the main earners usually are affected by lower income and restricted access to land and capital. Third, the combination between gender-related differences and household structure. For example, FHHs tend to be smaller and usually do not have other female adults. As consequence, the female head of household has to divide their time between labor market activities (job search or work) and home production. In any case, they face greater time and mobility constraint than male heads, (Buvinic and Gupta 1997).

This project studies the phenomenon of Female Headship and its characteristics in Colombia. It is well known Colombia is a middle income country with high wealth inequality. In this context, poverty could have important effects on welfare for the group of Female Headed Households. As other developing countries, female labor force participation has increased in Colombia. This fact has been accompanied by reductions in the number of children per-household and changes in the internal household composition, (Robbins 2009). In this context, Female Headed Households (FHH) have become more important in the analysis of poverty and vulnerability of Colombian households. Moreover, since 1991 the Colombian government has been interested in reduction of gender discrimination. The political constitution declares general and specific rights of women emphasizing on heads of households.

The main objective of this project is to investigate the characteristics of Female Headed Households and their relation to poverty and inequality in Colombia. In order to achieve this goal, a typology of households is defined taking into account their composition and headship. Two sources of information are used: Living Standards Measurement Study (Encuesta de Calidad de Vida, ECV2015) and the Demographic and Health Survey (DHS2015) for Colombia. The reason to use these surveys is because each one is specialised in the study of particular household and individual characteristics.

In the case of Living Standards Measurement Survey (ECV2015), it contains information about wealth, income and living standards. This survey is used to describe general characteristics of the household types and their poverty incidence according to the official measures. On the other side, the Demographic and Health Survey (DHS2015) focuses on socio-economic and health characteristics of households with special emphasis on women. This survey is composed by 5 questionnaires: household, women between 13 and 49 years old, men between 13 and 59 years old, Cancer prevention for women of 50 years old or more, and Cancer prevention for Men of 60 years old or more. For the purposes of this project, only questionnaires for households and women (13-49) are used. From the first, the general households characteristics are described and compared to the information obtained from ECV2015. This exercise allows to compare consistency between surveys. In the case of the female module, it contains only women information and allows to make direct comparisons between women in the different household types. The main aspects analysed are education, fertility, teenage pregnancy, domestic violence and wealth.

It is important to remark this study is based on a general measurements of poverty and inequality. The Colombian official poverty lines are used to determine the poverty incidence by household types. The Gini index provides an overall view of Colombian inequality. However, the description of income distribution among households is implemented by computing and comparing income deciles. Another consideration is that in both surveys, ECV2015 and DHS2015, the headship is self-reported. The definition of "head of household" is subjective since the assignment of headship status is defined by the household members. Despite this potential ambiguity, the identification of households headed by women is useful when economic maintenance is used as definitional criterion. In this case, households that depend on woman tend to be less well-off than households that depend on male (Buvinic 1991). As a final remark, the study of intra-household decisions is beyond the scope of this project.

This document is divided in five sections. First section is this introduction. Section 2 is the literature review of general aspects of the concept "feminization of poverty" and it presents studies for Colombia. Section 3 presents a general context of poverty and inequality in Colombia, defines a household typology, and provides a characterization of FHHs depending on basic poverty and inequality measures. Section 4 explores the special socio-economic characteristics of women living in different household types based on DHS2015. Finally, section 5 concludes.

2 Literature Review

Since 1970, there have been an increasing interest in the study of the relation between poverty and female headship specially in developing countries. Buvinic, Youssef and Von Elm (1978), in a pioneering study, found that Female Headed Households (FHH) represent a very special group among the poor. Moreover, they explore the main policy and methodology obstacles to identify this kind of households, and propose a strategy to overcome the lack of information about the specific characteristics of this group. This is the first report analysing three key aspects about FHH: 1) Evidence about the increase in this group of households; 2)The lack of common definitional standard for the precise meaning of "head of household"; 3) the critical evaluation of myths and stereotypes regrading the family structure.

After this pioneering report, there were a number of studies about the potential vulnerability of FHH in developing countries, especially in Latin America and the Caribbean. Buvinic (1991), provides answer to some questions about the concept of FHH, its social implications and the relevance of policy interventions. This report also presents a table summarizing 22 studies about FHH in Latin America and the Caribbean.

The concept "feminization of poverty" appears in the academic literature in the 1970s and the FHHs were designed as the "poorest of the poor". However, the lack of concluding and rigorous evidence considered difficult to target FHH for public policy purpose. Buvinic and Gupta (1997), almost 20 years later from the first report, take up the controversy by presenting an analysis of the definition and measurement of female headship. Also, they reviewed the empirical evidence and examined the potential costs and benefits of targeting FHH. Finally, they present the experience of Chile as an example of a country which explicitly targeted female headship through government intervention. As conclusion, they recommend targeting female headship to reduce poverty, but with some exceptions and constraints related to design and implementation of the interventions.

It seemed to be a consensus about that FHH are the most vulnerable type of households. However, some critical views rise at the beginning of the 2000. Chant (2003), challenges the link between "feminization of poverty" and the Female Headed Households. Moreover, the idea of FHH as the "poorest of the poor" is considered as weak and with no supportive evidence. In general terms, Chant (2003) is criticizing the formation of stereotypes around the FHH based on a preconception about the notion of "traditional" household structure. This paper summarizes the causes and reasons why FHHs were categorized as the "poorest of the poor" in the development literature. Then, it presents a discussion about the arguments and evidence against or in favor of this "stereotype". Finally, it explores the implications and outcomes related to the concept of FHH and its consequences for policy interventions.

In the same line, Moghadam (2005) argued that the increasing interest on women's poverty is not based on objective evidence, but it is rooted in demographic trends, cultural patterns and the neoliberal economy. The author claims that the available evidence is not conclusive for many countries and regions. He presents arguments in favour of a presumed poverty-inducing nature of neoliberal economic policies and their effects on women and girls. In other words, he argues that the implementation of neoliberal policies contributed to the increase of poverty and inequality, especially for women.

In general, there is a controversy about the concept of "feminization of poverty", its causes and consequences. Essentially, the discussion can be divided in two sides. On one hand, the "feminization of poverty" understood as a fact with roots on the economic vulnerability of women. In the other hand, the view identifying "feminization of poverty" as a stereotype constructed by demographic, cultural and political trends. In any case, the common factors are the weakness and the inconclusive evidence about poverty among women.

In response to the potential hidden aspects of "feminization of poverty", Chant (2006) describes the main issues and weaknesses of the standard measures reflecting gendered poverty and proposes directions for kinds of data and indicators that could improve the measures related to poverty and its gender aspects.

In the last decade there have been great improvements in the availability of information and measurements of female poverty. However, despite some recent attempts (For example, Lowe and MacKelway 2017 or Ngenzebuke, De Rock and Verwimp 2018) , the measure of intra-household inequalities remains as an empirical challenge. And of course, the standard poverty measurements cannot capture these intra-household gender differences, (World's Women 2015).

Given the world's extreme poverty reduction from 1.9 billion in 1990 to 1 billion in 2011 World's Women (2015), recent literature has focused on investigating if poverty reductions affect FHHs in the same proportion as other types of households. Milazzo and Van de Walle (2015) provide evidence on FHHs for Sub-Saharan Africa using the Demographic and Health Surveys for 24 countries. They examine changes in living standards and provide a breakdown of the total change in poverty into that contributed by male versus female headed households. The paper shows that the share of FHHs is rising over time. However, poverty is declining for both male and female headed households. In fact, FHHs are contributing nearly as much as male headed households, despite their smaller share in the population.

For the specific case of Cameroon, Ngah and Menjo (2016) decompose poverty-inequality linkages of sources of deprivation by men and women headed households. They analyse the determinants of economic welfare for both types of households, the marginal impact of poverty, inequality and the elasticity of poverty by different types of households, and the between and within components for the marginal impacts on poverty and inequality. The main results show that FHH face more human and household capital deprivation and higher inequality than male-headed households. But they also find that inequality reductions reduce human and physical capital deprivation between these households.

Finnoff (2015) presents a study of poverty and inequality in post-war Rwanda. He decomposes the "major" sources of poverty and inequality and finds differences in vulnerability by region, gender and widow status of the head of household. Specifically, the physical assets have a increasing importance in determining the inter-household distribution of income. In fact, physical assets explain a great part of total inequality. They conclude that female headed households are more likely to be poor.

Female Headed Households in Colombia

Colombia has been one of the pioneering countries in Latin America to implement actions in order to reduce poverty of FHHs. Since the beginning of 1990, the new political constitution declares the imperative need to defend and ensure the rights of Colombian women. All of this in a context of equal opportunities and gender equity. From here, there were created special policies and interventions addressed to women who are heads of household in Colombia, (Fuentes 2002).

The Colombian political constitution definition of a woman head of household is "any woman single or married that is economically or socially in charge of her own children or other incapacitated adults"[1]. Starting from this definition the Colombian government created a special office, "Consejería Presidencial para la Equidad de la Mujer", in charge of the design, promotion and administration of public programs and interventions targeted to these women.

The specific program for women heads of household was called "Mujer Cabeza de Familia Microempresaria" and it was a micro-credit program to finance productive projects presented by women heads of household. At the end of the 1980s, this program born as a private initiative. However, after the constitutional reform of 1991, the program adopted a mixed administration between public and private institutions (1992-1998). Finally, from 1999 to 2009 the program was entirely public. This implies Colombia has almost 20 years of policy interventions addressed to women heads of households.

Rico (2006) explores the relationship between FHHs, informality and poverty. In a descriptive exercise, she analyses some indicators to show that women heads of household in informal jobs are more vulnerable to poverty and this vulnerability implies some inter-generational consequences.

Robbins (2009) presents a detailed summary of the gender legislation, a description and evaluation of the public programs for women heads of household, and an analysis of the socio-economic characteristics of women heads of household in Colombia. There are several comparisons between eight types of households. The typology is defined taking into account the head's gender, presence of one or two parents and other adults in the household. The study is focused on urban areas and finds household type 8 (woman headship without partner and no other adults in the household) is the most important in terms of population size. The total income per-capita in these households was the lowest among all types. This type increased from 15% of total population in 1976 to 25% in 1999. Unfortunately, the public program to benefit FHHs was relatively small in comparison to the potential beneficiary population, it only had capacity to cover 0.14% of FHHs. In order to improve the program, the main recommendations were: increase loan amounts, redesign and improve the process and transparency in the selection of women, improve the system for collecting and monitoring information, formalize the monitoring process and revalue the subsidized credit system.

