October 12th, 2023

Bernoulli Society
for Mathematical Statistics
and Probability

Timoteo Carletti

Physics and hypergraphs. Synchronization on hypergraphs & simplicial complexes

Department of mathematics
UNamur

Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR

Acknowledgements

Master Students

Gwendoline Planchon

Marine Jamoulle

Alice Bellière

PhD Students

Martin Moriamé

Jean-François De Kemmeter

Lorenzo Giambagli

Cédric Simal

Sara Nicoletti

Marie Dorchain

PostDocts

Riccardo Muolo

Sarah De Nigris

Giulia Cencetti

Luca Gallo

Nikos Kouvaris

Thierry-Sainclair Njougouo

Collaborators

Malbor Asllani (Florida State University)

Federico Battiston (Central European University)

Ginestra Bianconi (Queen Mary University)

Duccio Fanelli (Università di Firenze)

Mattia Frasca (Università di Catania)

Valentina Gambuzza (Università di Catania)

Renaud Lambiotte (University of Oxford)

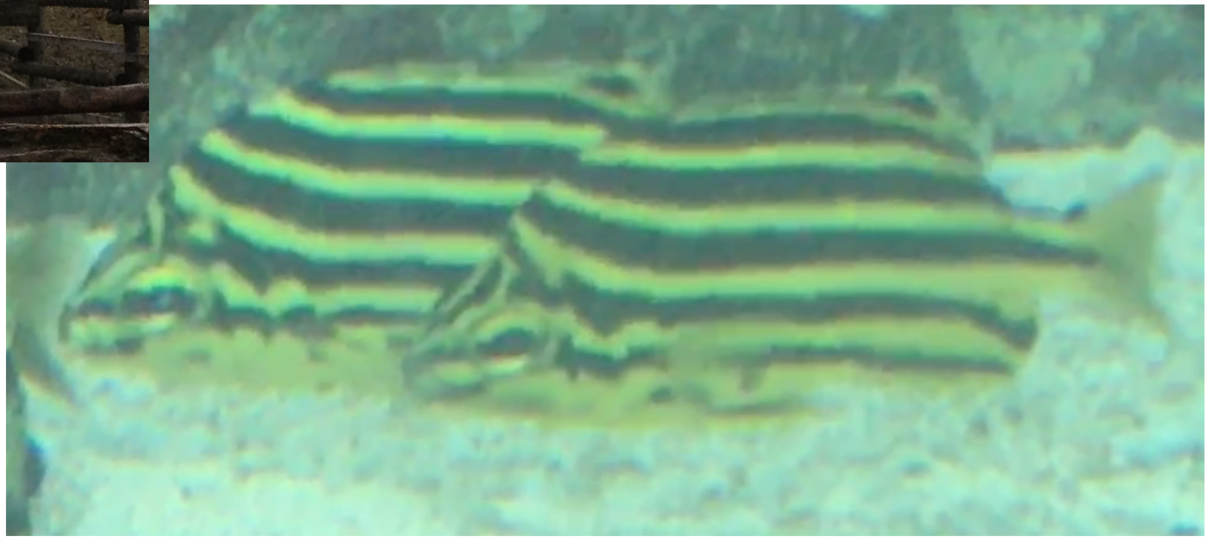
Vito Latora (Università di Catania & Queen Mary University)

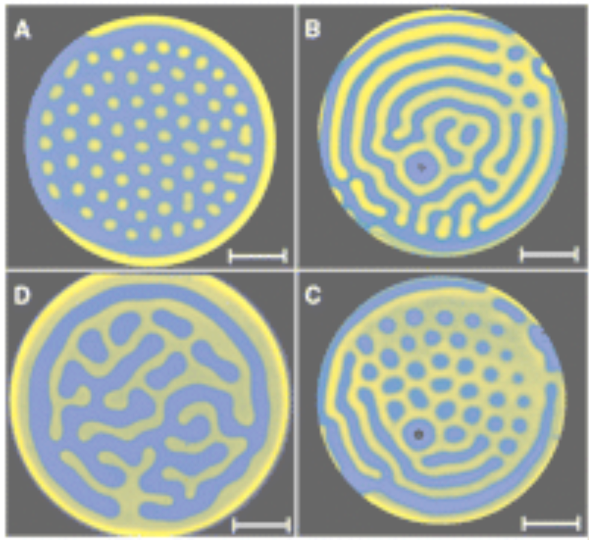
Hiroya Nakao (Tokyo Institute of Technology)

Julien Petit (Ecole Royale Militaire)

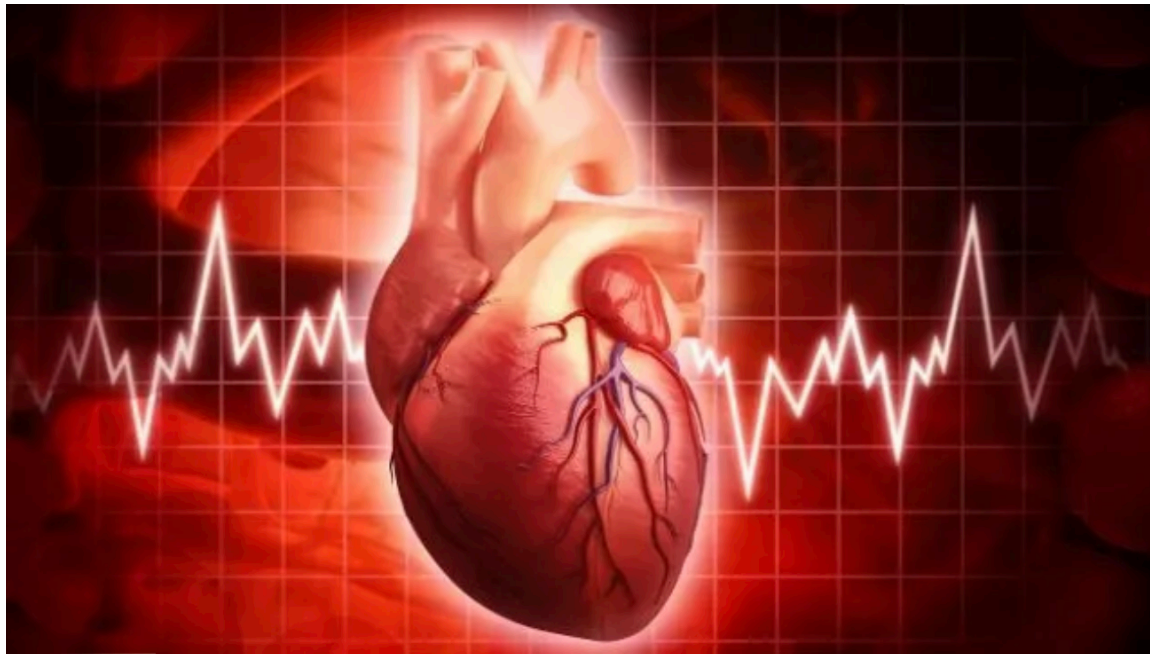
timoteo.carletti@unamur.be

Order from disorder ...



