27th September 2023

25-28 Sept , 2023

CAMEROON'S FIRST SCHOOL ON NON-LINEAR DYNAMICS AND COMPLEX SYSTEMS ON HYPERGRAPHS

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Global Topological Synchronisation on Simplicial Complexes

Namur Institute for Complex Systems

Department of mathematics

UNIVERSITÉ DE NAMUR

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Ginestra Bianconi

Lorenzo Giambagli

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PHYSICAL REVIEW LETTERS **130,** 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

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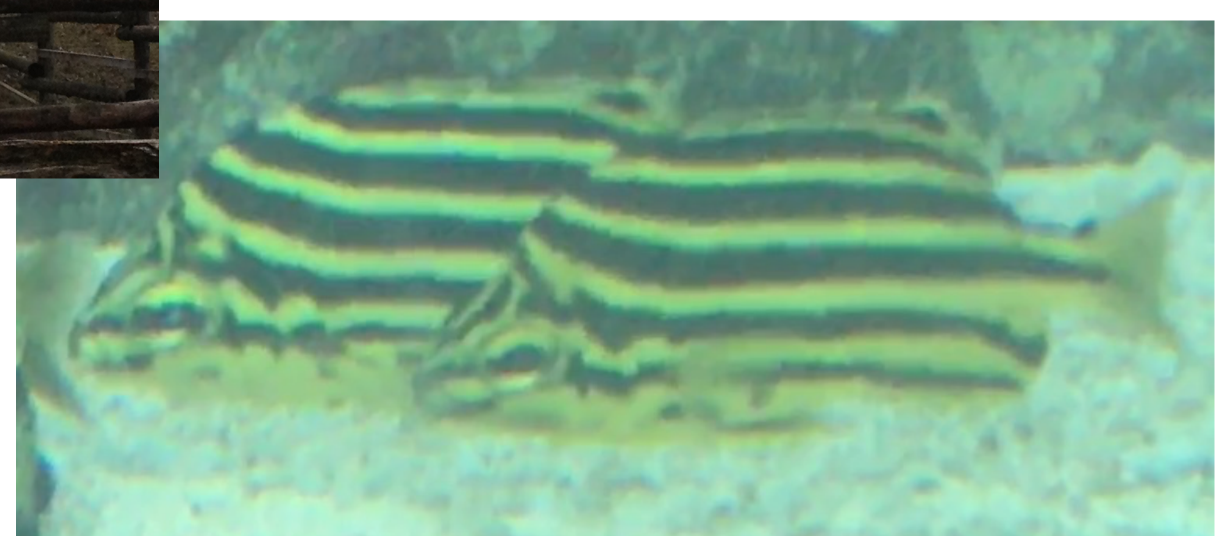
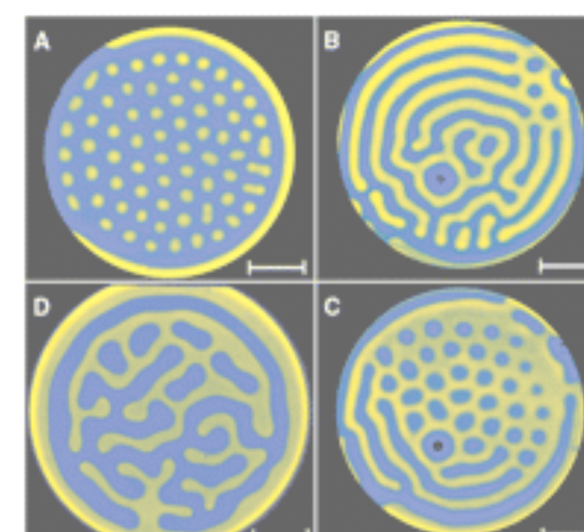
Topological signals, i.e., dynamical variables defined on nodes, links, triangles, etc. of higher-order networks, are attracting increasing attention. However, the investigation of their collective phenomena is only at its infancy. Here we combine topology and nonlinear dynamics to determine the conditions for global synchronization of topological signals defined on simplicial or cell complexes. On simplicial complexes we show that topological obstruction impedes odd dimensional signals to globally synchronize. On the other hand, we show that cell complexes can overcome topological obstruction and in some structures signals of any dimension can achieve global synchronization.