The Colombian long civil conflict has affected women in several dimensions. Romero and Chavez (2013) explore the consequences of forced displacement to women head of households. In specific, they study the case of women who arrive to Soacha[2] after a violent displacement process. The vulnerability of this women is evident given their lower levels of education, the high responsibility in taking care of their family, the affective and psychological problems as result of violence, and the poor access to good jobs. This paper reviews the social characteristics around

[1]The exact definition in original language is: "toda mujer que siendo soltera o casada tenga bajo su cargo económica o socialmente, en forma permanete, hijos menores propios u otras personas incapaces o incapacitadas."
[2]Soacha is a municipality close to the Capital city, Bogotá.

these women and gives a diagnostic about the social importance of these women and their role as fundamental member of their own families.

In general terms, Colombia has a long tradition of public interventions specific for women heads of household. However, the evaluation of these interventions has been difficult. It seems some type of female headed households increased in great magnitude between 1976 and 1999. This fact contrasts with the reduction of poverty and inequality in the last two decades. It is important to investigate what are the recent patterns for the relation between FHHs, poverty and inequality.

In order to provide an adequate context to the study of poverty, inequality and socio-economic characteristics of women heads of household, the next section presents an overview of the recent measures of poverty and inequality in Colombia. There is a definition of a household typology and proposes a characterization of FHH given poverty and inequality measures.

3 Poverty, Inequality and Female Headed Households

In Colombia, poverty is measured by two complementary methods: multidimensional poverty and monetary poverty. Each method shows different aspects related to the life conditions and privations of Colombian population. In the case of multidimensional poverty, this indicator has been implemented since 2012, and its design is an adaptation of Alkire and Forster's methodology. The monetary poverty measure defines a poverty line based on a minimum "acceptable" consumption bundle. In the other side, the official measure for inequality is the Gini index.

Despite the relatively long Colombian measures of poverty and inequality, there is a lack of gender related indicators. For this reason it is difficult to have a clear and recent picture of the behaviour of these two measures in relation to gender disparities. This chapter provides a general view of Colombian poverty and inequality, and presents a characterization of these measures focused on gender and Female Headed Households.

3.1 General context

In the last fifteen years, Colombian poverty measures have been decreasing. In 2002 the share of total population considered as poor, by the monetary poverty index, was 49.7%, while the same measure for 2017 was 26.9%. Certainly, this is a great poverty reduction and it is in line with poverty measurements provided by ECLAC(2017) for the whole Latin-American Region. Moreover, the extreme poverty has been reduced in a great proportion as well. In 2002 people in extreme poverty were 17.7% while in 2017 this proportion was 7.4%. Figure 1 shows the recent pattern of these indicators.

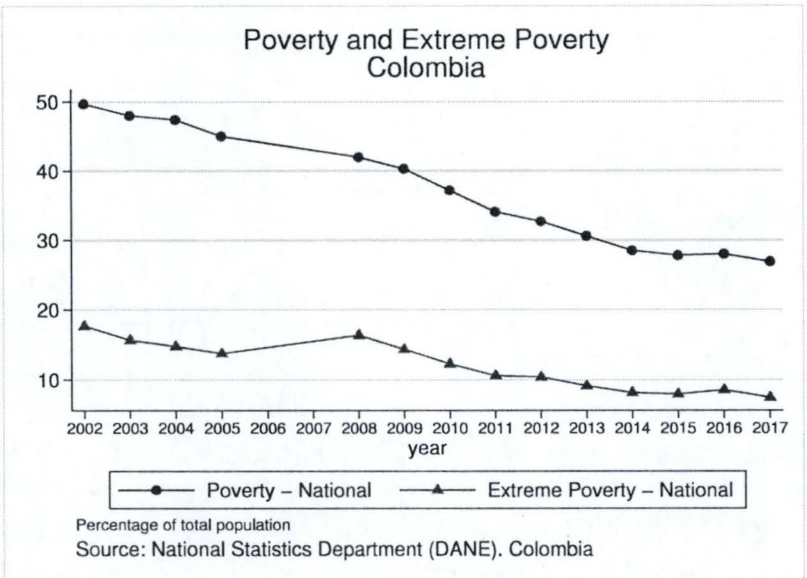

figure 1: Poverty and Extreme Poverty Colombia

National Poverty line is defined as the minimum monthly per-capita consumption expenditure required to satisfy the basic alimentary and non-alimentary needs. Individuals with income below this line are considered as poor. In 2015, the poverty line was 223,638 Colombian Pesos (COP) (68.9 U.S dollars[3]), then a household composed by 4 people was considered poor if their total income was below 894,552 COP (275.6 U.S dollars). In the case of extreme poverty line, it is defined as the minimum monthly per-capita consumption expenditure required to satisfy only the basic alimentary needs. In 2015, the extreme poverty line was 102.109 COP, (31.4 U.S dollars). Table 1, shows the poverty lines for years 2014 and 2015.

Table 1: Poverty Lines*

	2014	2015	Δ%
National Poverty line	211.807	223.638	5.6
National Extreme Poverty line	94.103	102.109	8.5
Minimum Wage	616.000	644.350	4.6

* Monthly per-capita values, Colombian Pesos.
Source: National Department of Statistics - DANE.

At this point it is interesting to compare the poverty lines with the minimum wage and some general household characteristics. Given that a 4 people household is considered poor if their total income is less than 894,552 COP, then the minimum wage seems to be low in comparison to this poverty line. Also it is interesting that the growth rate of the minimum wage is less than the growth rate of the poverty line. In some sense, this could implies a higher risk to fall into poverty for less qualified workers. According to the Colombian Living Standards Measurement Survey 2015 (Encuesta de Calidad de Vida - ECV2015), the average household size is 3.4 people, the average number of workers per household is 1.3 and the average income per household is around 2,226,033 COP (686.1 U.S dollars). See table 2.

[3]Monthly average of official Exchange Rate for December 2015: 3244.51 Colombian pesos per 1 U.S dollar.

Table 2: General Household Characteristics

Average Household Income (COP)	2,226,033
Average Household Size	3.38
Average Workers per Household	1.31
Total Number of Households	14,100,519

Source: Own computations based on Colombian Living Standard Measurement Survey 2015. National Department of Statistics.

The average household income seems good enough in comparison to the minimum wage and the poverty lines. However, more detailed analysis is needed to have a better understanding of poverty in Colombian households.

The multidimensional poverty index (MPI) was designed to bring a wider view of deprivations and living standards for a specific population. The index considers five dimensions: household education conditions, child and youth conditions, health, occupation and public services access. These five dimensions are represented by 15 indicators. If a household fails in at least 33% of the indicators it is considered as poor.

The MPI has been decreasing since its implementation in Colombia. In the last seven years, there was a reduction of 13% in the multidimensional poverty, in 2010 the index was 30.4%, while for 2017 it was 17%, As illustrated figure 2.

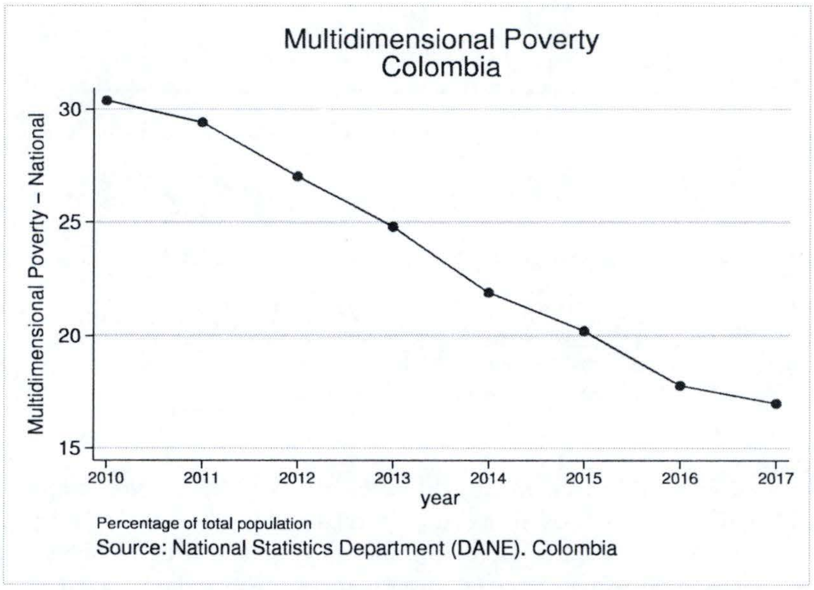

Figure 2: Multidimensional Poverty Index - Colombia

Despite the poverty reduction has been considerable in the last two decades, Colombia remains as one of the most unequal countries in Latin America. In 2002, the National Department of Statistics reported the Gini index was 57.2 and, fifteen years later, in 2017 the measure was 50.8. This is certainly a modest reduction given the relatively long time lapse. See figure 3.