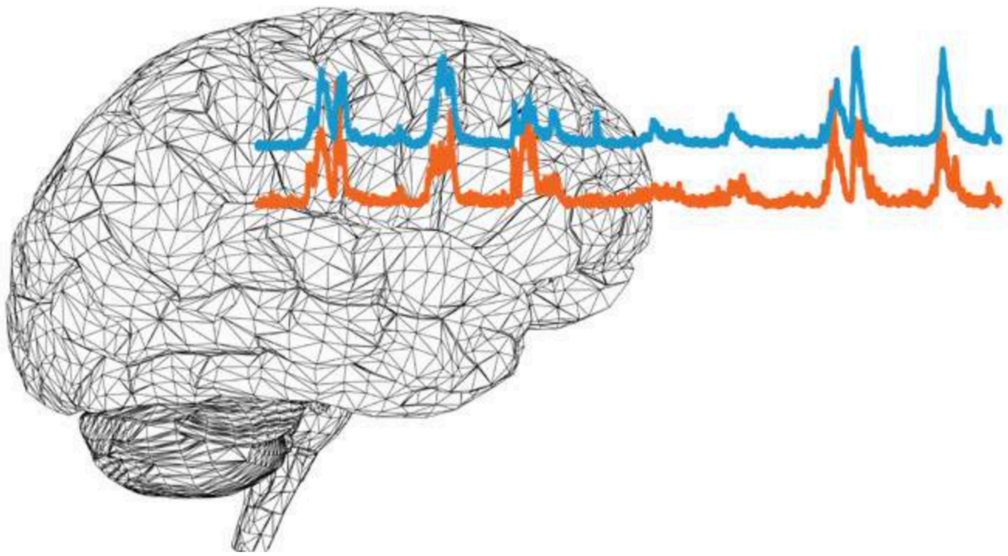


Synchronisation

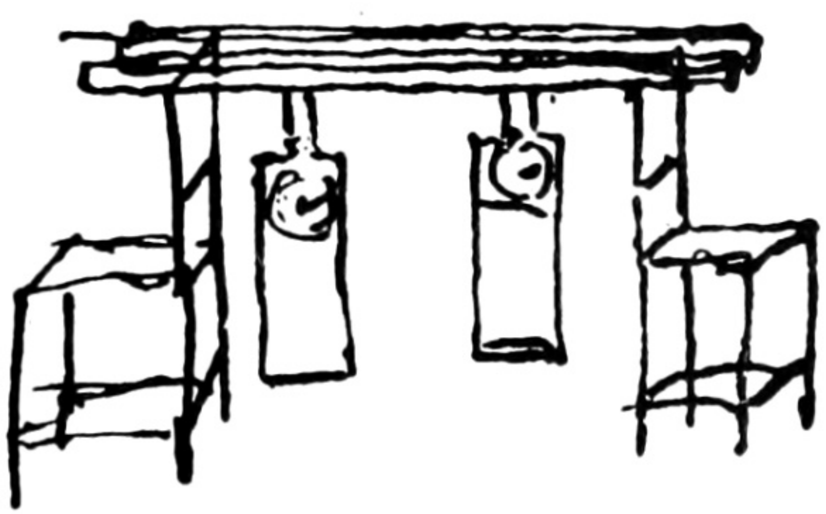


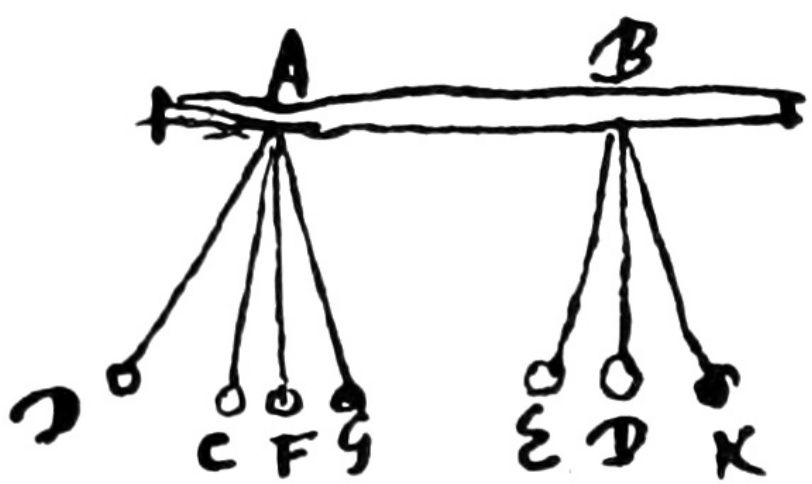
www.youtube.com

www.quantamagazine.org

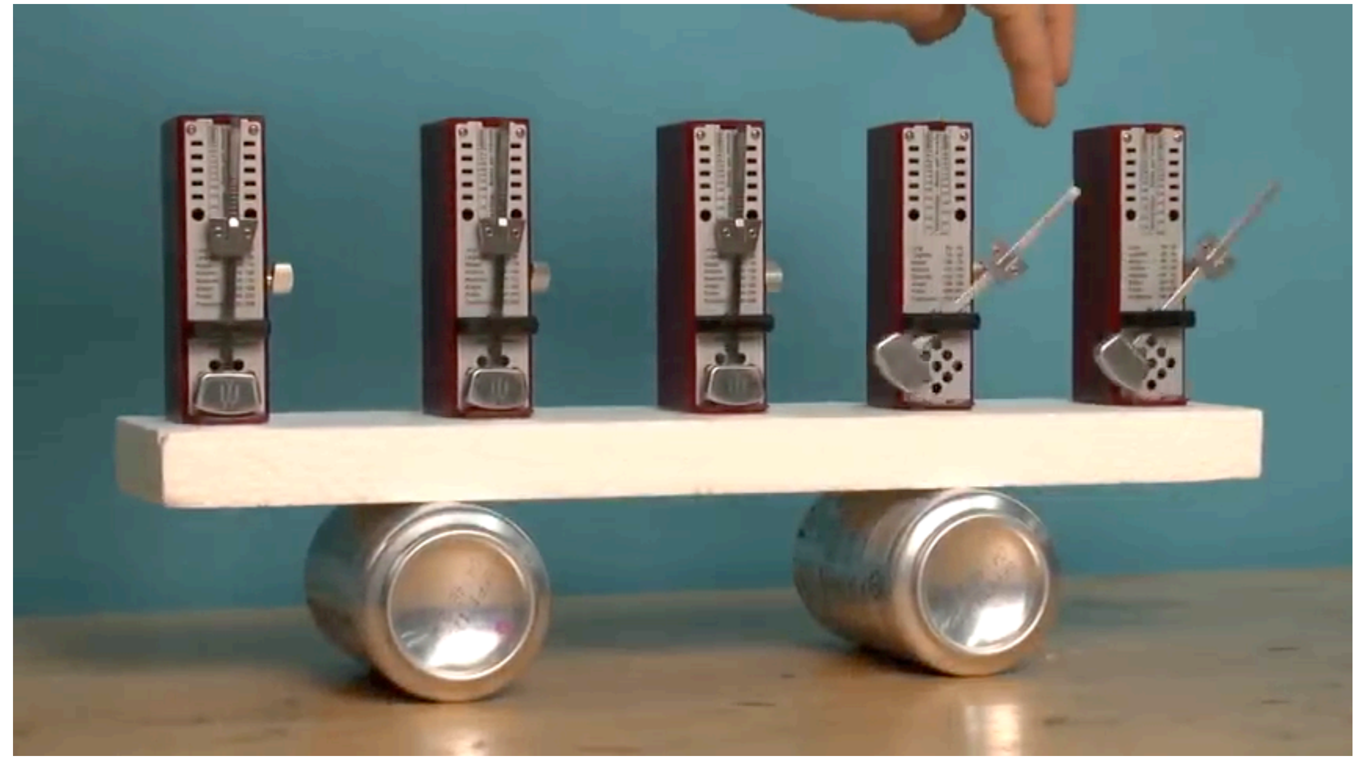


Synchronisation

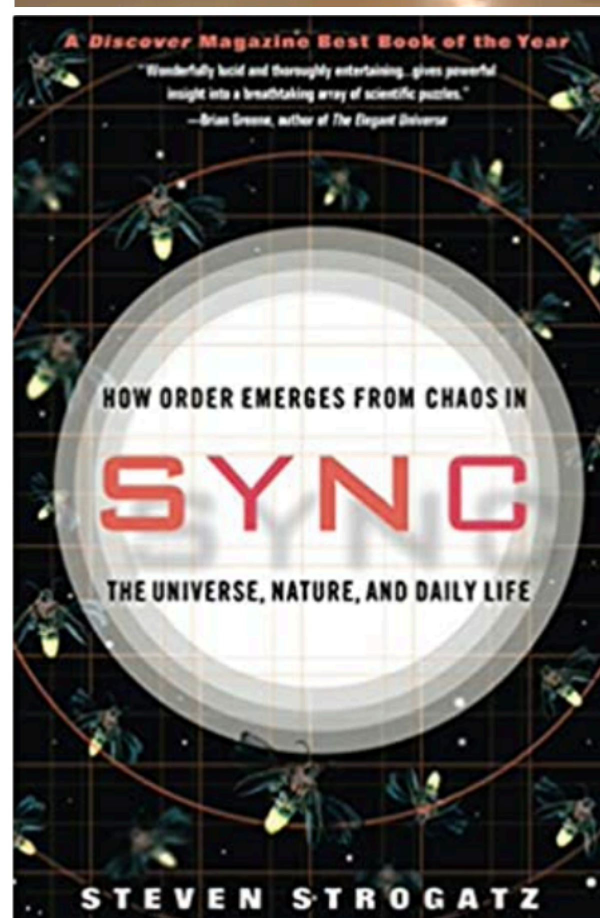




Huygen (1665)
"An odd kind of sympathy"

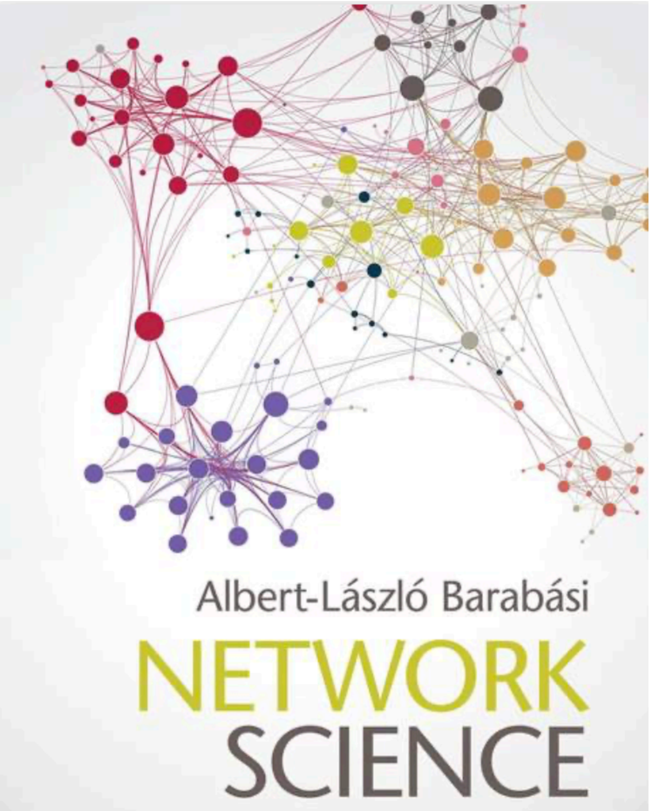


www.youtube.com



timoteo.carletti@unamur.be

We live in an interconnected world ...



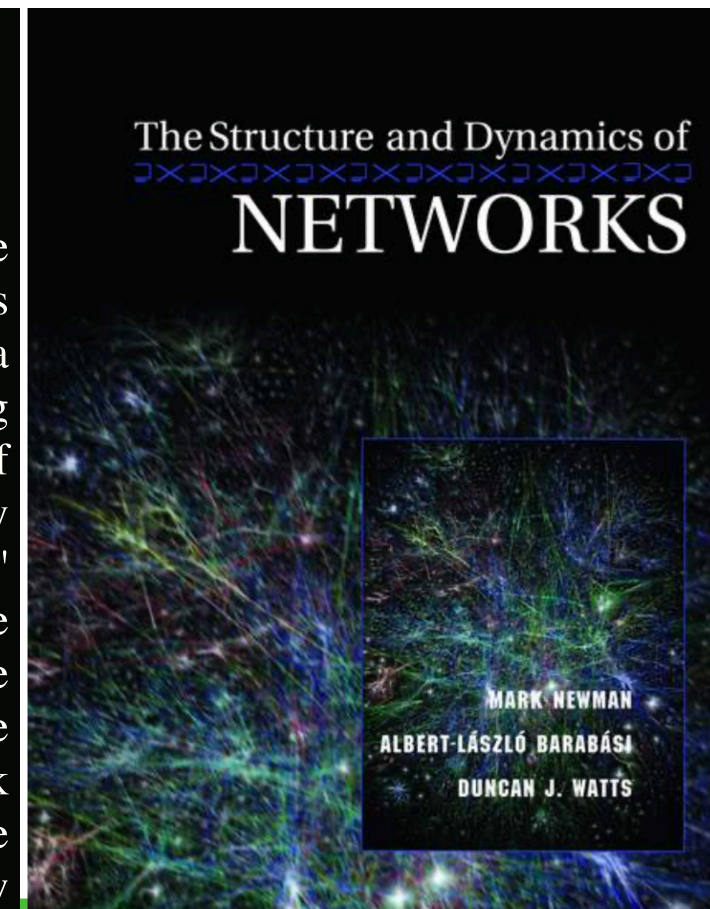
Network Science A.-L. Barabási

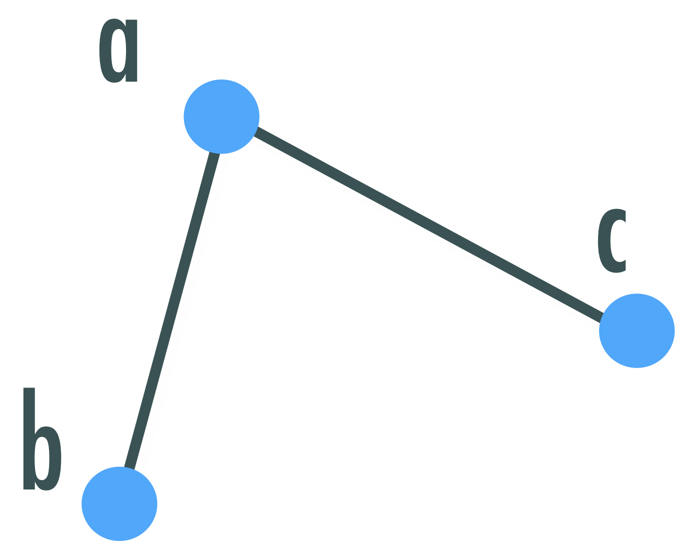
Networks are everywhere, from the Internet, to social networks, and the genetic networks that determine our biological existence. Illustrated throughout in full colour, this pioneering textbook, spanning a wide range of topics from physics to computer science, engineering, economics and the social sciences, introduces network science to an interdisciplinary audience. From the origins of the six degrees of separation to explaining why networks are robust to random failures, the author explores how viruses like Ebola and H1N1 spread, and why it is that our friends have more friends than we do. Using numerous real-world examples, this innovatively designed text includes clear delineation between undergraduate and graduate level material. The mathematical formulas and derivations are included within Advanced Topics sections, enabling use at a range of levels. Extensive online resources, including films and software for network analysis, make this a multifaceted companion for anyone with an interest in network science.