DOI: 10.1103/PhysRevLett.130.187401

Order from disorder …





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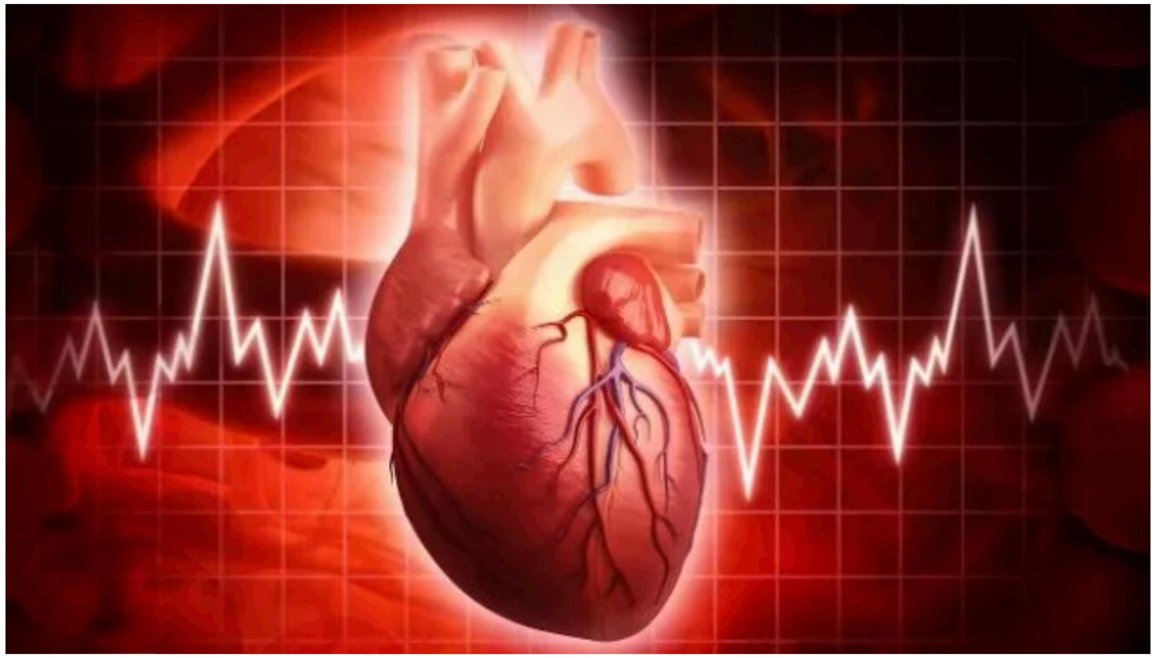
Synchronisation