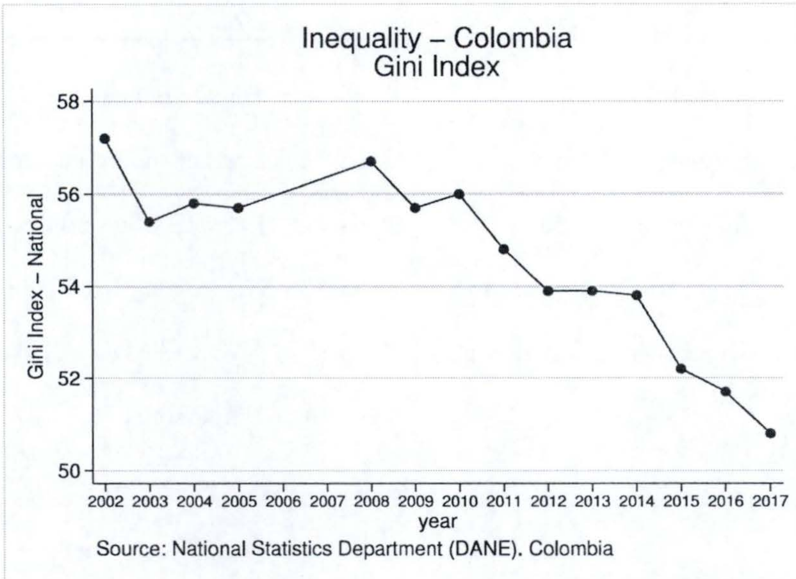

Figure 3: Gini Index - Colombia

In summary, the last fifteen years Colombia has had a great poverty reduction (Monetary and Multidimensional), that could be partly explained by the relatively good economic growth in the last two decades. Remember, Colombia was not directly affected by the crisis in 2008-2009. Then, there were good conditions for poverty indicators to decrease. Unfortunately, the income and wealth concentration have not been good enough in last decades. Despite a small reduction, the general income inequality measured by Gini index remains high. This could be associated to labor market issues, as high informality, a regressive tax structure and problems with distribution of land.

Given this general picture, it is interesting to study poverty and inequality in a gender perspective. The next parts of this document explore how the Female Headed Households are affected by poverty and inequality, and provide a general characterization of the main aspects of FHHs related to poverty, inequality and living standards.

3.2 Characterization of Female Headed Households - Poverty and Inequality

3.2.1 Data

The data used in this part of the document comes form the Colombian Living Standard Measurement Survey 2015, or "Encuesta de Calidad de Vida 2015" (ECV2015). This survey is carried out by the National Department of Statistics (DANE) and collects information about different aspects of households as housing physical characteristics, access to public services, health, living conditions and demographic variables. The sample is 23,005 households and it is representative at national level as well as 9 regions, main cities and rural areas.

3.2.2 Typology of Households

To study female headship is not enough to differentiate between households with male and female head. The household composition matters, in special for analysis regarding poverty and inequality. In this project the definition of household types is based on Robbins(2009), where each type depends on three characteristics: number of parents, other adults in the household, and head's gender. In order to provide a more detailed analysis, two more types were added to take into account uni-personal households. Table 3 describes the eight types of household to be study in the following sections.

Table 3: Typology of Households

	Male Headship		Female Headship
Type 1:	Partner, no other adults	Type 2:	Partner, no other adults
Type 3:	Partner, other adults	Type 4:	Partner, other adults
Type 5:	No partner, other adults	Type 6:	No partner, other adults
Type 7:	No partner, no other adults	Type 8:	No partner, no other adults
Type 9:	Unipersonal	Type 10:	Unipersonal

This typology is useful to study poverty and inequality among male and female-headed households because the composition of types emphasizes the presence of individuals in working ages. Of course, this typology is flexible enough to take into account presence of children and other important aspects related to determinants of poverty. Moreover, each female headed household type has a similar male-headed counterpart. This allows to make direct gender comparisons among similar household compositions. In order to make ease the identification of households in this typology, the even numbers (2, 4, 6, 8 and 10) are always referring to a type with female headship.

3.2.3 Descriptive Statistics

Table 4 presents a general descriptive statistics to give a picture of each household type in terms of the share of total households, and head's attributes. All indicators were calculated using weights. For this reason, the number of households and population are expanded to represent the total Colombian population. The observation unit is the household and, all numbers presented in table 4 are totals and means computed by each household type.

According to ECV2015, Colombian total population was 47,764,755 and the number of households was 14,100,519. This implies the average households size is 3.4 people. The share of Female Headed Households is 35.3%[4] and the share of total population living in FHH is 33.1%.

Columns (1) and (2) show the total number of households and its share by type. The most common household (31.5%) is type 1 (Male head with partner and no other adults present in the household), also called nuclear household. Without taking into account uni-personal households, Male headed households type 3 represent 19.7%, while type 6 (15.4%) and type 8 (6.5%) are the most common female headed household types. These four types represents more than 73% of total Colombian households. It is interesting to remark that for male headed households usually the most common types are those where the partner is present (households with 2 parents), while female headed types are more common when there is not male parent.

Columns (3) and (4) show the average of household income per capita and the percentage of poor households by type. A particular household is considered poor if its total income is less than the monetary poverty line times the number of household members. The greater percentage of poor households is found in type 8. However the income per capita for this type is not the lowest.

[4]Note the number reported by the World Development Indicators (based on Demographic and Health Surveys) is 36.4% of FHHs for 2015.

le 4: Descriptive Statistics

	Total HH	Share type	Income* per capita	Share Poor	Size*	Head*				Children under 13
								Occupation		
						Age	Edu	Employed	Housework	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
pe 1	4,442,803	31.5%	722,051	26.3%	3.4	42	8.5	86.5%	3.9%	1.1
pe 2	592,754	4.2%	790,447	25.1%	3.3	39	9.0	43.8%	48.5%	1.0
pe 3	2,782,895	19.7%	724,572	22.1%	4.8	53	7.7	76.1%	8.8%	0.8
pe 4	482,894	3.4%	679,153	23.5%	5.2	48	8.2	41.1%	53.2%	1.0
pe 5	599,406	4.3%	964,415	17.2%	3.2	51	8.3	69.0%	11.5%	0.4
pe 6	2,168,143	15.4%	685,237	25.0%	3.7	55	7.2	42.9%	47.3%	0.6
pe 7	101,661	0.7%	420,485	28.9%	2.4	41	8.1	81.1%	4.8%	0.6
pe 8	919,404	6.5%	471,084	42.1%	2.7	38	9.4	63.5%	29.1%	1.2
pe 9	1,197,286	8.5%	1,499,997	12.4%	1	49	7.4	78.4%	7.0%	–
pe 10	813,270	5.8%	1,518,665	19.4%	1	56	7.9	44.1%	42.8%	–
al	14,100,519	100%	822,061	24.2%	3.4	48	8.1	68.9%	19.6%	0.8

rce: Own computations based on Colombian Living Standard Measurement Survey 2015.

verage value per household type.

es: Column (4) "Share of Poor", presents the percentage of poor households within type. So, the sum of column (4) is not 100%.
total of 24.2% corresponds to the total share of poor households respect to the total number of Households.

usehold types: Type 1: Male head, partner no other adults; Type 2: Female head, partner, no other adults; Type 3: Male head, tner, other adults; Type 4: Female head, partner, other adults; Type 5: Male head, no partner, other adults; Type 6: Female d, no partner, other adults; Type 7: Male head, no partner, no other adults; Type 8: Female head, no partner, no other adults; e 9: Male head, unipersonal; Type 10: Female head, unipersonal

Columns (6) to (9) present the average age, years of education, the proportion of head workers and the share of houseworker heads, respectively. It is interesting to note that female types have heads with more average education than their similar male types. Of course, the percentage of male heads working is greater than female heads, while the opposite happens for the housework.

Finally, column (10) shows average number of children under 13 years old. Households type 8 tend to have more children than other types.

It seems female headed household types are poorer than their male counterpart. However, these female households have more educated heads and lower shares of workers. The descriptive data provides a very general view of main characteristics for each household type and its potential relation to poverty. Next section provides econometric exercises to explore the correlations between poverty and household type characteristics.

3.2.4 Poor Households

The poverty determinants have been widely studied and there is a consensus about what kind of variables can play a central role in poverty incidence. (Haughton and Khandker 2009) present a detailed list of variables taking into account regional, municipal district, household and individual characteristics.

Following Haughton and Khandker (2009), this section presents an exercise to explore how the main household characteristics are correlated with poverty. As in previous section, the computations and regressions are presented at household level.

Using regression techniques for discrete dependent variable this exercise attempts to identify variables potentially related to the household's poverty. If the household, j, is considered poor[5], then the dependent variable $y_j = 1$ and $y_j = 0$ otherwise. The independent variables can be organized in groups represented by the following vectors: X_j household j characteristics included their typology, Z_j communal variables and W_j regional variables. Then, the household probability to be poor is given by the following equation:

$$P(y_j = 1|X_j, Z_j, W_j) = G(X_j\beta_1 + Z_j\beta_2 + W_j\beta_3) \tag{1}$$

The function G can take several functional forms. For instance, if takes a logistic functional form the model is known as Logit. If this function follows a normal distribution, the model is probit. Also, G can be linear, i.e Linear Probability Model (LPM), but this specification presents some problems related to the efficiency and the possibility to produce estimated coefficients outside the unit interval.

In this project, the equation (1) is estimated using both the LPM and the probit model, for which the marginal effects are reported in tables 6 and 7. Results for the LPM and the standard coefficients for the probit are reported in the annex at the end this document.

The independent variables are listed in the table below:

Table 5: Independent Variables

Variable	Description
Education	Years of education - Head of Household
Age	Age - Head of Household
Worker	Head of Household occupation: Worker
Father's Educ	Head's Father Education: Secondary or higher
Mother's Educ	Head's Mother Education: Secondary or higher
Children under 13	Children in the Household: 0 to 13 years
Area	Urban of Rural Area
Region	Country Regions

In order to study the relation between poverty and Female Headship, there are two set of regressions: The first set includes a discrete independent variable that takes the value 1 if head of household is female, 0 otherwise. This regressions capture the general relation between female headship and poverty. The second set of regressions uses the typology of households to explore in detail the relation between each type of household and poverty. Both set of regressions are controlling by the same variables.

Table 6 presents the marginal effects of a Probit model of household poverty and female headship. In all specifications, the coefficient for female headed household is positive and statistically different from zero. Column (1) presents the basic equation all estimated coefficients have the expected signs. Negative for head's education and age. Both signs and magnitudes are robust for the 4 specifications.

Column (2) show results for the specification including if the head of household has a job. The sign is negative, as expected, but the marginal effect of FHH decrease in magnitude. This implies

[5]The household is considered poor if its total income is less than the monetary poverty line times the number of household members

that a worker head of household reduces the marginal effect of been female head of household and poor. The estimated coefficient of FHH is reduced by the middle.