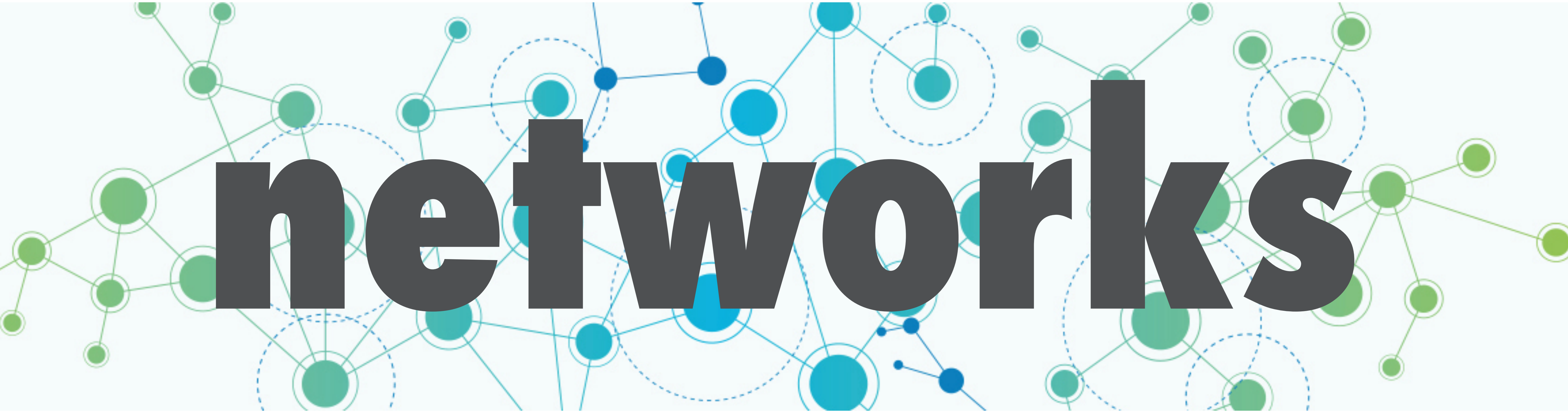
The Structure and Dynamics of Networks

A.-L. Barabási, M. Newman, D.J.Watts

From the Internet to networks of friendship, disease transmission, and even terrorism, the concept-and the reality-of networks has come to pervade modern society. But what exactly is a network? What different types of networks are there? Why are they interesting, and what can they tell us? In recent years, scientists from a range of fields-including mathematics, physics, computer science, sociology, and biology-have been pursuing these questions and building a new "science of networks." This book brings together for the first time a set of seminal articles representing research from across these disciplines. It is an ideal sourcebook for the key research in this fast-growing field. The book is organized into four sections, each preceded by an editors' introduction summarizing its contents and general theme. The first section sets the stage by discussing some of the historical antecedents of contemporary research in the area. From there the book moves to the empirical side of the science of networks before turning to the foundational modeling ideas that have been the focus of much subsequent activity. The book closes by taking the reader to the cutting edge of network science--the relationship between network structure and system dynamics. From network robustness to the spread of disease, this section offers a potpourri of topics on this rapidly expanding frontier of the new science.







Network = finite set of nodes pairwise connected, i.e., there is a link (edge) among the two nodes if there is some interaction among them

Dynamics

Structure

Global Synchronisation on networks

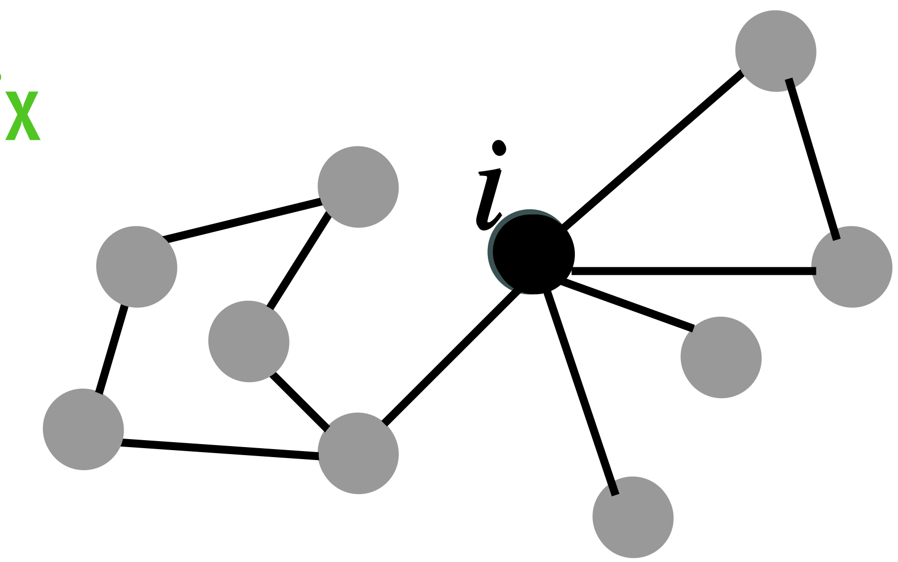
$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \ldots, n$$

A_{ij} Adjacency matrix



2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} A_{ij}\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

$$\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \mathbf{h}(\mathbf{x}^{(j)}) - \mathbf{h}(\mathbf{x}^{(i)})$$

Diffusive-like coupling

$$= \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})$$

L_{ij} Laplace matrix

Global Synchronisation on networks

Reference orbit $\mathbf{s}(t)$ solution of $$\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$$

Global synchronisation : $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

- Does the whole system admit such (spatially) homogeneous solution?

- Is it stable?

Global Synchronisation: Pecora et al.

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

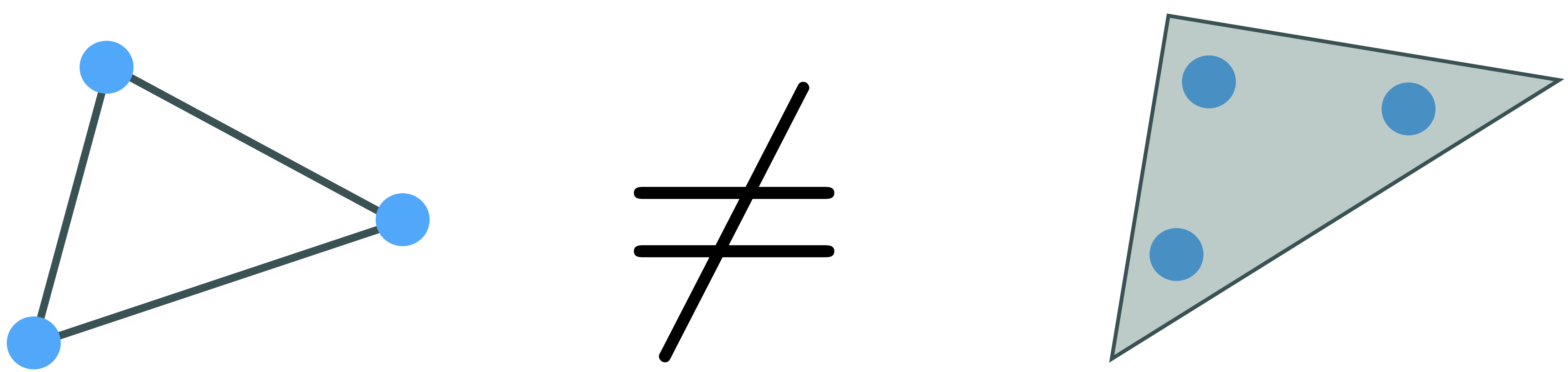
Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

PHYSICAL REVIEW E **80**, 036204 (2009)

Generic behavior of master-stability functions in coupled nonlinear dynamical systems

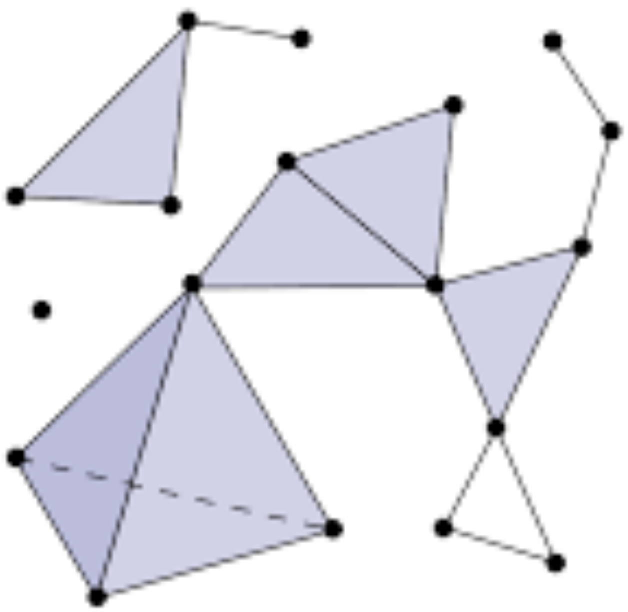
Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]
[1]*School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[2]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Code 6362, Naval Research Laboratory, Washington, DC 20375, USA*
(Received 9 June 2009; published 15 September 2009)

limitation



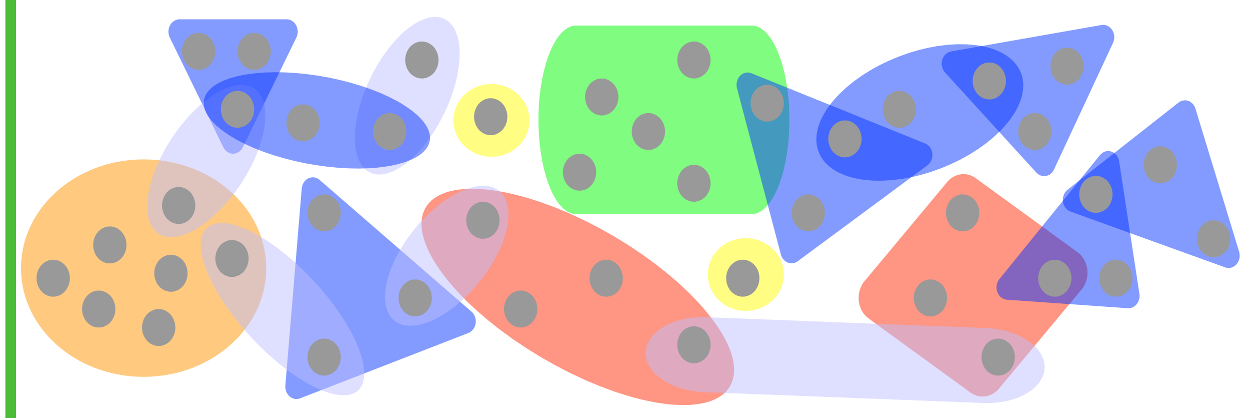
Simplicial complexes and Hypergraphs

Simplicial complexes



d-simplex = d+1 nodes
(all linked together)
0-simplex = node
1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