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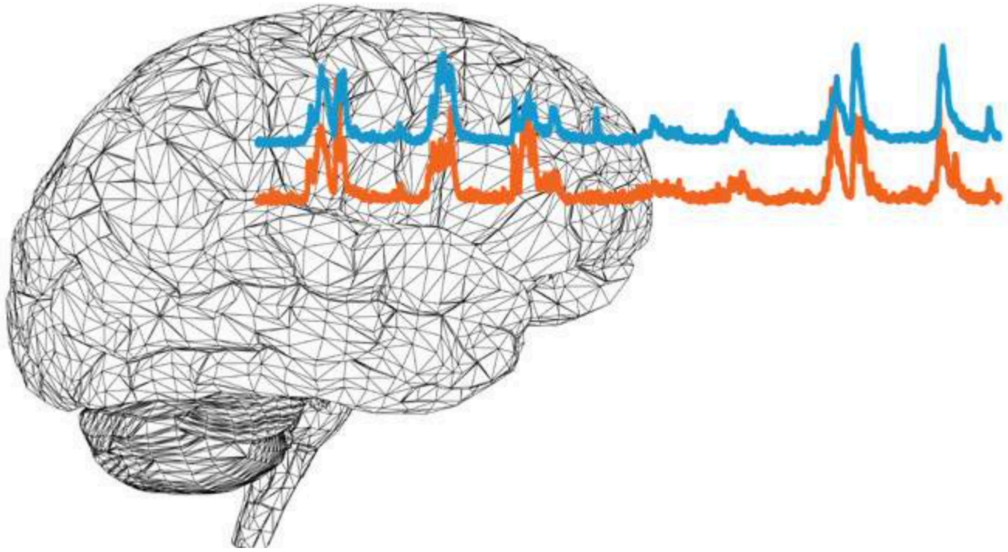