Columns (3) and (4) introduce variables for head's parent education (discrete variable equals 1 if parents has more than secondary education) and presence of children in the household. Signs are as expected, negative for parent's education and positive for kids under 13 in the household. The magnitude of the marginal effect of FHH decreases.

In summary, table (6) shows all expected signs for the whole set of variables. It is important to remark the marginal effect for FHH decreases from the basic equation to the full specification. This result is interesting because it implies the effect of female headship becomes relatively small with controls for household composition and characteristics.

Table 6: Poverty and Female Headship - Probit Regressions: Marginal Effects

	(1)	(2)	(3)	(4)
	Poor	Poor	Poor	Poor
VARIABLES	dy/dx	dy/dx	dy/dx	dy/dx
Education (years)	-0.0243***	-0.0233***	-0.0218***	-0.0208***
	(2.56e-05)	(2.57e-05)	(2.72e-05)	(2.70e-05)
Age	-0.00321***	-0.00442***	-0.00452***	-0.00257***
	(7.25e-06)	(7.52e-06)	(7.54e-06)	(8.16e-06)
Worker		-0.124***	-0.126***	-0.120***
		(0.000255)	(0.000254)	(0.000250)
Father's Educ: sec+			-0.0376***	-0.0389***
			(0.000412)	(0.000405)
Mother's Educ: sec+			-0.0355***	-0.0302***
			(0.000394)	(0.000389)
Children under 13				0.138***
				(0.000222)
Female Head	0.0787***	0.0340***	0.0324***	0.0336***
	(0.000221)	(0.000239)	(0.000239)	(0.000236)
Observations	22,144	22,144	22,144	22,144

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Source: Regressions based on data from Colombian Living Standard Measurement Survey 2015.

Notes: All regressions control by urban and rural areas and regions.

Table 7 presents results for the marginal effects from a probit model where the main objective is to analyse the relation between household typology and poverty. The specifications used are the same as in table 6, but instead the variable for FHH there are seven dummy variables to indicate the household types (Types 2 to 10). The dummy for type 1 is excluded.

For all specifications the estimated parameters show the expected signs and the marginal effects show similar magnitudes than regressions in table 6. The basic specification is showed in column (1) where dummies for types 2, 4, 6 and 8 have positive sign. This implies these household types are poorer than type 1. However, as soon as more controls are added the estimated parameter change their sign for all types but type 8. Then, the only type with positive sign for all specifications is type 8. It seems that households type 8 are consistently poorer than the other types of household.

These results suggest that poverty is concentrated in households type 8 and 1. This last conjecture is made because the negative signs for all other types of households.

Table 7: Poverty and Household Types - Probit Regressions: Marginal Effects

VARIABLES	(1) Poor dy/dx	(2) Poor dy/dx	(3) Poor dy/dx	(4) Poor dy/dx
Education (years)	-0.0245***	-0.0235***	-0.0221***	-0.0212***
	(2.55e-05)	(2.55e-05)	(2.70e-05)	(2.69e-05)
Age	-0.00220***	-0.00353***	-0.00362***	-0.00220***
	(7.85e-06)	(8.15e-06)	(8.17e-06)	(8.65e-06)
Worker		-0.127***	-0.128***	-0.124***
		(0.000253)	(0.000253)	(0.000251)
Father's Educ: sec+			-0.0380***	-0.0405***
			(0.000409)	(0.000405)
Mother's Educ: sec+			-0.0309***	-0.0286***
			(0.000391)	(0.000388)
Children under 13				0.119***
				(0.000242)
Type 2	0.0207***	-0.0446***	-0.0457***	-0.0349***
	(0.000574)	(0.000540)	(0.000541)	(0.000531)
Type 3	-0.0276***	-0.0238***	-0.0253***	-0.0250***
	(0.000309)	(0.000317)	(0.000317)	(0.000305)
Type 4	0.0104***	-0.0439***	-0.0472***	-0.0453***
	(0.000614)	(0.000579)	(0.000576)	(0.000558)
Type 5	-0.0626***	-0.0704***	-0.0714***	-0.0416***
	(0.000525)	(0.000531)	(0.000531)	(0.000557)
Type 6	0.0234***	-0.0181***	-0.0205***	-0.0117***
	(0.000353)	(0.000353)	(0.000352)	(0.000346)
Type 7	-0.00603***	-0.0168***	-0.0170***	-0.000477
	(0.00120)	(0.00121)	(0.00121)	(0.00120)
Type 8	0.189***	0.144***	0.141***	0.129***
	(0.000515)	(0.000520)	(0.000519)	(0.000503)
Type 9	-0.141***	-0.144***	-0.144***	-0.0884***
	(0.000330)	(0.000347)	(0.000349)	(0.000415)
Type 10	-0.0174***	-0.0547***	-0.0552***	0.00313***
	(0.000513)	(0.000497)	(0.000497)	(0.000548)
Observations	22,144	22,144	22,144	22,144

Standard errors in parentheses

*** $p<0.01$, ** $p<0.05$, * $p<0.1$

Source: Regressions based on data from Colombian Living Standard Measurement Survey 2015.

Notes: All regressions control by urban and rural areas and regions.

Household types: Type 1: Male head, partner, no other adults; Type 2: Female head, partner, no other adults; Type 3: Male head, partner, other adults; Type 4: Female head, partner, other adults; Type 5: Male head, no partner, other adults; Type 6: Female head, no partner, other adults; Type 7: Male head, no partner, no other adults; Type 8: Female head, no partner, no other adults; Type 9: Male head, unipersonal; Type 10: Female head, unipersonal.

From tables 6 and 7, it is possible to argue that, in the Colombian context, female headship does not directly implies poverty. Despite the first set of regressions (table 6) show a positive relation between FHH and poverty, this relation becomes weaker as more controls are added[6]. Table 7 shows that poverty seems to be concentrated in two household types: one type with male headship (type 1) and the other with female headship (type 8).

In the following section are explored some general characteristics of income inequality among types of households.

[6]This could implies the more unobserved characteristics are accounted for the smaller becomes this relation.

3.2.5 Inequality among Household Types

Colombian income inequality measured by th Gini index is one of the greatest in Latin-America. As it was mentioned before, this inequality index has been decreasing in the last two decades. However, it is still to high and this fact has important implications for social welfare and living standards in general.

A general picture is provided by table 8, it describes the income distribution in Colombia. The population is organized by deciles taking into account the monthly household income per-capita.

Table 8: Income Deciles*

Percentile	HH's Income per-capita* (1)	Average Income per-capita within decile (2)	Cumulative Percentage of Income (3)
10	116,389	73,655	1.2%
20	175,000	145,957	3.5%
30	229,286	199,452	6.7%
40	290,714	254,756	10.8%
50	360,950	317,158	15.9%
60	453,700	394,182	22.2%
70	583,333	499,538	30.2%
80	800,000	653,488	40.7%
90	1,298,313	956,839	56.0%

* Monthly Household per-capita Income (2015) in percentile 'p', Colombian Pesos.
Note: Column (2) shows the average per-capita income for percentiles 0-10 in the first row, 11-20 in the second row and so on. The last row shows the average for percentiles 80-90
Source: National Department of Statistics - DANE.
Own Computations

Remember that the national poverty line in 2015 was 223.638 COP, so almost all individuals in the first three deciles are considered poor. In terms of inequality, the information in table 8 suggests the richest 10% of population owns 44% of total income in Colombia. This is a very illustrative number about how big is the income inequality in this country.

Table 9 shows the income distribution within household types. The columns are representing the household types and the sum of percentages in each of them is 100%. Then, this table shows the share of people living in a household type 'i' and located in the percentiles x to y in the total income distribution. For example, 10.66% of people living in households type 1 is located in percentiles 0 to 10 of total income distribution. This exercise provides a general picture of the inequality within household types and allows to observe the concentration of households in different percentiles of income distribution.

Table 9: Income Distribution within Household Types*

	Household Type									
Percentiles	1	2	3	4	5	6	7	8	9	10
0-p10	10.66	9.90	8.01	7.92	7.20	9.94	8.53	23.19	5.89	10.49
p10-p20	10.74	9.74	9.67	9.96	9.03	10.24	11.38	14.54	3.63	4.84
p20-p30	10.74	12.36	9.38	10.82	7.27	10.11	10.84	10.29	3.22	4.69
p30-p40	11.35	10.03	9.16	11.54	5.81	10.09	14.16	11.72	4.40	4.77
p40-p50	9.77	11.21	11.20	9.66	8.10	9.80	16.59	9.04	6.14	5.52
p50-p60	9.48	9.41	11.17	11.42	9.49	10.22	10.69	9.64	5.08	5.03
p60-p70	9.88	8.57	10.33	13.68	8.38	10.88	8.07	7.17	7.75	6.11
p70-p80	9.13	10.12	10.48	8.81	13.52	10.18	13.77	5.54	16.84	15.09
p80-p90	8.96	8.46	11.08	8.48	13.57	10.35	2.90	3.92	18.52	15.53
p90+	9.30	10.20	9.52	7.71	17.62	8.18	3.07	4.95	28.52	27.93
Total	100	100	100	100	100	100	100	100	100	100

* Income distribution based on Monthly Household per-capita Income (2015), Colombian Pesos.
Source: National Department of Statistics - DANE.
Own Computations

Figure 4 shows the data from table 9 to provide a general view of the distributions within household types. Note the distribution in types 1, 2, 3, 4, and 6 is relatively uniform. There is not a clear concentration of people in any of these types. In contrast, households type 8 are concentrated in the first deciles. This implies that a great share of people living in this type of household is poor. This result is according to the findings in previous section.

It is interesting the case of households type 9 and 10, which seems to concentrate more people in the higher deciles. Taking into account this types corresponds to uni-personal households, it is clear why these types have higher income.