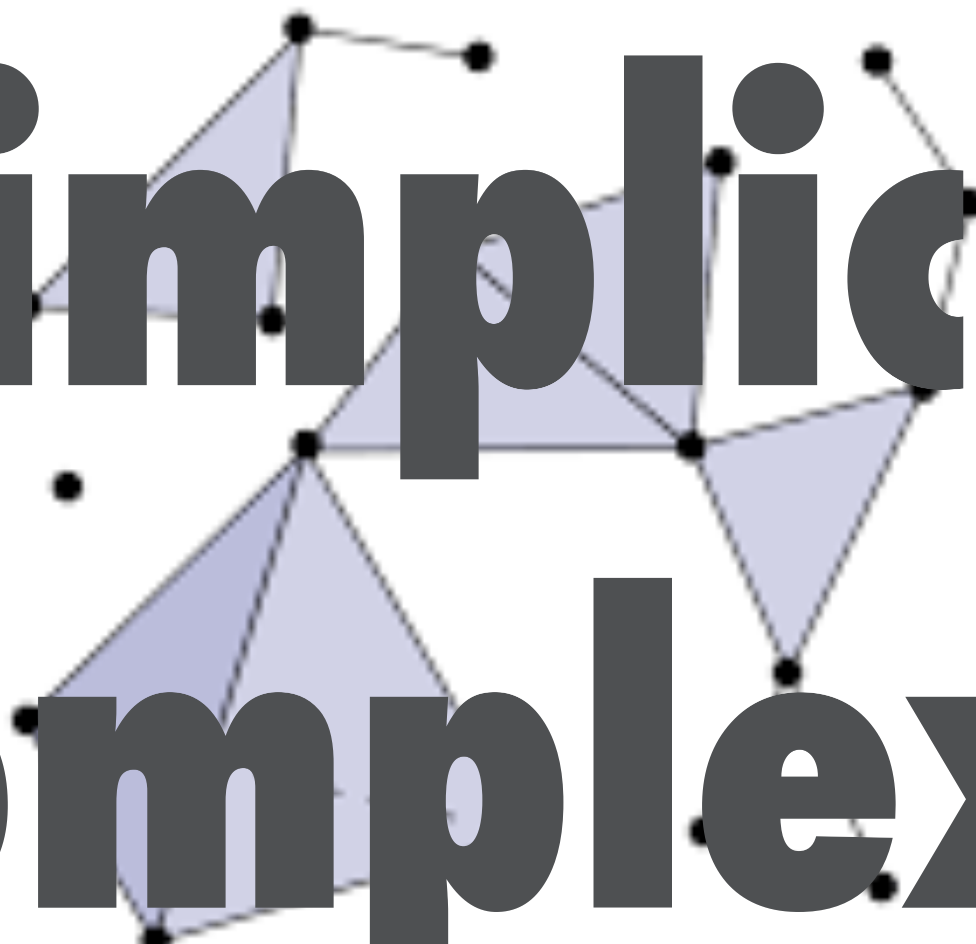
Hypergraphs



hyperedge = set of nodes

timoteo.carletti@unamur.be

Simplicial complexes



Global Synchronisation : simplicial complexes

nature COMMUNICATIONS

NATURE COMMUNICATIONS | (2021)12:1255 | https://doi.org/10.1038/s41467-021-21486-9 | www.nature.com/naturecommunications

ARTICLE

Check for updates

https://doi.org/10.1038/s41467-021-21486-9 OPEN

Stability of synchronization in simplicial complexes

L. V. Gambuzza[1,12], F. Di Patti[2,12], L. Gallo[3,4,12], S. Lepri[2], M. Romance[5], R. Criado[5], M. Frasca[1,6,13]✉, V. Latora[3,4,7,8,13]✉ & S. Boccaletti[2,9,10,11,13]✉

Pairwise interaction

(D+1)-body interaction

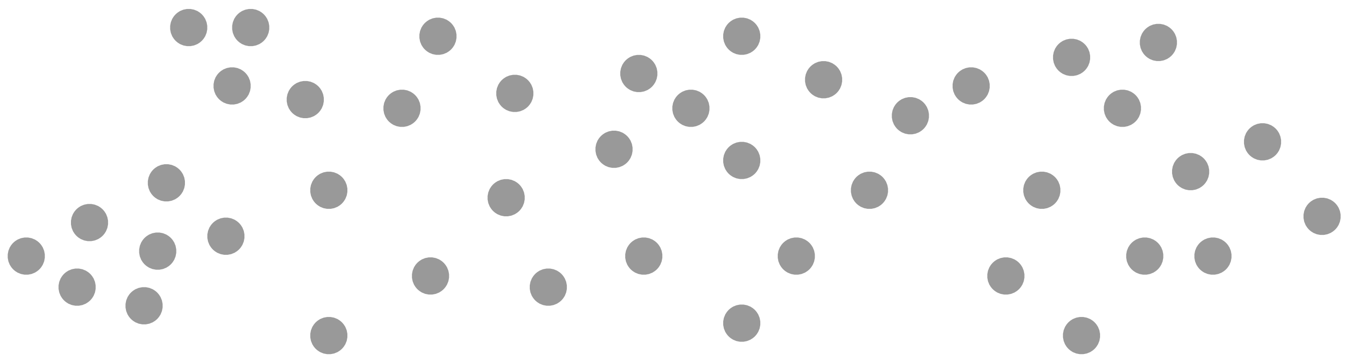
$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma_1 \sum_{j_1=1}^{N} a^{(1)}_{ij_1} \mathbf{g}^{(1)}(\mathbf{x}_i, \mathbf{x}_{j_1}) + \sigma_2 \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} a^{(2)}_{ij_1 j_2} \mathbf{g}^{(2)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \mathbf{x}_{j_2}) + \ldots + \sigma_D \sum_{j_1=1}^{N} \cdots \sum_{j_D=1}^{N} a^{(D)}_{ij_1 \ldots j_D} \mathbf{g}^{(D)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_D}),$$

Single node interaction

3-body interaction

Hypergraphs

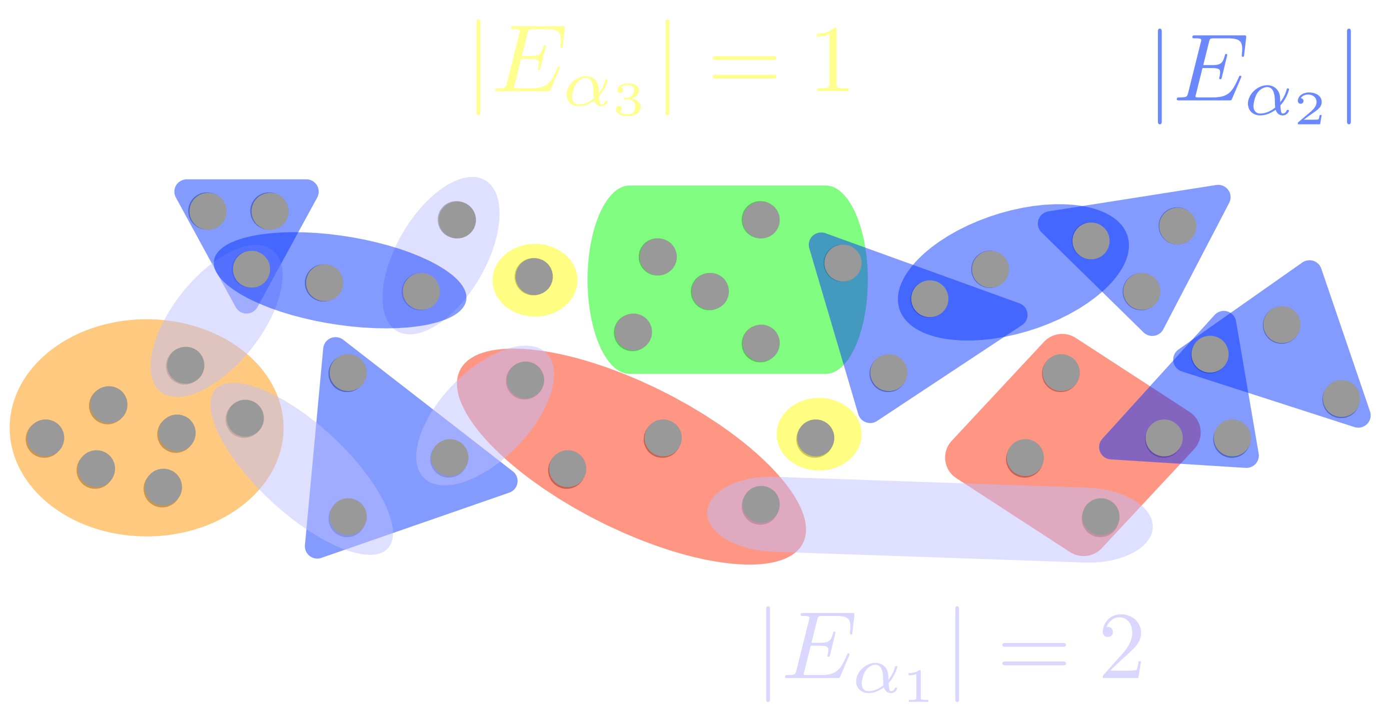
Hypergraphs. Some definitions.