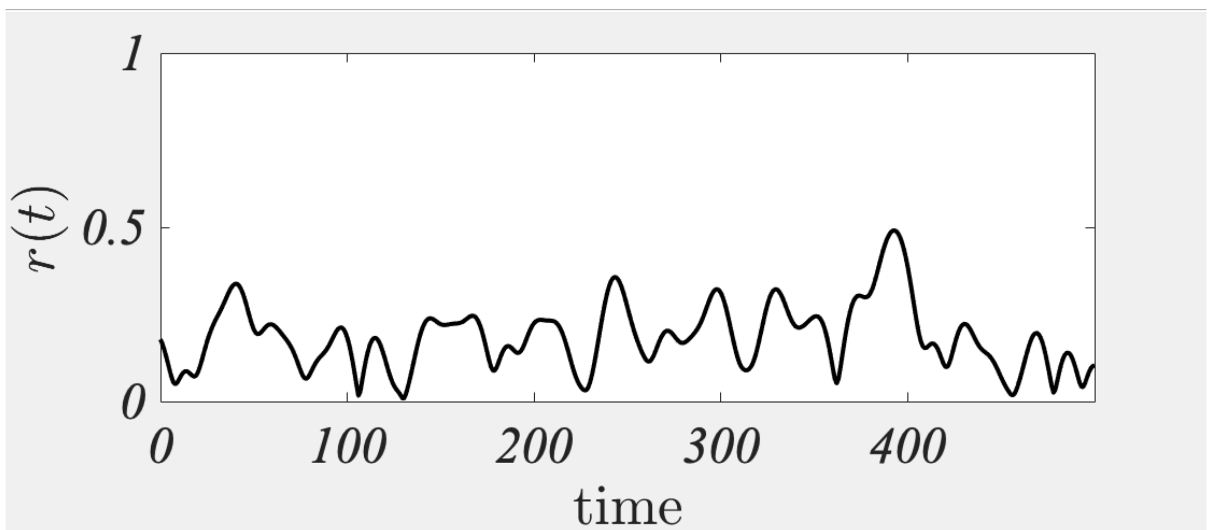
Attention: Kuramoto model

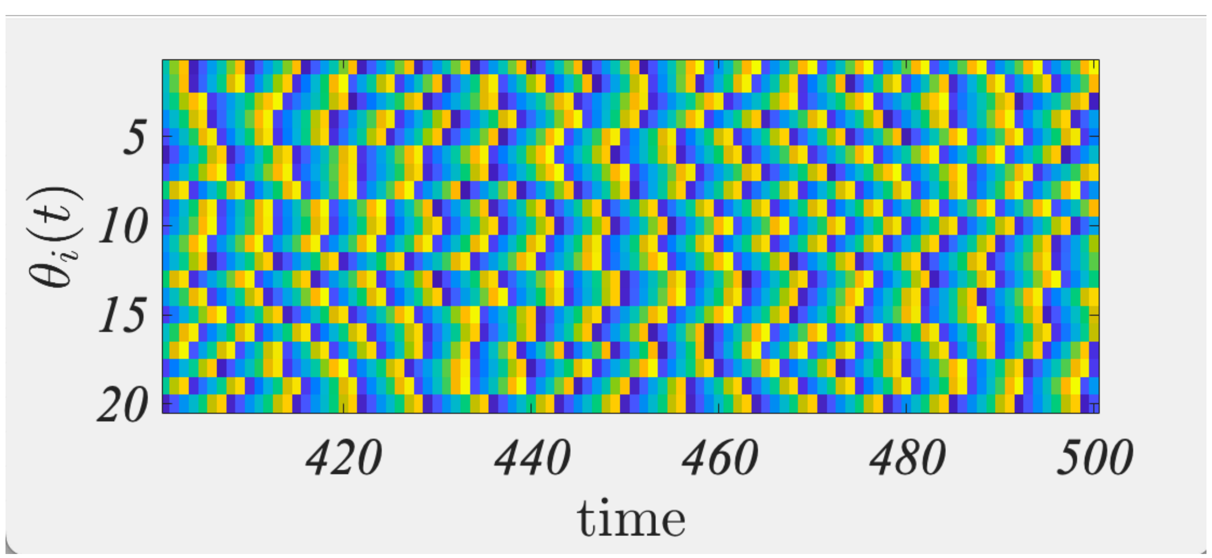
order parameter

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

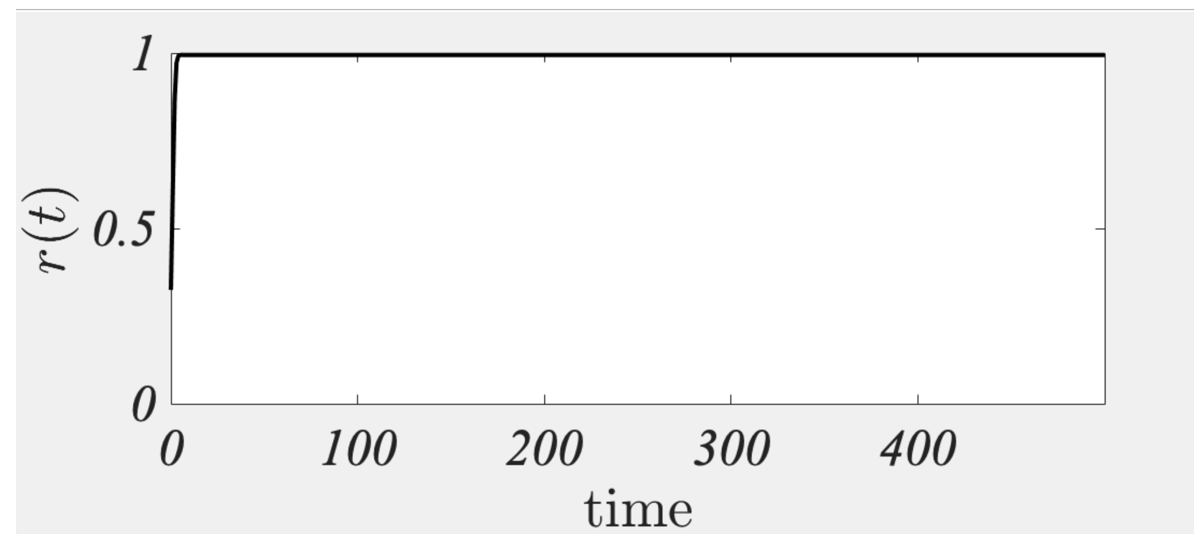
$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