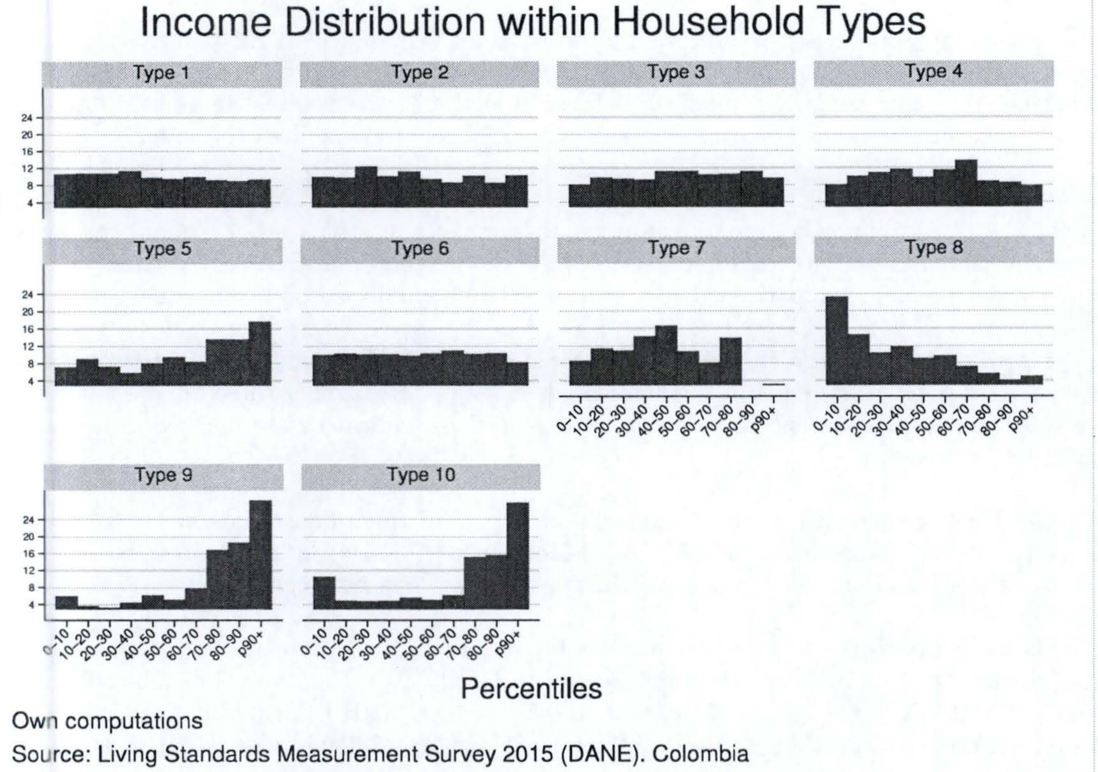

figure 4: Income Distribution within Household Types - Colombia

4 Socio-economic aspects of Female Headed Households

This section focus on the characterization of FHH given specific gender aspects based on the main findings of previous sections. The identified characteristics of female headed households and its typology are the starting points to provide a wider characterization of women well-being. The Colombian Demographic and Health Survey for 2015 is the natural source of information to achieve this goal.

The descriptive exercises in this section are focused on aspects as education, fertility, teenage pregnancy, domestic violence, and wealth. The main objective in this section is to provide an analysis of correlations between the typology and some characteristics of households, in order to provide a general characterization. The identification of causal relations is beyond the scope of this project.

4.1 Data

The main objective of the Colombian Demographic and Health Survey 2015 is to identify demographic changes in the last five years, and to obtain information about sexual and reproductive health among women and men. Also, the DHS provides information to describe special gender characteristics of FHHs. For 2015, the sample was 44,614 households, 38,718 women between 13 and 49 years old, 13,761 women between 50 and 59, 35,783 men between 13 and 59 and 4,517 men between 50 and 59. The survey is representative at national, regional and residence levels[7].

[7]Regional level: 9 regions and 16 subregions. Residence level: Urban-Rural

4.2 General Characteristics of Households

This section presents a set of indicators in order to provide a general characterization of Female Headed Households taking into account the main findings in section 3. As the DHS allows to construct the same household typology defined in section3, it is possible to make comparisons of similar measures between surveys. This can give an idea about the data consistency.

All indicators are calculated using weights to control by the actual distribution of the total population. It is important to keep in mind that weights in DHS do not expand the sample to have the same numbers as total population, while the weights from ECV2015 certainly do so. For this reason the main comparisons between surveys are doing based on percentages.

According to DHS2015, the average household size is 3.6, the total share of FHHs is 36.4% and the share of population living in this households is 34.3%. Remember these numbers in ECV2015 were 3.4, 35.3% and 33.1%, respectively. It seems these indicators are not too different among these two surveys.

The FHHs are concentrated in urban areas, 84.2% of these households are located in cities. In contrast, the percentage of Male Headed Households (MHHs) in urban areas is 73.6%. This indicates that rural areas has a predominant patriarchal household composition.

The DHS considers six great regions: North (Atlantica), East (Oriental), West (Pacifica), Central, South (Orinoquía/Amazonía), and Bogota (Capital City). Column (1) of table 10 shows the share of Total FHH per region. The greatest share of total FHH is located in the Central region, 28.1%. However, if we take into account the total number of households (FHH and MHH), and check its distribution within regions, it is clear the share of FHHs is relatively the same for all regions. Then, from this table we can argue that there is not a clear difference of household headship among Colombian regions, and the apparently high share of FHHs concentrated in Central region is given the more share of total population living there.

Table 10: Female Headed Households by Region

Region	FHH Total Share (1)	Share of MHH and FHH Within Regions: MHH (2)	FHH (3)
Atlantica (North)	17.96	65.25	34.75
Oriental (East)	16.56	65.65	34.35
Central	28.1	61.35	38.65
Pacifica (West)	17.85	63.29	36.71
Bogota (Capital City)	16.98	63.32	36.68
Orinoquia/Amazonia (South)	2.55	63.83	36.17

Source: Own computations based on Demographic and Health Survey 2015.

The DHS allows to categorize households using the typology defined in section 3. Table 11 presents some indicators based on this typology. This table is comparable to Table 4 constructed in base of ECV2015. In first place, we observe a similar shares of household types between DHS2015 and ECV2015. In both surveys the most common is type 1, followed by types 3, 6 and 8. In the case of type 8 in DHS2015 its share is 6.2% and 6.5% in ECV2015.

Types have also similar average household size, head's age, share of worker heads, and affiliation to health system. However, the average years of education do not have the same patterns among types. The DHS2015 does not have income information, this is the reason why this table is not report about household per-capita income.

In general terms, despite their different design, both surveys provide similar and consistent patterns for almost all the basic indicators related to the proposed household typology.

A point to remark is the similar percentage of households type 8, 6.5% in ECV2015 and 6.2% in DHS2015. This is interesting finding because the pattern presented in Robbins(2009) was increasing for this type of households in the period (1976-1999): 16.7% in the period 1976-1979, 19.8% in 1986-1989 and 22.9% in 1996-1999. Then, the measures presented in this project for 2015 could imply a great change in composition of households in Colombia. With no doubt, this is a good topic to explore in further research.

Table 11: Descriptive Statistics based on Household Typology

	Total HH	Share type	Size*	Head*				Children under 13
						Occupation		
				Age	Edu	Working	Housework	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Type 1	13,258	29.7%	3.4	44	8.3	85.1%	2.0%	1.0
Type 2	1,537	3.5%	3.3	40	8.8	48.0%	44.7%	0.89
Type 3	10,561	23.7%	4.9	54	7.7	76.9%	3.6%	0.74
Type 4	1,621	3.6%	5.2	50	7.6	47.4%	43.8%	0.89
Type 5	1,733	3.9%	3.4	55	7.3	68.7%	6.7%	0.43
Type 6	8,173	18.3%	3.9	56	7.1	42.4%	36.4%	0.63
Type 7	308	0.69%	2.5	46	7.1	85.5%	2.3%	0.56
Type 8	2,751	6.17%	2.9	40	8.8	62.3%	27.8%	1.07
Type 9	2,505	5.61%	1	53	7.0	75.3%	5.35%	-
Type 10	2,167	4.86%	1	58	7.3	42.3%	32.0%	-
Total	44,614	100%	3.6	50	7.8	68.1%	15.1%	0.74

Source: Own computations based on Demographic and Health Survey 2015.

* Average value per household type.

Household types: Type 1: Male head, partner no other adults; Type 2: Female head, partner, no other adults; Type 3: Male head, partner, other adults; Type 4: Female head, partner, other adults; Type 5: Male head, no partner, other adults; Type 6: Female head, no partner, other adults; Type 7: Male head, no partner, no other adults; Type 8: Female head, no partner, no other adults; Type 9: Male head, unipersonal; Type 10: Female head, unipersonal.

The greater differences between ECV2015 and DHS2015 seem to be the educational indicators. Next section explores the main characteristics of households related to this aspect.

4.3 Education

The lack of education is one of the main determinants of poverty. Then, the analysis of education patterns in female headed households is a relevant aspect to determine its potential influence on poverty vulnerability among this group of households.

Table 12 shows some general indicators related to the average years of education for groups of household members. The main objective is to make comparisons between the education levels for the Head of household, head's partner, and other adults. Also, for households with members between 6 and 18 years old, the percentage of children attending to school is calculated.

First row of table 12 presents the Head's years of education depending on the household composition. In specific, it is interesting to differentiate if the head has partner living in the household or not. This is specially important for FHH given that usually partners are missing for this group of households. Then, it is interesting to explore if this characteristic could have influence in poverty vulnerability for these households.

Column (1) and (2) compare the average years of education for different household members in FHH and MHH. In the case of heads of household, women have more years of education than men. However, households without the head of household's partner, have less education than those where both are present.

Table 12: Education and Female Headed Households

	Household Composition	Average Years of Education		
		FHH (1)	MHH (2)	Total (3)
Head	Alone	7.5	7.1	7.4
	Head and Partner	8.2	8.0	8.1
Head's Partner		7.5	8.4	8.3
Other Adults		9.6	9.8	9.7
Children Attending to School*	Alone	88.0%	81.7%	87.3%
	Head and Partner	88.2%	90.0%	89.8%

Source: Own computations based on Demographic and Health Survey 2015.
*Share of Children attending to school by household.

For partners, it is interesting for both, FHH and MHH, women always have more education than men. In the case of FHHs, the partner's average years of education is 7.5 while the head has in average 8.2 years of education. For MHHs, partners have 8.4 years of education while the heads only have 8.0 years.