ensemble of nodes

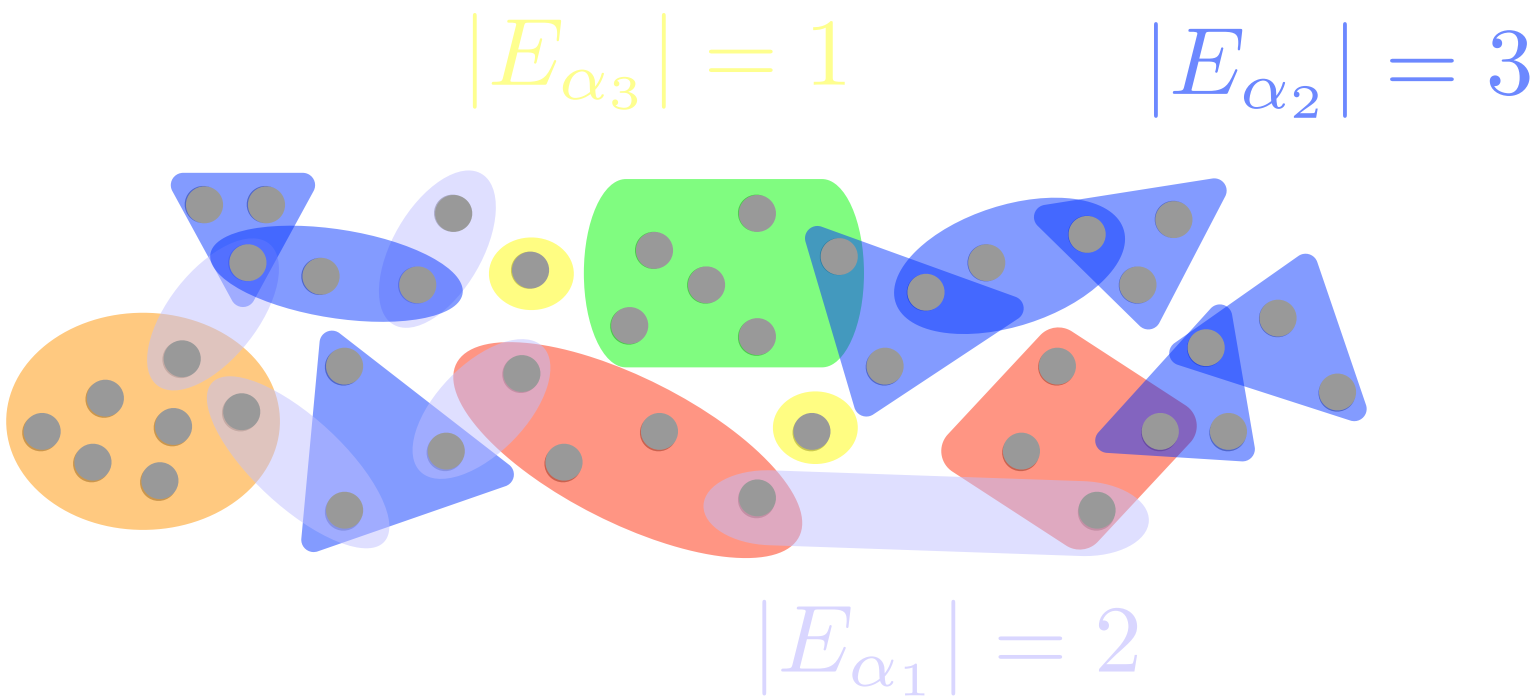
naxys Namur Center for Complex Systems

Hypergraphs. Some definitions.



ensemble of nodes
=
hyperedges

Hypergraphs. Some definitions.



ensemble of nodes
=
hyperedges

Incidence matrix

$$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$$

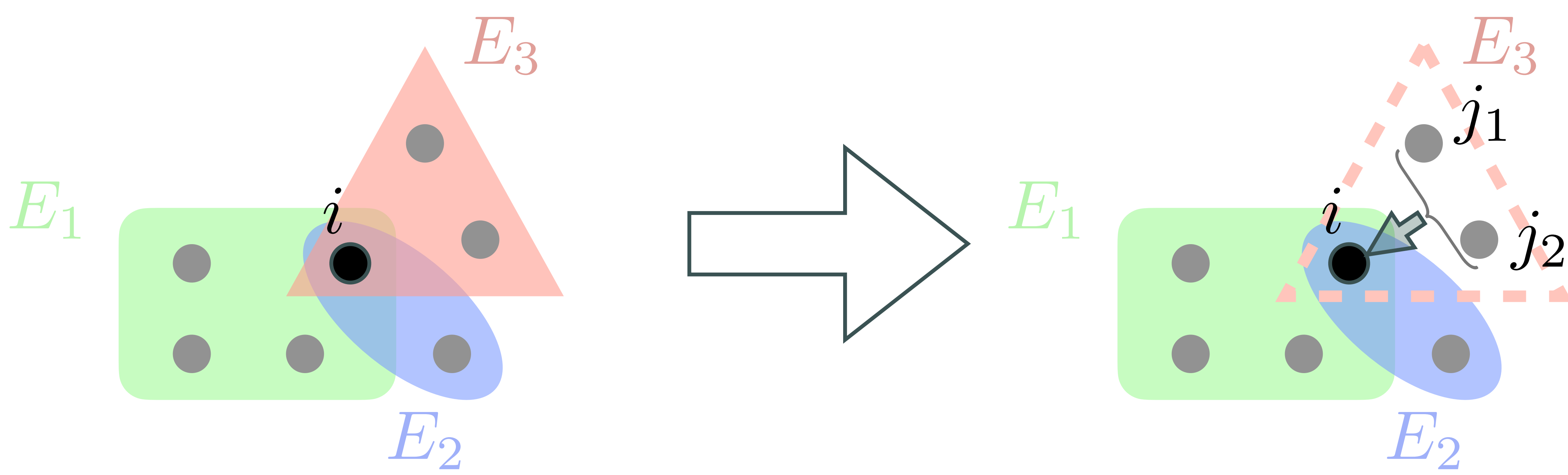
Hyperadjacency matrix

$$A = ee^{\mathrm{T}}$$

Hyperedge matrix

$$C = e^{\mathrm{T}}e$$

Hyperedge Mean Field



non-linearity

$$k_{ij}^H = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

Global Synchronisation : hypergraphs

IOP Publishing   J.Phys.Complex. 1 (2020) 035006 (16pp)   https://doi.org/10.1088/2632-072X/aba8e1

Journal of Physics: Complexity

OPEN ACCESS

PAPER

Dynamical systems on hypergraphs

RECEIVED
2 June 2020

REVISED
9 July 2020

ACCEPTED FOR PUBLICATION
23 July 2020

PUBLISHED
17 August 2020

Timoteo Carletti[1,4], **Duccio Fanelli**[2] **and Sara Nicoletti**[2,3]

[1] naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge, 8 B5000 Namur, Belgium
[2] Università degli Studi di Firenze, Dipartimento di Fisica e Astronomia, CSDC and INFN, via G. Sansone 1, 50019 Sesto Fiorentino, Italy
[3] Dipartimento di Ingegneria dell'Informazione, Università di Firenze, Via S. Marta 3, 50139 Florence, Italy
[4] Author to whom any correspondence should be addressed.

E-mail: timoteo.carletti@unamur.be

Keywords: hypergraphs, master stability function, synchronisation, Turing patterns, dynamical systems

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{\alpha,j} e_{i\alpha} e_{j\alpha} (C_{\alpha\,\alpha} - 1) \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right)$$

$$\tau = 1$$

$$= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j k^H_{ij} \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right) = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j \left(\delta_{ij} k^H_i - k^H_{ij}\right) \mathbf{G}(\mathbf{x}_j)$$

$$= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j L^H_{ij} \mathbf{G}(\mathbf{x}_j),$$

L^H_{ij} Higher-order Laplace matrix

Global Synchronisation : hypergraphs

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j L^H_{ij} \mathbf{G}(\mathbf{x}_j)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{F}(\mathbf{s})$

- $\sum_j L^H_{ij} = 0$ ⇒ $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

  is a solution of the coupled system

- Is it stable? $\delta \mathbf{x}_i = \mathbf{x}_i - \mathbf{s}$

Linearize : $$\frac{d\delta\mathbf{x}_i}{dt} = D\mathbf{F}(\mathbf{s}(t))\delta\mathbf{x}_i - \varepsilon \sum_j L^H_{ij} D\mathbf{G}(\mathbf{s}(t))\delta\mathbf{x}_j$$

timoteo.carletti@unamur.be

Global Synchronisation : hypergraphs

$$\frac{\mathrm{d}\delta\mathbf{x}_i}{\mathrm{d}t} = D\mathbf{F}(\mathbf{s}(t))\delta\mathbf{x}_i - \varepsilon\sum_j L_{ij}^H D\mathbf{G}(\mathbf{s}(t))\delta\mathbf{x}_j;$$

- $\mathbf{L}^H\phi^{(\alpha)} = \Lambda^{(\alpha)}\phi^{(\alpha)}$ $\quad \Lambda^{(1)} = 0 \quad \vec{\phi}^{(1)} = (1,\dots,1)^\top/\sqrt{n}$

  $\phi^{(\alpha)}\cdot\phi^{(\beta)} = \delta_{\alpha\beta} \quad \Lambda^{(\alpha)} > 0 \quad \alpha \geq 2$

- $\delta\mathbf{x}^{(i)} = \sum_\alpha \delta\mathbf{x}_\alpha \phi_i^{(\alpha)}$

- $\frac{d\delta\mathbf{x}_\alpha}{dt} = D\mathbf{F}(\mathbf{s}(t))\delta\mathbf{x}_\alpha - \varepsilon\Lambda^{(\alpha)} D\mathbf{G}(\mathbf{s}(t))\delta\mathbf{x}_\alpha$

Global Synchronisation : hypergraphs

$$\mathbf{J}_\alpha(t) = D\mathbf{F}(\mathbf{s}(t)) - \varepsilon\Lambda^{(\alpha)} D\mathbf{G}(\mathbf{s}(t))$$

- $\lambda_\alpha = \lambda(\Lambda^{(\alpha)})$ Lyapunov exponent / Floquet exponent/ Real part of eigenvalues

Master Stability Function

- if $\lambda_\alpha < 0 \;\; \forall\alpha \geq 2$ then <u>synchronisation</u>
- if $\exists\alpha \geq 2 \;\; \lambda_\alpha > 0$ then <u>desynchronisation</u>

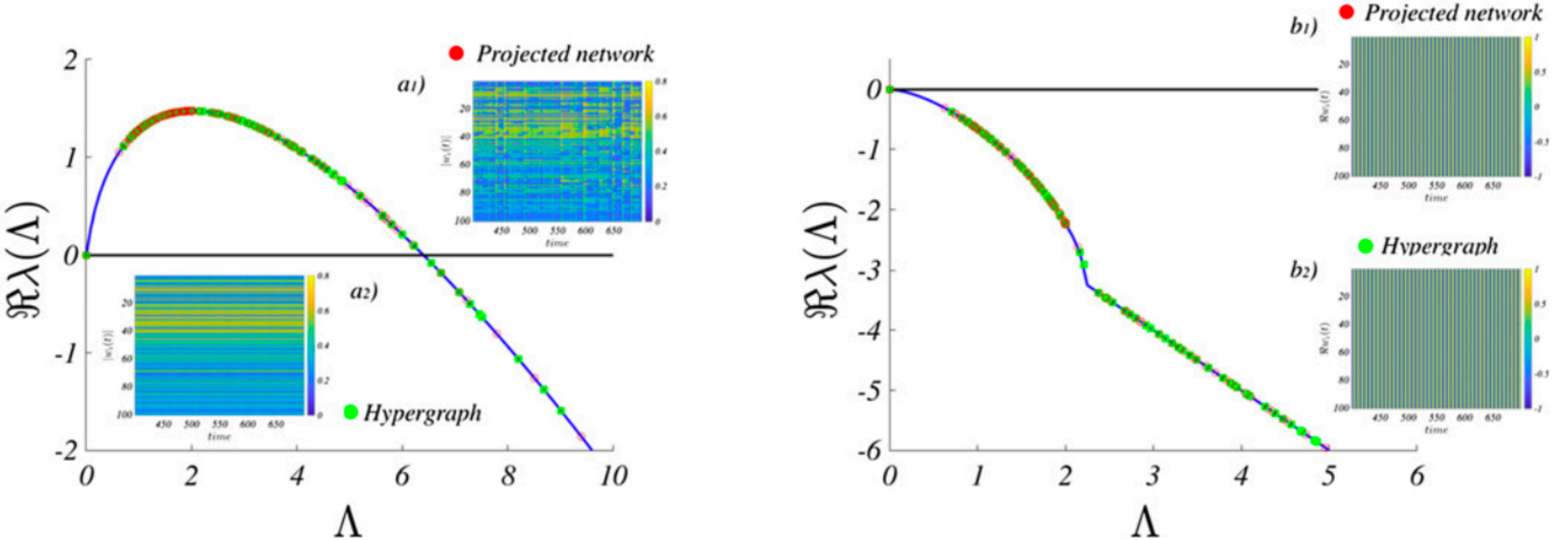
Stuart - Landau oscillator

$$\frac{\mathrm{d}}{\mathrm{d}t}W_j = W_j - (1 + \mathrm{i}c_2)|W_j|^2 W_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}W_j = W_j - (1 + \mathrm{i}c_2)|W_j|^2 W_j - (1 + \mathrm{i}c_1)K\sum_k L^H_{jk} W_k$$

a) $K = 1, c_1 = 0.5, c_2 = -10$

b) $K = 1, c_1 = 1, c_2 = -0.9$



Global Topological Synchronisation

PHYSICAL REVIEW LETTERS **130,** 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