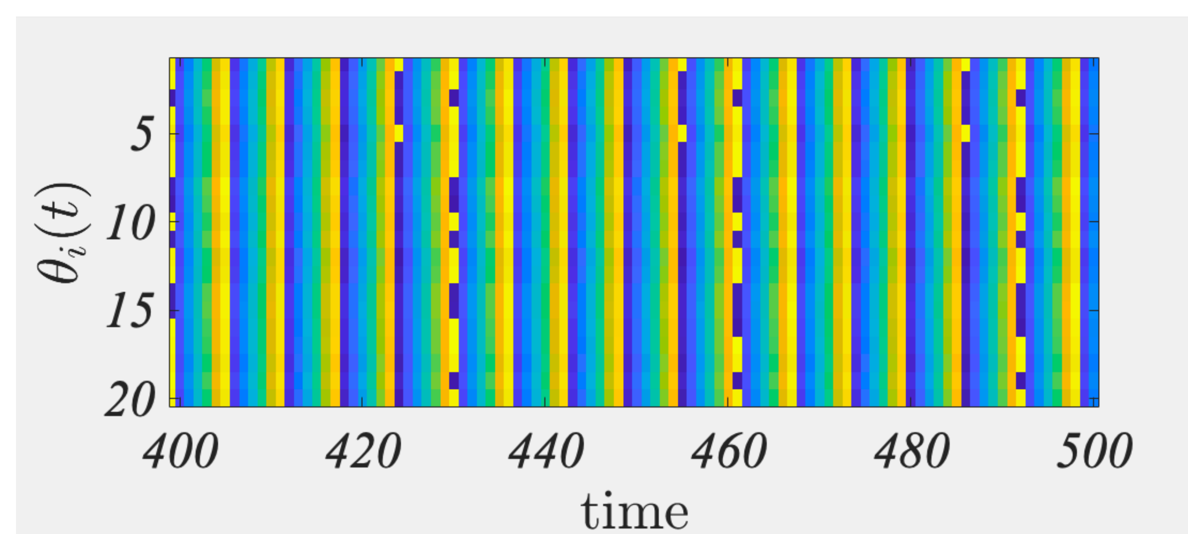
$K = 0.01$





$K = 1.0$





Global Synchronisation: Pecora et al.

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

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Generic behavior of master-stability functions in coupled nonlinear dynamical systems

Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]
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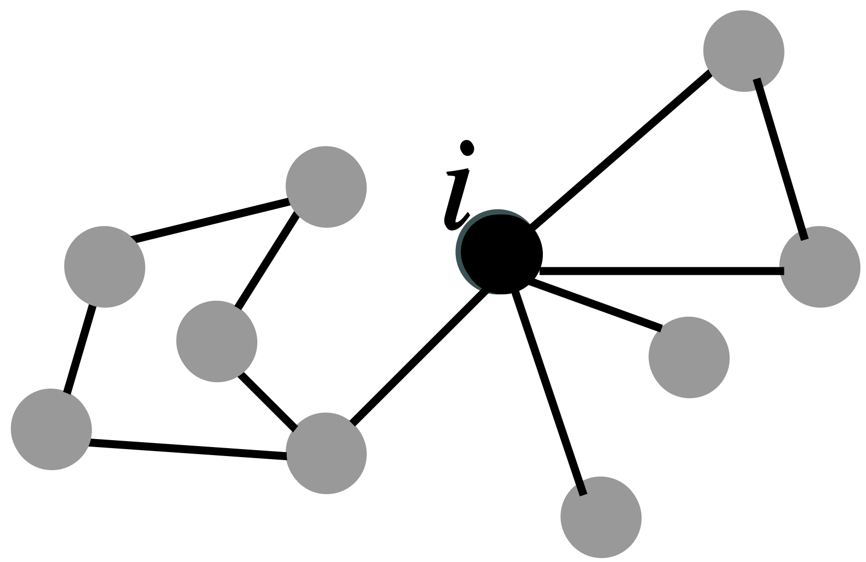
Global Synchronisation on networks

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} A_{ij}\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

$$\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \mathbf{h}(\mathbf{x}^{(j)}) - \mathbf{h}(\mathbf{x}^{(i)})$$

$$= \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})$$

Diffusive-like coupling

Global Synchronisation on networks

Reference orbit $\mathbf{s}(t)$ solution of $$\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$$

Synchronisation : $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

Does the <u>whole</u> system admit such (spatially) <u>homogeneous</u> solution?