When there are other adults in the household, the average years of education of these adults is higher than the head and partner's years of education.

The percentage of children school attendance is always higher than the 80%, this implies in a household with five children, at least four are attending to school. The highest attendance is in households with two parents (head and partner) and male head, 90%. The lowest is for households with male head alone, 81.7%. For female headed households, the percentage is around 88% in both cases (Head alone, Head and partner).

More detailed educational levels are presented in table 13, where the average years of education are computed by household types. Column (1) reports years of education for head of households by type. This information is the same as reported in column (5) table 11. Type 2 has the highest average years of education, 8.8. It is important to remark the smallest number of years of education is for type 7. As in table 12, It seems the head of household in FHH has in average relatively more education than MHH.

Table 13: Average Education and Children School Attendance

	Head (1)	Head's Partner (2)	Other Adults (3)	Children School Attendance* (4)
Type 1	8.3	8.8		92.2%
Type 2	8.8	8.1		91.2%
Type 3	7.7	7.8	10.0	87.4%
Type 4	7.6	7.0	9.7	85.9%
Type 5	7.3		8.2	82.1%
Type 6	7.1		9.6	85.9%
Type 7	7.1			84.2%
Type 8	8.8			91.9%
Type 9	7.0	-	-	-
Type 10	7.3	-	-	-

Source: Own computations based on DHS2015.
*Share of Children attending to school by household

The patterns for head's partner and other adults are similar than in table 12. Partners has more years of education in MHH, while the opposite happens for FHH, partners have less average years of education than head of households. Other adults always have more average years of education.

Children school attendance is specially high for types 1, 2 and 8. As in table 12, the children attendance is higher in households with 2 parents present.

In summary, it seems women have more education than men. The most important groups of households are types 1, 2 and 8 given their high educational levels. However, the results should be read carefully because these exercises are not identifying neither statistical correlations nor causal implications. This is a raw descriptive picture of the educational levels between FHH and MHH.

4.4 Fertility, Teenage Pregnancy and Sexual Violence

Fertility, teenage pregnancy and domestic violence are usually related to poverty and female households, Baker(1997). The DHS2015 is a rich source of information to study these aspects in the Colombian context.

4.4.1 Fertility

Table 15 presents basic indicators to provide a general picture of fertility, number of mothers in the household, age at first birth and abortion. Column (1) shows the average number of babies ever born by household types[8]. There are not great differences among types. However, it is interesting

[8]Including households with more than one women with children.

type 8 has the same average than types 1 or 2, both with two parents. This implies women in one parent households have as many children as women in households with two parents and then, there could be implications for poverty vulnerability of these women in households type 8. Households type 7 do not have measures because there are not observations enough.

The average number of women with children by household type (column 2), is close to 1 for all types. In other words, in average all Colombian households has at least one women who is mother. For the percentage of children under 5, types (1) and (2) have the highest values, 46.7% 40.9%. The lowest percentage is for type 8, 34.1%.

Table 15: Fertility, Number of Mother in the HH, Abortion

	Number babies ever born (1)	Number of mothers in HH (2)	Households with Children under 5 (3)	Age at First Birth (4)	Percentage of Abortion (5)
Type 1	1.7	0.88	46.7%	21	19.0%
Type 2	1.7	0.84	40.9%	20	24.1%
Type 3	1.4	0.82	33.1%	21	15.3%
Type 4	1.5	0.86	37.4%	20	17.9%
Type 5	1.0	0.57	36.0%	20	11.9%
Type 6	1.2	0.79	32.2%	21	14.9%
Type 7	0.7	0.07	6.78%	15	-
Type 8	1.7	0.91	34.1%	21	19.9%
Type 9	-	-	-	-	-
Type 10	0.8	0.39	-	19	20.2%
Total	1.5	1.1	37.6%	21	17%

Source: Own computations based on DHS2015.

For all types, the age at first birth is around 20 years old, which is a low age compared with the whole region of Latin America and the Caribbean where the age at first birth was 22.2 for the period 2010-2014, Bongaarts, Mensch and Blanc (2017). Finally, the percentage of abortion is 17% for the whole population. However, this percentage is 24% and 19.9% in households type 2 and 8 respectively. Both types of households with female headship. These high percentages could have some relation to domestic violence experiences, health issues or even it can be related to poverty. This potential causes should be studied in further research.

4.4.2 Teenage Pregnancy

Related to the relatively low age at first birth, it is logical to infer the incidence of teenage pregnancy could be high. Defining teenage as women who were pregnant at age between 13 and 18. This definition includes all women in the household who satisfy this definition. Table 16 presents basic indicators by household types. Column (1) shows the number of observations by type. Column (2) presents the distribution of cases by types in total population. Of course, types with more population have higher percentages. For this reason column (3) shows percentages within types. Type 4 has the highest percentage, 19.8%. Types 1, 2 and 8 have also high percentages.

Column (4) shows the percentage of grandmothers who were pregnant at ages between 13 and 18. In other words, it is interesting to explore if there is some kind of intergenerational effect of teenage pregnancy. Academic literature propose the hypothesis that daughters of teenage mothers have more risk of teenage childbearing. The greater percentages are for types 4 and 5. Type 7 has only 4 observations, so its percentage is not relevant.

Table 16: Teenage Pregnancy

	Observations	Total Percentage	Percentage Within Types	Percentage Teenage Mothers
	(1)	(2)	(3)	(4)
Type 1	2,184	34.4%	19.3%	8.8%
Type 2	254	4.0%	19.3%	9.3%
Type 3	1,659	26.1%	14.1%	9.4%
Type 4	371	5.8%	19.8%	11.8%
Type 5	153	2.4%	15.2%	17.1%
Type 6	1,146	18.0%	14.2%	9.2%
Type 7	4	0.06%	6.9%	29.6%
Type 8	509	8.0%	18.4%	8.7%
Type 9	-	-	-	
Type 10	72	1.1%	14.1%	6.8%
Total	6,357	100%	16.4%	9.4%

Source: Own computations based on DHS2015.

The descriptive statistics from table 16 are illustrative but difficult to analyse because the number of potential causes of teenage pregnancy. In order to get a better idea of this phenomenon, table 17 presents a set of probit regressions, and their respective marginal effects, to study the correlations between teenage pregnancy and the types of households.

The discrete dependent variable is 1 if the woman was pregnant between age 13 to 18. The independent variables are birth cohorts to control by age, years of education, Household size, daughters of teenage mother, and either a dummy for FHH or dummies for household typologies. It is important to remark that the objective of this exercise is to provide a better picture of correlations between some selected variables and teenage pregnancy. Given there is not a theoretical structure behind the econometric equation, the results cannot be understood as causal relations.

Table 17: Teenage Pregnancy - Probit Regressions

VARIABLES	(1) Teenage Pregnancy Probit	(2) Teenage Pregnancy dy/dx	(3) Teenage Pregnancy Probit	(4) Teenage Pregnancy dy/dx
Age Cohort: 1975-1984	0.423***	0.0884***	0.386***	0.0805***
	(0.0253)	(0.00525)	(0.0258)	(0.00536)
Age Cohort: 1985-1994	0.426***	0.0890***	0.400***	0.0835***
	(0.0260)	(0.00541)	(0.0263)	(0.00546)
Age Cohort: 1995+	-0.406***	-0.0850***	-0.424***	-0.0885***
	(0.0303)	(0.00631)	(0.0305)	(0.00634)
Education (Years)	-0.0988***	-0.0207***	-0.0968***	-0.0202***
	(0.00268)	(0.000541)	(0.00270)	(0.000544)
Household size	0.0411***	0.00860***	0.0557***	0.0116***
	(0.00434)	(0.000906)	(0.00501)	(0.00104)
Teenage Mother	0.271***	0.0566***	0.271***	0.0565***
	(0.0314)	(0.00656)	(0.0316)	(0.00658)
Female Headed Household	0.0641***	0.0134***		
	(0.0178)	(0.00372)		
Type 2			0.0195	0.00433
			(0.0472)	(0.0105)
Type 3			-0.158***	-0.0324***
			(0.0245)	(0.00503)
Type 4			0.0453	0.0102
			(0.0412)	(0.00937)
Type 5			-0.136**	-0.0283**
			(0.0564)	(0.0111)
Type 6			-0.0762***	-0.0162***
			(0.0256)	(0.00544)
Type 7			-1.996	-0.162***
			(1.792)	(0.0155)
Type 8			0.0657*	0.0149*
			(0.0355)	(0.00817)
Type 10			0.139*	0.0324
			(0.0818)	(0.0201)
Constant	-0.452***		-0.447***	
	(0.0414)		(0.0423)	
Observations	34,708	34,708	34,708	34,708

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Source: Regressions based on data from Colombian Demographic and Health Survey 2015.

Notes: All regressions control by urban and rural areas and regions.

Household types: Type 1: Male head, partner, no other adults; Type 2: Female head, partner, no other adults; Type 3: Male head, partner, other adults; Type 4: Female head, partner, other adults; Type 5: Male head, no partner, other adults; Type 6: Female head, no partner, other adults; Type 7: Male head, no partner, no other adults; Type 8: Female head, no partner, no other adults; Type 10: Female head, unipersonal.

The dummies for ten years birth cohorts are controlling by the woman current age. The dummy for women born before 1974 was excluded. The estimated parameters are positive for birth cohorts 1975-1984 and 1985-1994. This seems to describe an increase in teenage pregnancy in the last 30 years of the last century. For the birth cohort (1995 -) parameters are negative. However, this sign does not necessary implies a reduction in teenage pregnancy. As this cohort begins in 1995, there are teenagers in this group and they still can get pregnant.

Years of education and Household size have the expected signs, negative and positive respectively. More education is related to lower incidence of teenage pregnancy, while greater households tend to have more pregnant teenagers . The variable to capture the intergenerational effect of teenage pregnancy is "Teenage Mother". In this case the sign is positive. This implies daughters of teenage mothers has more possibilities to get pregnant at early ages.

The discrete variable indicating FHHs produced coefficients with positive sign. This means the incidence of teenage pregnancy tends to affect FHHs more than MHHs. When types of households are included in the regression, with type 1 as omitted dummy, the coefficients for type 8 and 10 are positive and statistically different from zero. Then the possibilities to find a women who was teenager mother in households type 8 and 10 are greater.