Timoteo Carletti,[1] Lorenzo Giambagli,[1,2] and Ginestra Bianconi[3,4]

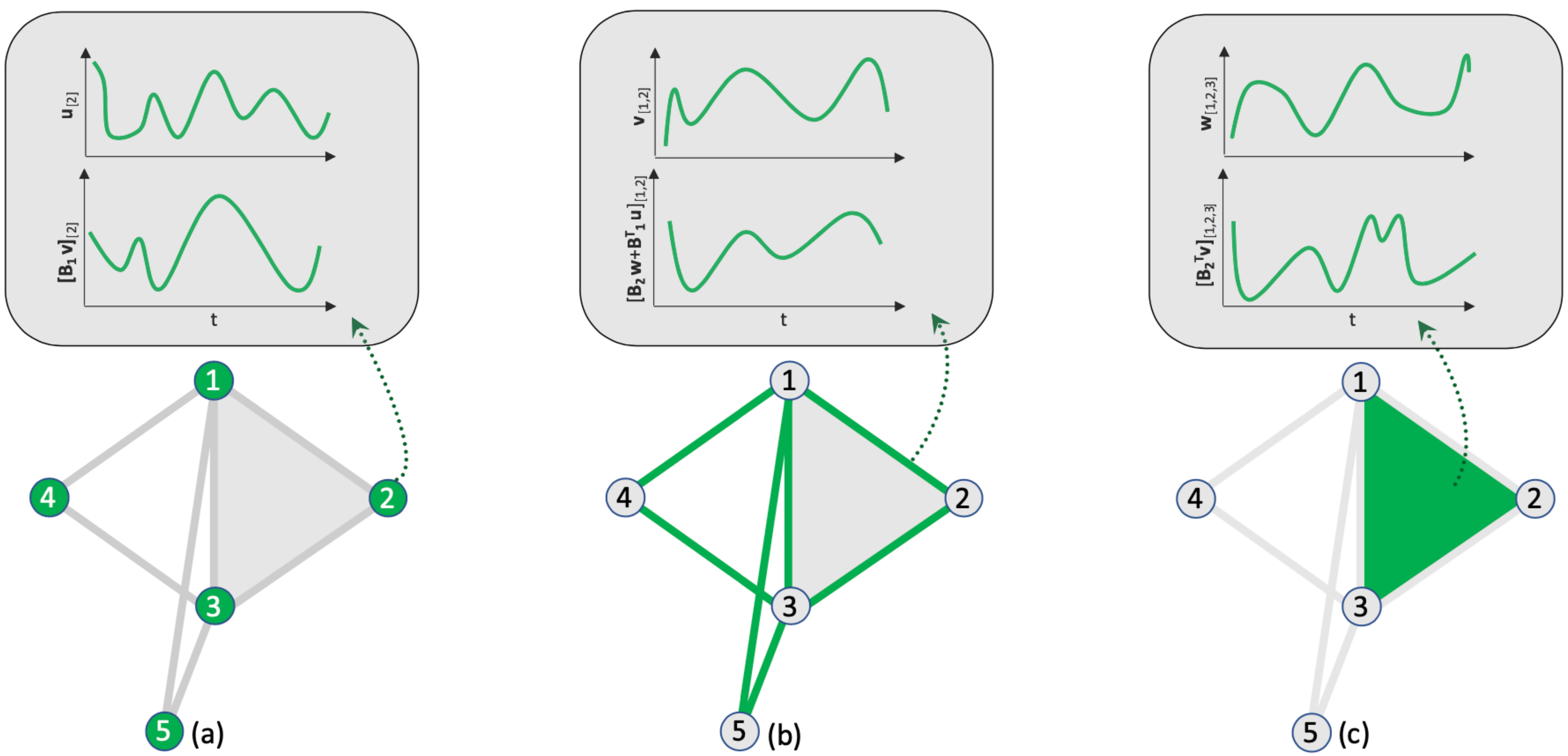
[1]*Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, Rue Grafé 2, B5000 Namur, Belgium*

[2]*Department of Physics and Astronomy, University of Florence, INFN and CSDC, 50019 Sesto Fiorentino, Italy*

[3]*School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, United Kingdom*

[4]*The Alan Turing Institute, 96 Euston Road, London, NW1 2DB, United Kingdom*

(Received 31 August 2022; revised 17 February 2023; accepted 11 April 2023; published 3 May 2023)



Simplicial complex

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

Incidence matrix

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

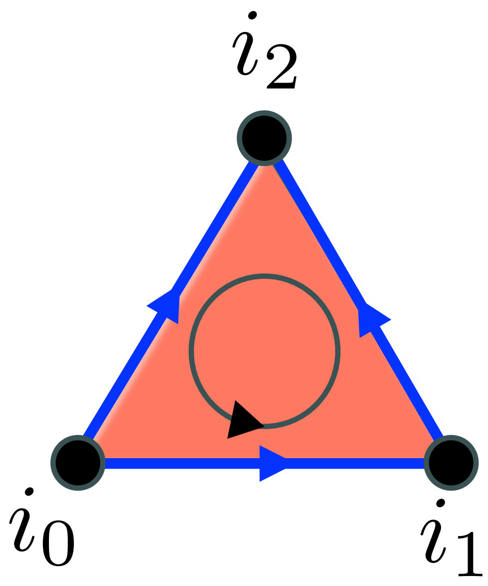
$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

Simplicial complex: an example

$k = 2$ Three nodes, hence a triangle $\sigma^{(2)} = [i_0, i_1, i_2]$



$$\sigma_1^{(1)} = [i_0, i_1] \qquad \sigma_2^{(1)} = [i_1, i_2] \qquad \sigma_3^{(1)} = [i_0, i_2]$$

Incidence matrices

$$\mathbf{B}_1 \in M^{N_0 \times N_1}$$

$$\mathbf{B}_2 \in M^{N_1 \times N_2}$$

$$\mathbf{B}_1(\sigma_i^{(0)}, \sigma_j^{(1)}) = \begin{array}{c} \\ i_0 \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{ccc} [i_0, i_1] & [i_1, i_2] & [i_0, i_2] \end{array} \\ \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{B}_2(\sigma_i^{(1)}, \sigma_j^{(2)}) = \begin{array}{c} \\ [i_0, i_1] \\ [i_1, i_2] \\ [i_0, i_2] \end{array} \begin{array}{c} [i_0, i_1, i_2] \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array}$$

Global Topological Synchronisation

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{x} : C_k \to \mathbb{R}^d$$ k-cochain

$$\mathbf{x}(-\sigma_i^{(k)}) = -\mathbf{x}(\sigma_i^{(k)})$$

$$\mathbf{x}_i = \mathbf{x}(\sigma_i^{(k)}) = (x_i^1, \dots, x_i^d)$$

Dynamical system on a simplex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i)$$

$$\mathbf{f}(-\mathbf{x}_i) = -\mathbf{f}(\mathbf{x}_i)$$

Global Topological Synchronisation

Dynamical system on a simplicial complex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N_k} L_k(i,j)\mathbf{h}(\mathbf{x}_j) \qquad \mathbf{h}(-\mathbf{x}_i) = -\mathbf{h}(\mathbf{x}_i)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

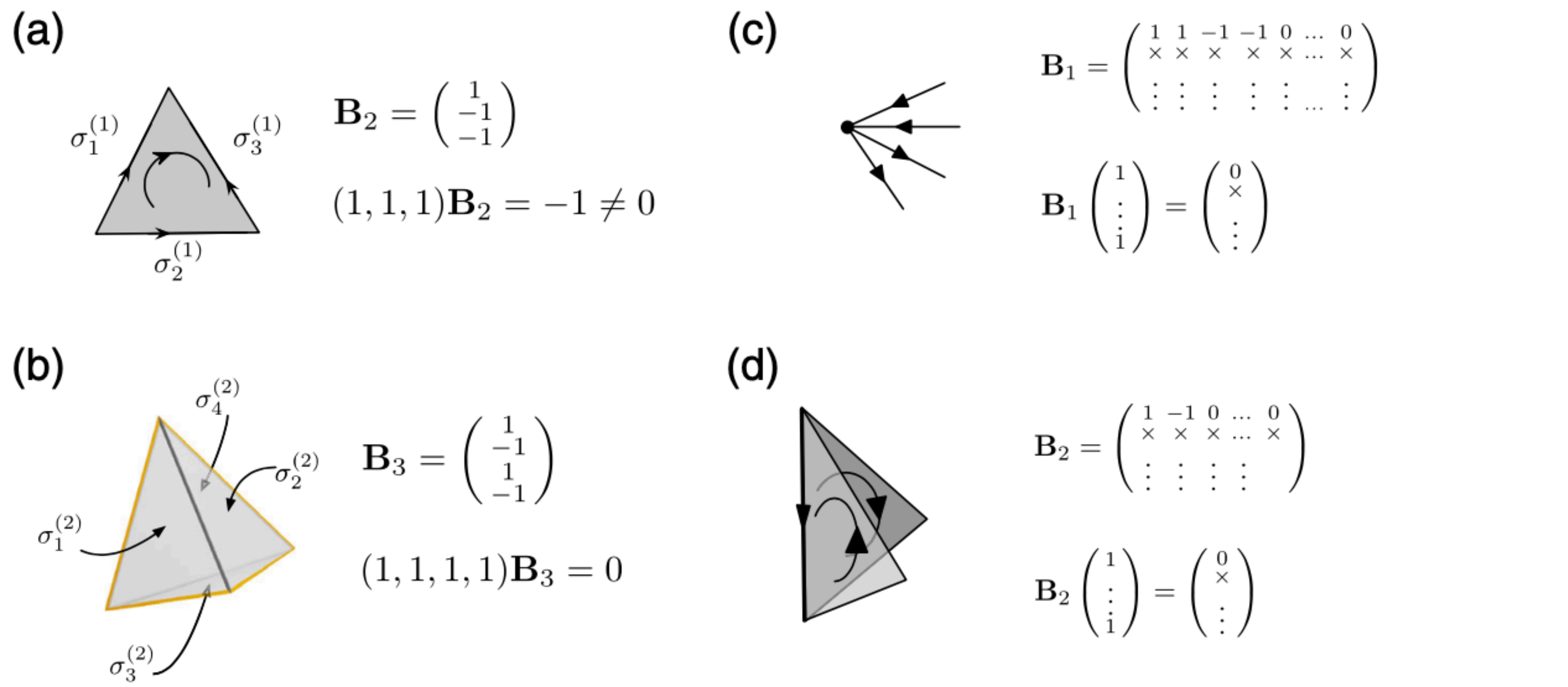
Synchronisation : $\mathbf{x}_i(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, N_k$

$$\left.\frac{d\mathbf{x}_i}{dt}\right|_{\mathbf{x}_i=\mathbf{s}} = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}_j)\Big|_{\mathbf{x}_i=\mathbf{s}} \overset{?}{\neq} 0$$