- $$\left.\frac{d\mathbf{x}^{(i)}}{dt}\right|_{\mathbf{x}^{(i)}=\mathbf{s}} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = 0$$

$$\mathbf{L}\mathbf{u} = 0 \qquad \mathbf{u} = (1, \dots, 1)^\top$$

Laplace matrix

Global Synchronisation on networks

Is $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$ stable?

- $\delta\mathbf{x}^{(i)}(t) = \mathbf{x}^{(i)}(t) - \mathbf{s}(t) \quad \forall i = 1, \dots, n$

- $$\frac{d\delta\mathbf{x}^{(i)}}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}^{(i)} + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}^{(j)}$$

Time dependent linear system

Global Synchronisation on networks

- $\mathbf{L}\phi^{(\alpha)} = \Lambda^{(\alpha)}\phi^{(\alpha)} \qquad \phi^{(\alpha)} \cdot \phi^{(\beta)} = \delta_{\alpha\beta} \qquad \Lambda^{(1)} = 0 \quad \Lambda^{(\alpha)} < 0$

- $\delta\mathbf{x}^{(i)} = \sum_{\alpha} \delta\mathbf{x}_\alpha \phi_i^{(\alpha)}$

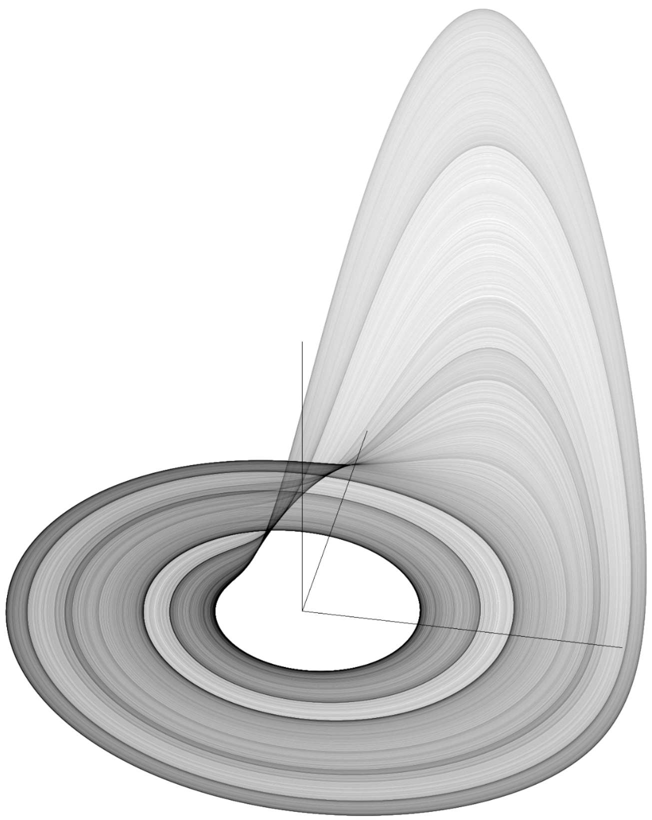
- $\frac{d\delta\mathbf{x}_\alpha}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}_\alpha + \sigma\Lambda^{(\alpha)}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}_\alpha := \mathbf{J}_\alpha(\mathbf{s}(t))\delta\mathbf{x}_\alpha$