4.4.3 Domestic Violence

Domestic violence is another factor that usually is closely related to poverty and inequality. Table 18 presents general indicators for the average percentage of women affected by three types of domestic violence: Psychological, Physical and Sexual. Also there are a general domestic violence indicator, that calculates the percentage of women who have been affected by one or more types of violence. All percentages are calculated by household types.

Psychological violence is configured if one or more of the following situations is suffered by woman:

- Husband/partner jealous if respondent talks with other men.
- Husband/partner accuses respondent of unfaithfulness.
- Husband/partner does not permit respondent to meet female friends.
- Husband/partner tries to limit respondent's contact with family.
- Husband/partner insists on knowing where respondent is.
- Husband/partner doesn't trust respondent with money.
- Partner ignores/don't address her.
- Hasn't request opinion for family/social gatherings.
- Hasn't request opinion on important family matters.
- Ever been insulted or made to feel bad by husband/partner.
- Has threaten to leave her.
- Has threaten to take away children.

Column (1) of table 18 presents the percentage of women affected by psychological violence by type of households. The half of women living in households type 1 and 2 are affected by psychological violence. Also women in type 8 have been highly affected by this type of violence. In general, this violence is the most common, the total share of women affected by psychological violence is 41.3%.

There is physical violence if woman has been affected by one or more of the following situations:

- Ever been pushed, shook or had something thrown by husband/partner.
- Ever been slapped by husband/partner.

- Ever been punched with fist or hit by something harmful by husband/partner.
- Ever been kicked or dragged by husband/partner.
- Ever been strangled or burnt by husband/partner.
- Ever been threatened with knife/gun or other weapon by husband/partner.
- Ever been attacked with knife/gun or other weapon by husband/partner.

In this case, women in households type 8 are the most affected by this violence, 31.2%. This percentage is relatively high compared to the total percentage of women suffering this violence, 20.8%.

Finally, sexual violence is considered if women have "ever been physically forced into unwanted sex by husband/partner". This is the most unreported type of violence. In part, given this reason the percentages of women affected by this violence are relatively small compared to other types of violence. The total percentage of women who suffered of sexual violence is 4.9%. Again the most affected women are those living in households type 8 with 10.7%.

Table 18: Domestic Violence

	Psychological Violence*	Physical Violence*	Sexual Violence*	Domestic Violence (1)+(2)+(3)*
	(1)	(2)	(3)	(4)
Type 1	50.3	22.2	3.5	51.7
Type 2	49.8	23.2	6.3	50.7
Type 3	35.2	17.0	3.7	36.2
Type 4	38.5	18.2	3.8	39.5
Type 5	32.3	18.5	5.3	33.6
Type 6	36.6	21.2	6.6	37.3
Type 7	7.3	4.7	-	7.3
Type 8	47.9	31.2	10.7	49.6
Type 9	-	-	-	-
Type 10	31.8	19.6	4.5	32.1
Total	41.3	20.8	4.9	42.37

Source: Own computations based on DHS2015.
* all reported values are in percentages

Summarizing, the households type 8 and 1 are the most affected by all types of violence. In specific, physical and sexual violence are most common among women in type 8.

4.5 Income, Wealth and Labor Market

Income, poverty and inequality aspects have been studied in section 3. However, DHS2015 has not only specific information focused on women, as land onwship, but also has an specific wealth index constructed to be internationally comparable. This index involve a number of questions about assets and services available to the household.

The DHS wealth index is used to complete the characterization of FHHs. Based the wealth index, table 19 presents percentages of poorest and poor women by household types (poorest and poor categories correspond to the first and second quintiles of the distribution: Poorest = women in the First quintile, Poor = women in the first and second quintile). Column (1) shows percentages for "poorer" women. Types 1, 2 and 8 have more women in this category. Column (2) presents the percentages for the first quintile, in this case types 1, 2 and 5 have the greatest percentages.

Column (3) presents the average percentage of working women by household types. As expected, the highest share of working women is found in types 10 and 8. For ownership of land or house, columns (4) and (5) respectively, the greater percentages are for types 1, 2, 10 and 8.

The general picture of wealth aspects provided by table 19, identifies women in households type 1, 2 and 8 as poorer but partial or total owners of house or land, and workers in a great proportion.

Table 19: Wealth Aspects for Women

	Poor*	Poorest*	Working*	House Owner*	Land Owner*
	(1)	(2)	(3)	(4)	(5)
Type 1	62.1	33.1	49.5	31.7	13.0
Type 2	63.1	29.7	55.1	28.9	12.0
Type 3	50.7	25.7	51.4	23.9	9.8
Type 4	52.8	20.2	52.9	22.6	8.4
Type 5	57.9	28.5	53.4	18.3	11.6
Type 6	47.6	15.1	58.7	18.9	7.8
Type 7	57.6	27.1	12.5	8.8	7.0
Type 8	62.5	21.3	59.9	27.4	9.82
Type 9	-	-	-	-	-
Type 10	60.1	15.4	80.5	26.0	11.8
Total	55.4	25.3	53.6	25.4	10.4

Source: Own computations based on DHS2015.
* All reported values are in percentages

Figure 5 provides a general view of the wealth distribution of women by household types. Each graph can be understand as the proportion of women in households type 'i' by each quintile of wealth. Here it is possible to observe the importance of types 1 and 8 from another perspective. In the case of type 1, is clear the concentration of women is located in quintiles 1 and 2 (poorest and poorer). 60% of women in this type are in these two quintiles.

Women living in households type 8 are located in the second quintile (poorer), 40%. The first and third quintiles (poorest and middle) concentrated other 40% of women and only the remaining 20% are women in the top of the wealth distribution.

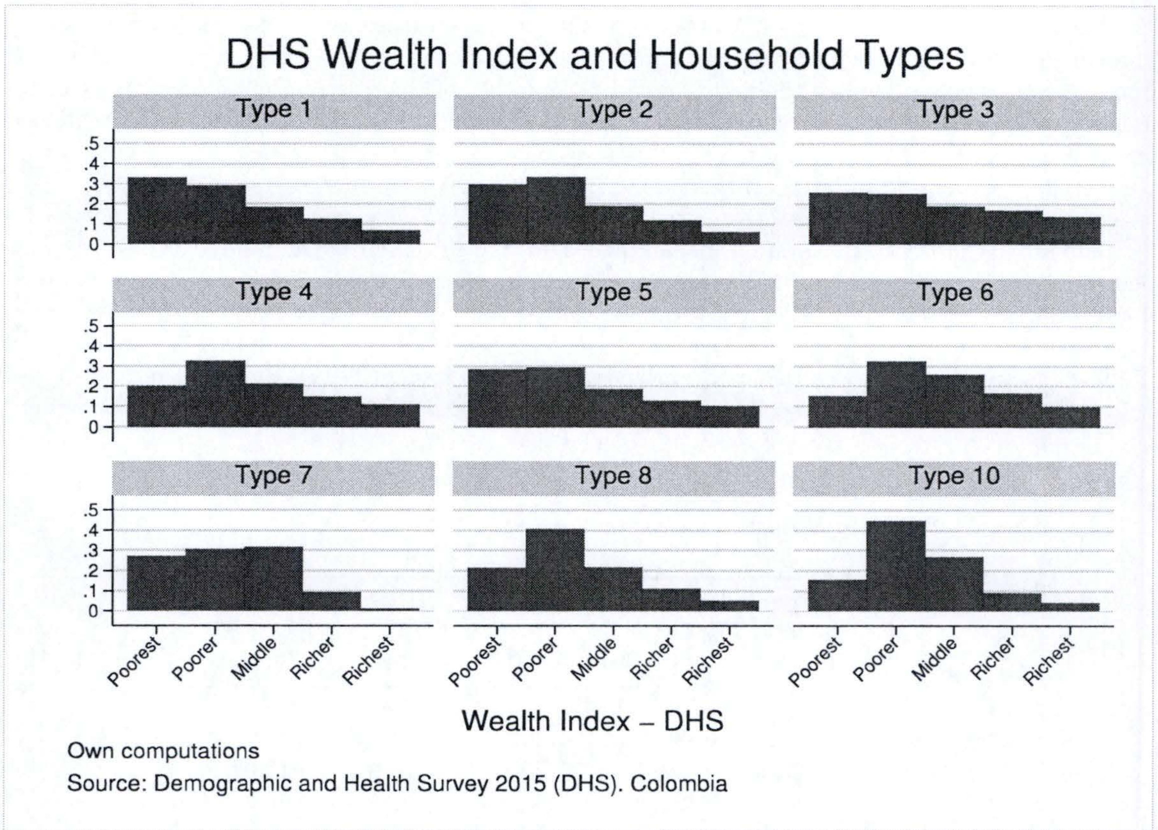

Figure 5: DHS Wealth Index Household Types - Colombia

5 Conclusion

This project investigated the main characteristics of Female Headed Households and their relation to poverty, inequality, education, fertility and other women-related aspects in Colombia. Since the 1970s, there have been a great interest in the study of the relation of poverty and female headship especially in developing countries. The general hypothesis was that female headed households are more affected by poverty than other types of households. After 3 decades of research, deep study and many policy interventions, there is still controversy about this hypothesis. There is not conclusive evidence to generalize the idea of "poverty feminization".

Since 1991, Colombia has been implemented public policies targeted women and female headed households. This intervention acts as channels to protect this group from poverty vulnerability. However, these interventions have not been consistently evaluated and then there are not enough clarity about their results.

The main objective of this project is to investigate the characteristics of female headed households and their relation to poverty, inequality and gender aspects in Colombia. This goal has been achieved using the Living Standards Measurement Survey (2015) and the Demographic and Health Survey (2015). As the household composition matters for poverty and inequality, a typology of households was defined based on three characteristics: number of parents, other adults and head's gender. This typology is based on Robbins(2009).