Global Topological Synchronisation

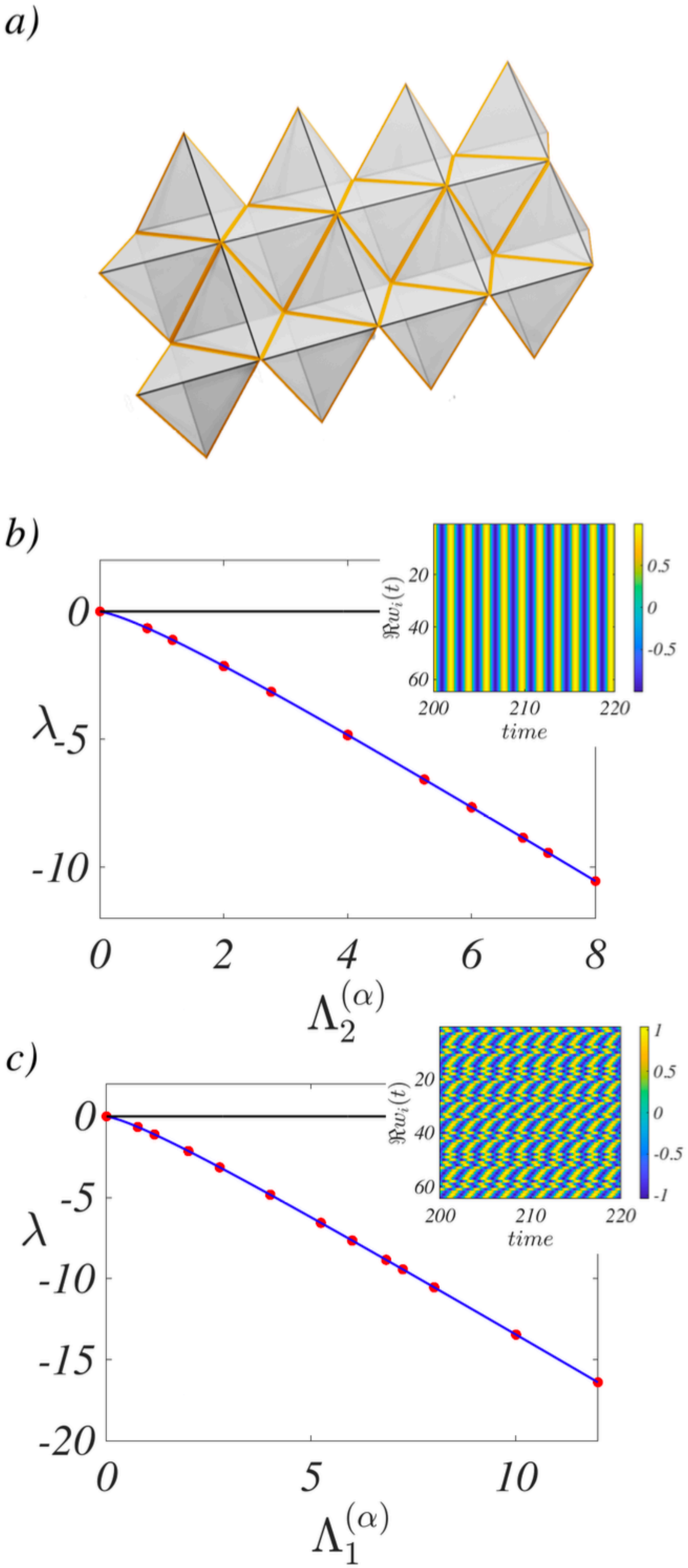
Necessary condition $\mathbf{L}_k u = 0 \quad \Leftrightarrow \quad \mathbf{B}_k u = 0$ and $\mathbf{B}_{k+1}^\top u = 0$

odd dim = non global synch



even dim = global synch if <u>balanced</u>

Global Topological Synchronisation : Stuart-Landau



global synch
for faces (k=2)

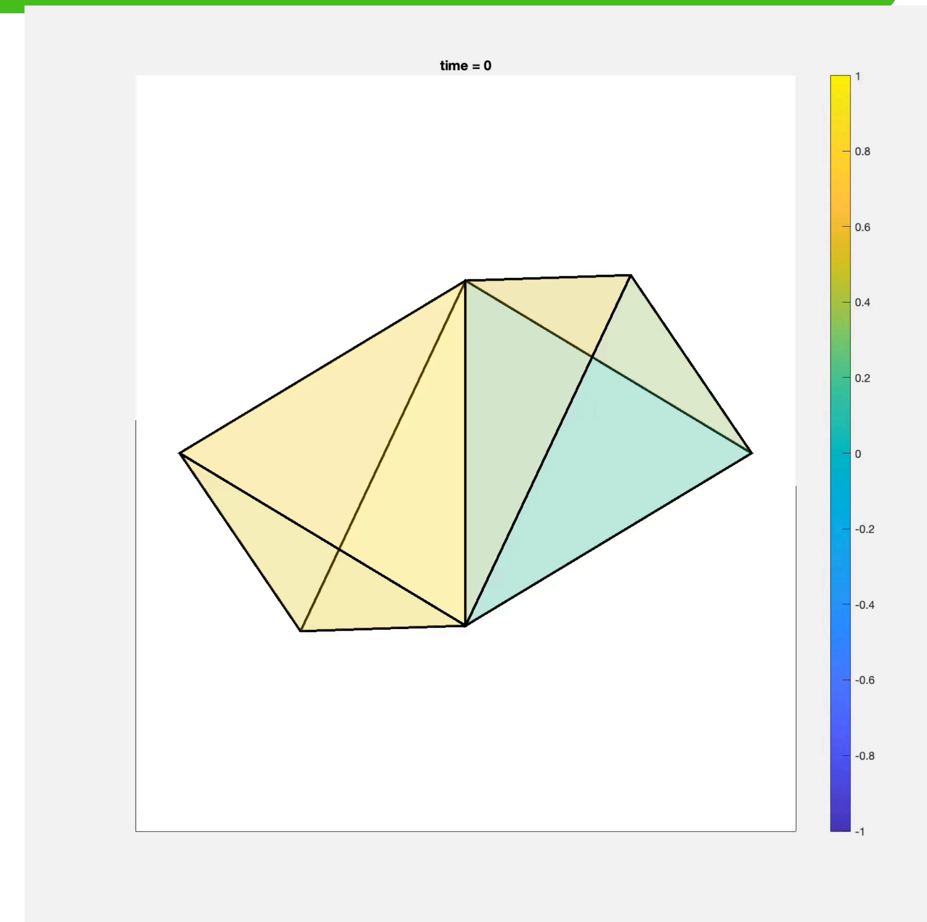
$$\mathbf{B}_2(1,\ldots,1)^\top = 0$$

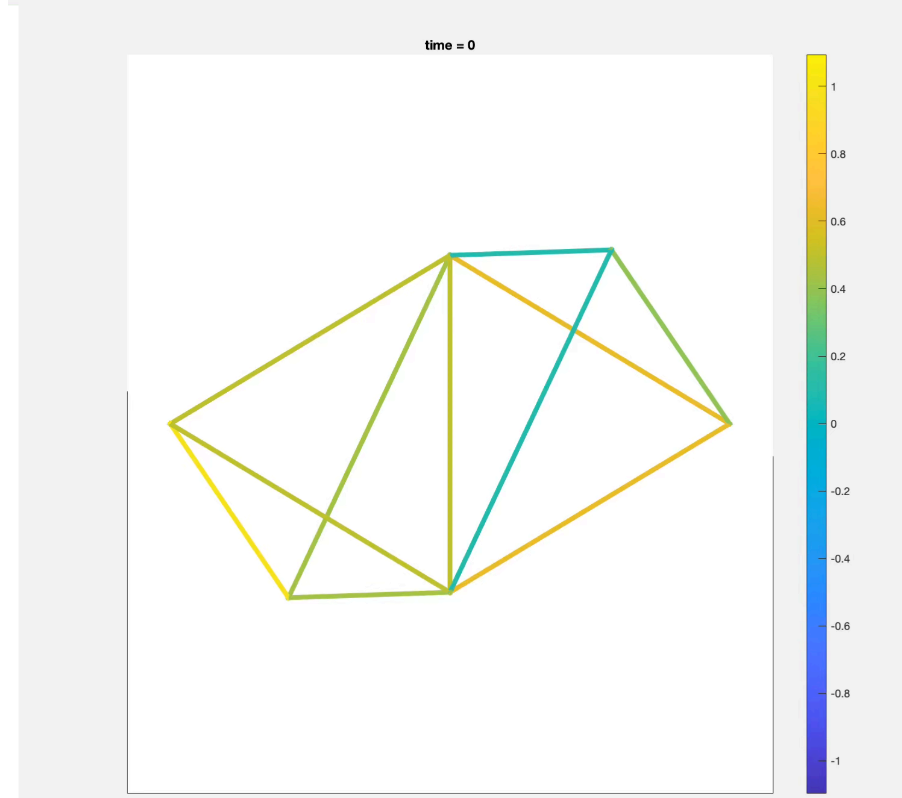
$$\mathbf{B}_3^\top(1,\ldots,1)^\top = 0$$

no global synch
for links (k=1)

$$\mathbf{B}_1(1,\ldots,1)^\top = 0$$

$$\mathbf{B}_2^\top(1,\ldots,1)^\top \neq 0$$



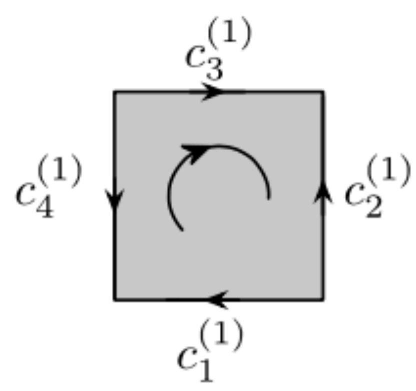


timoteo.carletti@unamur.be

Global Topological Synchronisation: cell complexes

The topological obstruction
does not exist for cell complexes

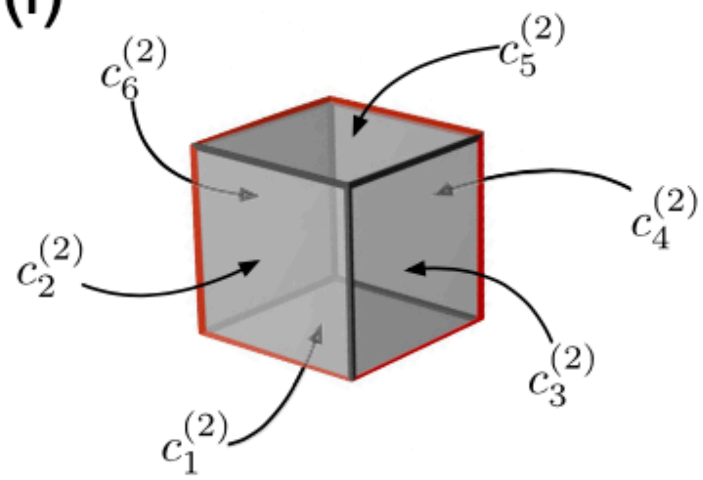
(e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