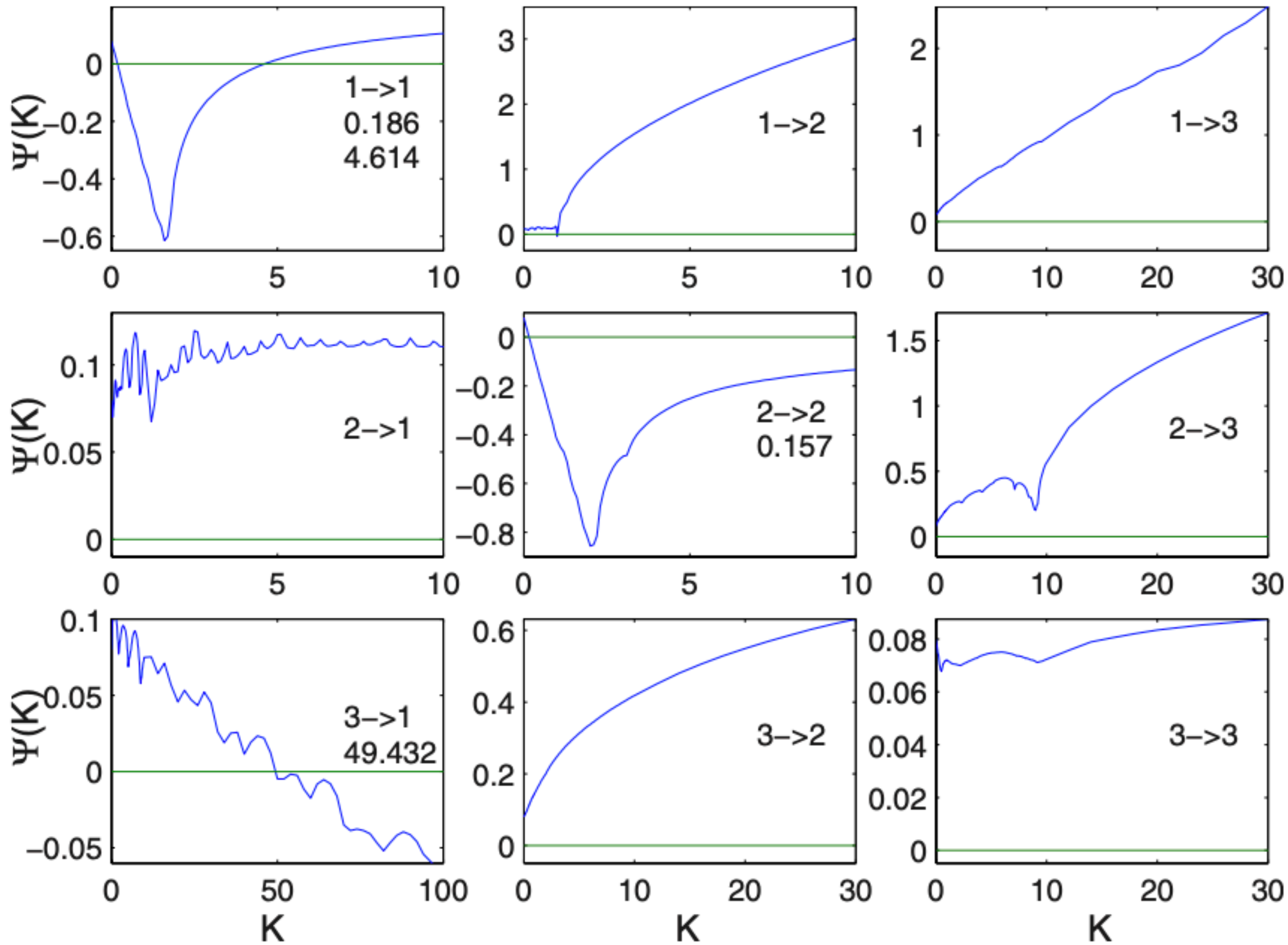
$\lambda(\Lambda^{(\alpha)})$ Master Stability Function = largest Lyapunov exponent of $\mathbf{J}_\alpha(\mathbf{s}(t))$