The Living Standards Measurement Survey was used to study poverty and inequality. The main findings show that female headed households are about 35% of total households in Colombia. The most common types of household are types 1, 3, 6 and 8. Types 6 and 8 are female headed households where the head's partner or husband is missing. Type 8 has important characteristics related to poverty and vulnerability. In fact, a within type analysis sows that type 8 has the greater proportion of poor households, 42.1%. This percentage is high in comparison to the total share of poor households, 24.2%.

Following Haughton and Khandker (2009), it is proposed an exercise to identify the potential correlates of poverty given female headship and the typology of households. The relation between poor households and female headship is explored using a model with discrete dependent variable (Probit). A positive correlation between poverty and female headship is found. However, the marginal effect for FHH decreases in magnitude when more household characteristics are included. This implies the effects of female headship become relatively small when controlling for household composition. When types of household are included in the regressions, the results suggests that poverty is concentrated in households type 1 and 8. The last type is the most vulnerable to poverty.

In terms of the income inequality, the main findings results show households type 1 and 8 are the most common in the botton of the distribution.

Once established the main characteristics of household types regarding poverty and inequality, the Demographic and Health Survey 2015 is used to provide a wider analysis of household characteristics including dimensions as education, fertility, teenage pregnancy, domestic violence and wealth aspects among women.

About education, it seems women have more education than men. The most important groups of households are types 1, 2 and 8 given the relatively high educational levels of the heads of households. However, the results should be read carefully because these exercises are not identifying neither statistical correlations nor causal implications.

In terms of fertility, there are no clear differences in the number of babies ever born among types of households. The average is 1.5 babies ever born per woman. Also the age at first birth is similar for all types, it is 20 years old. And the percentage of abortions is around 17%. however, for households type 2 and 8 this percentage is higher, 24.1% and 19.9%, respectively.

Teenage pregnancy is analysed using some descriptive statistics but also an exercise to clarify correlations between women who get pregnant at ages 13 to 18 and types of households. The results shows that correlation between FHHs and teenage pregnancy is positive. This means the incidence of teenage pregnancy tends to affect FHHs more than MHHs. When types of households are included in the regression, only the coefficients for types 8 and 10 are positive and statistically different from zero. Then the possibilities to find a women who was teenager mother in households type 8 are greater.

Domestic violence is divided between three types: Psychological, Physical and Sexual. Households type 8 and 1 are the most affected by all types of violence. In specific, physical and sexual violence are most common among women in type 8.

In the women's wealth analysis, the main findings are that women in households type 1, 2 and 8 are poorer, but also they are partial or total owners of house or land, and workers in a great proportion. Similar than the results from the Living and Measurement Survey, types 1 and 8 are located in the botton of the distribution.

As a general conclusion, our results do not give strong support for the hypothesis of "feminization of poverty". Given the household typology proposed, two types of households were identified as more vulnerable: type 1 and Type 8. Given their different composition, probably these types do not have the same causes for poverty vulnerability. However, despite households type 8 face many difficulties and deprivations not all FHHs have high poverty vulnerability, then it is not correct to suggest a direct relation between poverty and Female Headship in Colombia. The specific household composition and other socio-economic factors could be more important than the Female Household Headship by itself.

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7 Annex

Poverty and Female Headship: LPM

VARIABLES	(1)	(2)	(3)	(4)
Education (years)	-0.0222***	-0.0210***	-0.0200***	-0.0190***
	(2.47e-05)	(2.46e-05)	(2.63e-05)	(2.60e-05)
Age	-0.00289***	-0.00430***	-0.00439***	-0.00243***
	(7.32e-06)	(7.80e-06)	(7.84e-06)	(8.37e-06)
Worker		-0.134***	-0.135***	-0.128***
		(0.000271)	(0.000271)	(0.000268)
Father's Educ: sec+			-0.0142***	-0.0148***
			(0.000367)	(0.000362)
Mother's Educ: sec+			-0.0256***	-0.0195***
			(0.000360)	(0.000355)
Children under 13				0.141***
				(0.000229)
Female Head	0.0761***	0.0343***	0.0331***	0.0374***
	(0.000227)	(0.000240)	(0.000241)	(0.000237)
Constant	0.562***	0.725***	0.730***	0.540***
	(0.000537)	(0.000627)	(0.000628)	(0.000692)
Observations	13,498,923	13,498,923	13,498,923	13,498,923
R-squared	0.172	0.187	0.188	0.210

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Poverty and Female Headship - Probit Regressions

VARIABLES	(1)	(2)	(3)	(4)
Education (years)	-0.0931***	-0.0910***	-0.0853***	-0.0838***
	(0.000106)	(0.000107)	(0.000112)	(0.000114)
Age	-0.0123***	-0.0172***	-0.0177***	-0.0103***
	(2.82e-05)	(3.02e-05)	(3.04e-05)	(3.32e-05)
Worker		-0.485***	-0.491***	-0.483***
		(0.00102)	(0.00102)	(0.00103)
Father's Educ: sec+			-0.147***	-0.157***
			(0.00161)	(0.00163)
Mother's Educ: sec+			-0.139***	-0.122***
			(0.00154)	(0.00157)
Children under 13				0.555***
				(0.000926)
Female Head	0.301***	0.133***	0.127***	0.135***
	(0.000857)	(0.000936)	(0.000937)	(0.000952)
Constant	0.550***	1.142***	1.173***	0.460***
	(0.00205)	(0.00242)	(0.00243)	(0.00273)
Observations	22,144	22,144	22,144	22,144

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Poverty and Household Types - LPM

VARIABLES	(1)	(2)	(3)	(4)
Education (years)	-0.0224***	-0.0212***	-0.0204***	-0.0194***
	(2.45e-05)	(2.44e-05)	(2.61e-05)	(2.60e-05)
Age	-0.00196***	-0.00345***	-0.00352***	-0.00213***
	(7.86e-06)	(8.34e-06)	(8.38e-06)	(8.77e-06)
Worker		-0.135***	-0.136***	-0.131***
		(0.000270)	(0.000270)	(0.000268)
Father's Educ: sec+			-0.0119***	-0.0142***
			(0.000364)	(0.000360)
Mother's Educ: sec+			-0.0209***	-0.0178***
			(0.000357)	(0.000354)
Children under 13				0.122***
				(0.000249)
Type = 2, Type 2	0.0162***	-0.0436***	-0.0441***	-0.0325***
	(0.000561)	(0.000568)	(0.000568)	(0.000564)
Type = 3, Type 3	-0.0265***	-0.0222***	-0.0231***	-0.0197***
	(0.000313)	(0.000310)	(0.000310)	(0.000308)
Type = 4, Type 4	0.00644***	-0.0437***	-0.0461***	-0.0389***
	(0.000602)	(0.000605)	(0.000605)	(0.000600)
Type = 5, Type 5	-0.0599***	-0.0677***	-0.0683***	-0.0286***
	(0.000554)	(0.000549)	(0.000549)	(0.000550)
Type = 6, Type 6	0.0188***	-0.0180***	-0.0196***	-0.00531***
	(0.000345)	(0.000350)	(0.000350)	(0.000348)
Type = 7, Type 7	-0.0114***	-0.0161***	-0.0163***	-0.000316
	(0.00125)	(0.00124)	(0.00124)	(0.00123)
Type = 8, Type 8	0.186***	0.148***	0.146***	0.140***
	(0.000463)	(0.000465)	(0.000465)	(0.000461)
Type = 9, Type 9	-0.152***	-0.146***	-0.145***	-0.0753***
	(0.000418)	(0.000415)	(0.000415)	(0.000435)
Type = 10, Type 10	-0.0192***	-0.0515***	-0.0511***	0.00851***
	(0.000505)	(0.000504)	(0.000504)	(0.000514)
Constant	0.548***	0.715***	0.719***	0.561***
	(0.000544)	(0.000634)	(0.000636)	(0.000708)
Observations	13,498,923	13,498,923	13,498,923	13,498,923
R-squared	0.188	0.203	0.204	0.218

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Poverty and Household Types - Probit Regressions

VARIABLES	(1)	(2)	(3)	(4)
Education (years)	-0.0957***	-0.0936***	-0.0882***	-0.0861***
	(0.000107)	(0.000109)	(0.000114)	(0.000115)
Age	-0.00857***	-0.0141***	-0.0144***	-0.00895***
	(3.09e-05)	(3.30e-05)	(3.32e-05)	(3.54e-05)
Worker		-0.505***	-0.510***	-0.505***
		(0.00103)	(0.00104)	(0.00105)
Father's Educ: sec+			-0.151***	-0.165***
			(0.00163)	(0.00165)
Mother's Educ: sec+			-0.123***	-0.116***
			(0.00156)	(0.00158)
Children under 13				0.486***
				(0.00101)
Type = 2, Type 2	0.0771***	-0.176***	-0.180***	-0.143***
	(0.00211)	(0.00221)	(0.00222)	(0.00224)
Type = 3, Type 3	-0.108***	-0.0916***	-0.0976***	-0.101***
	(0.00122)	(0.00123)	(0.00123)	(0.00124)
Type = 4, Type 4	0.0392***	-0.173***	-0.186***	-0.188***
	(0.00230)	(0.00237)	(0.00237)	(0.00241)
Type = 5, Type 5	-0.257***	-0.287***	-0.291***	-0.171***
	(0.00230)	(0.00231)	(0.00232)	(0.00238)
Type = 6, Type 6	0.0869***	-0.0693***	-0.0785***	-0.0468***
	(0.00131)	(0.00136)	(0.00136)	(0.00138)
Type = 7, Type 7	-0.0231***	-0.0642***	-0.0652***	-0.00188
	(0.00461)	(0.00469)	(0.00469)	(0.00474)
Type = 8, Type 8	0.630***	0.491***	0.483***	0.462***
	(0.00165)	(0.00170)	(0.00171)	(0.00172)
Type = 9, Type 9	-0.670***	-0.666***	-0.664***	-0.389***
	(0.00185)	(0.00187)	(0.00188)	(0.00198)
Type = 10, Type 10	-0.0675***	-0.218***	-0.220***	0.0123***
	(0.00201)	(0.00207)	(0.00207)	(0.00215)
Constant	0.513***	1.141***	1.171***	0.560***
	(0.00211)	(0.00250)	(0.00251)	(0.00283)
Observations	22,144	22,144	22,144	22,144

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1