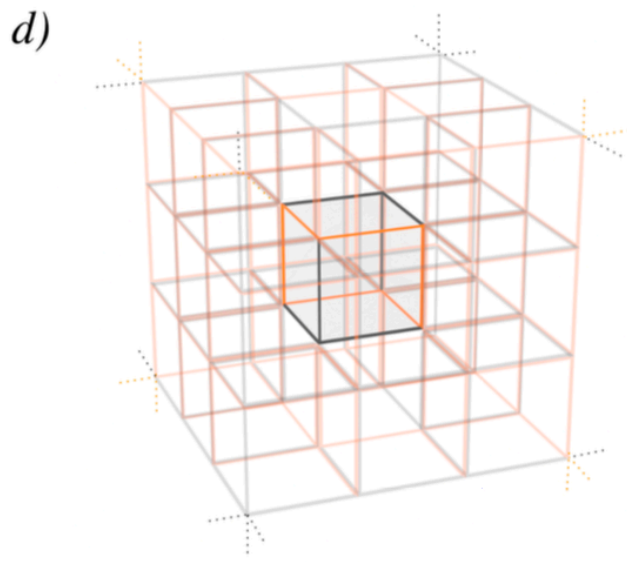
$$(1,1,1,1)\mathbf{B}_2 = 0$$

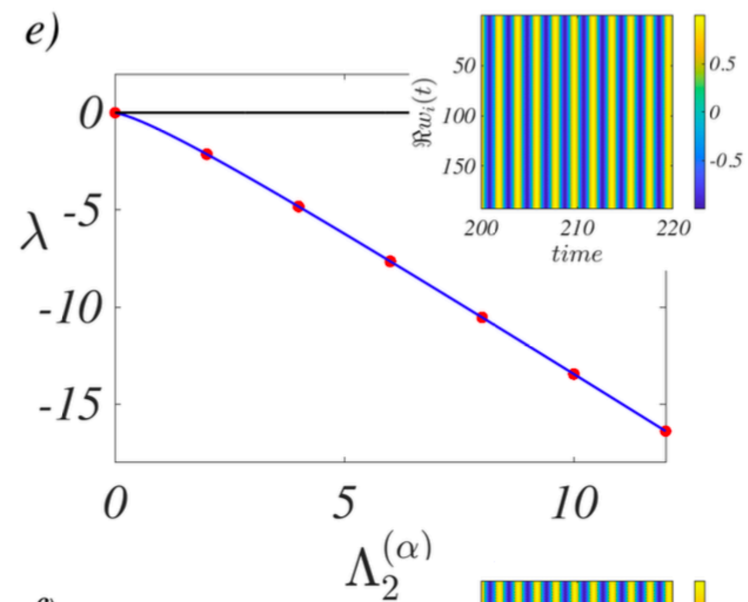
(f)



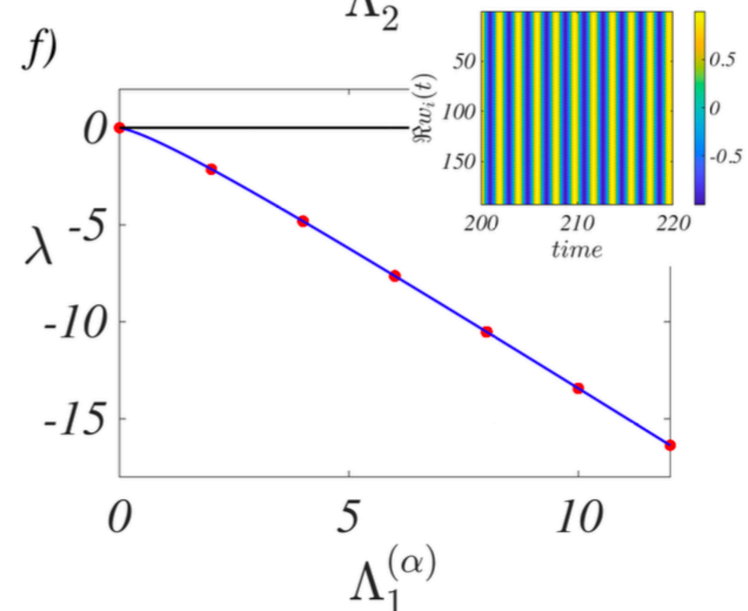
$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$





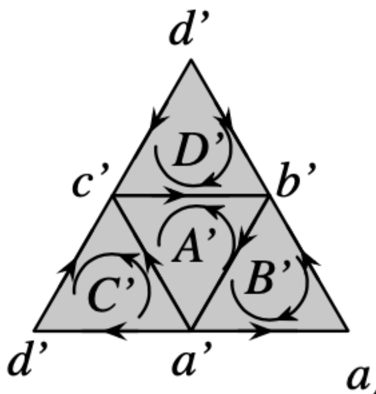
global synch
for faces



global synch
for links

The "waffle" 3-simplicial complex

a)



A' = [a'b'c']

B' = [a'b'd']

C' = [a'c'd']

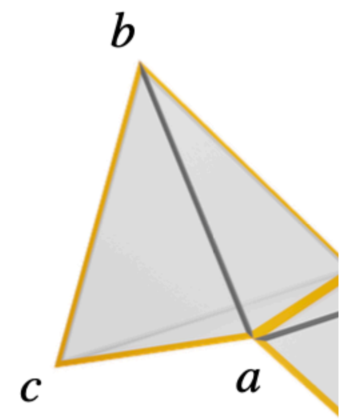
D' = [b'c'd']

b)

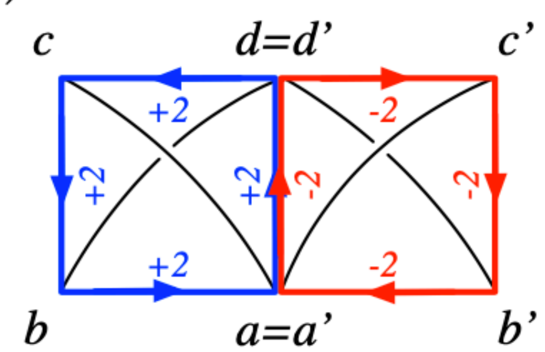
$B_2' =$

	A'	B'	C'	D'
a'd'	0	-1	-1	0
a'c'	-1	0	1	0
b'a'	-1	-1	0	0
c'b'	-1	0	0	-1
d'b'	0	-1	0	1
d'c'	0	0	[illegible]	[illegible]

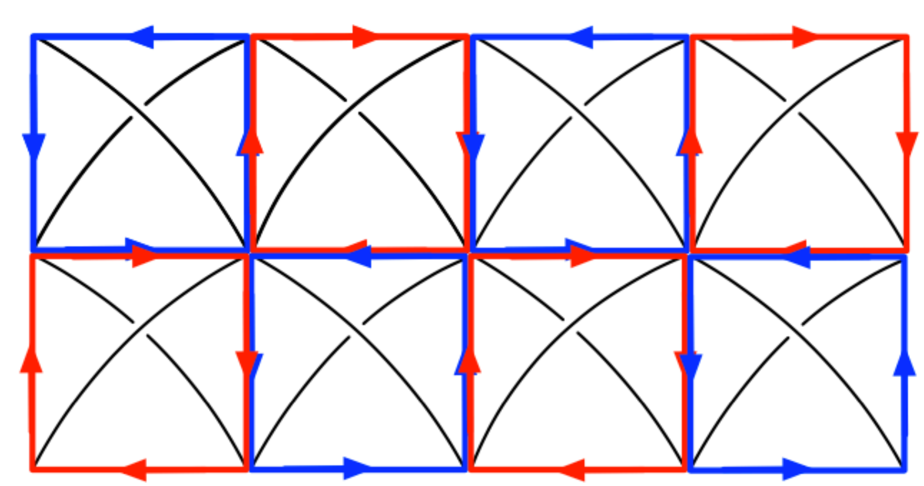
c)



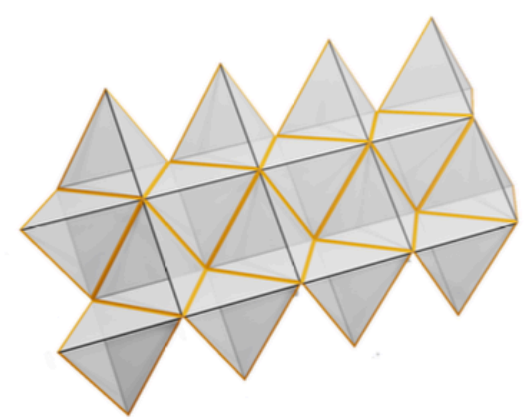
a)



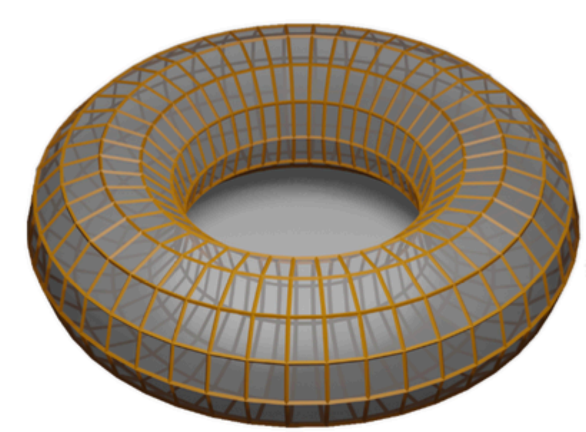
b)



c)



d)



October 12th, 2023

Bernoulli Society
for Mathematical Statistics
and Probability

Timoteo Carletti

Thank you

Any questions?

Department of mathematics
UNamur

Namur Institute for Complex

UNIVERSITÉ DE NAMUR