(function of $\Lambda^{(\alpha)}$)

Global Synchronisation: Rössler oscillator

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + \alpha y, \\ \dot{z} = \beta + (x - \gamma) z, \end{cases}$$





$$K = -\sigma \Lambda^{(\alpha)}$$

PHYSICAL REVIEW E **80**, 036204 (2009)

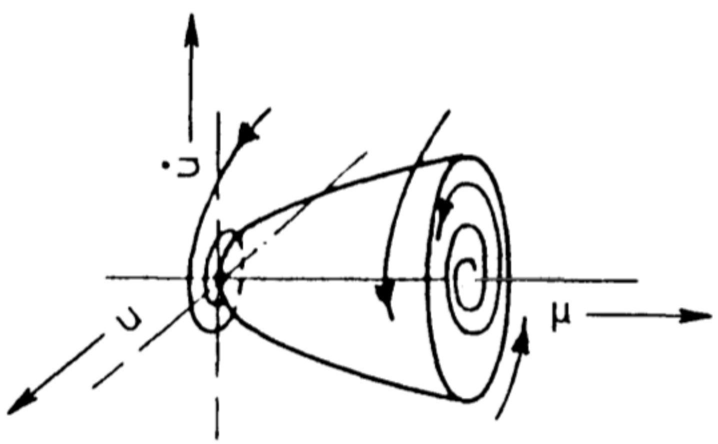
Generic behavior of master-stability functions in coupled nonlinear dynamical systems

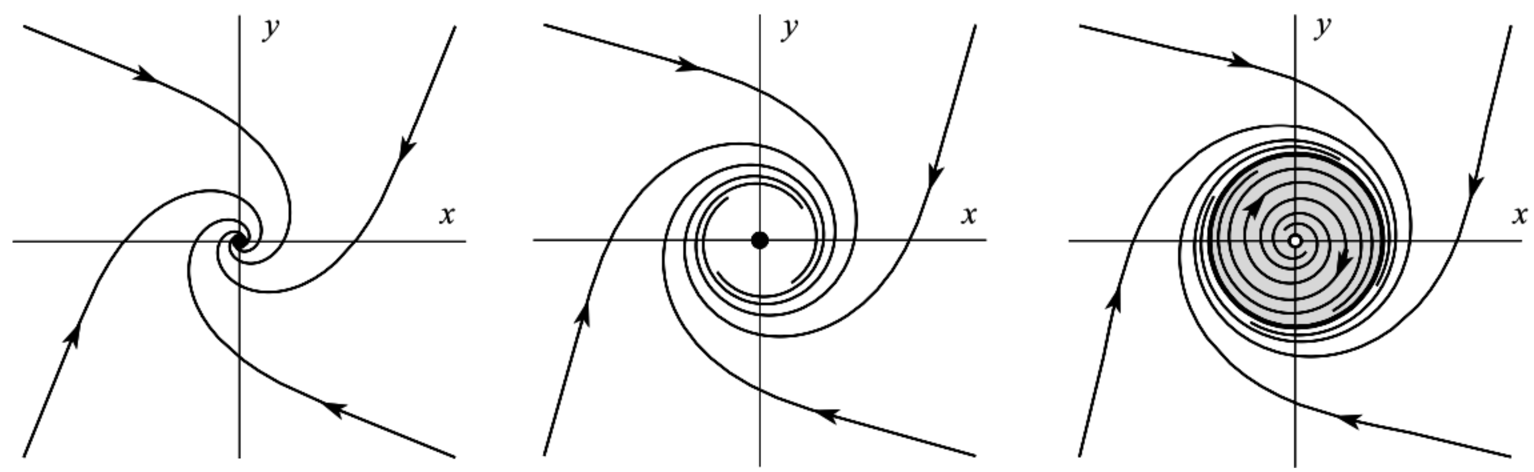
Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

Hopf Bifurcation

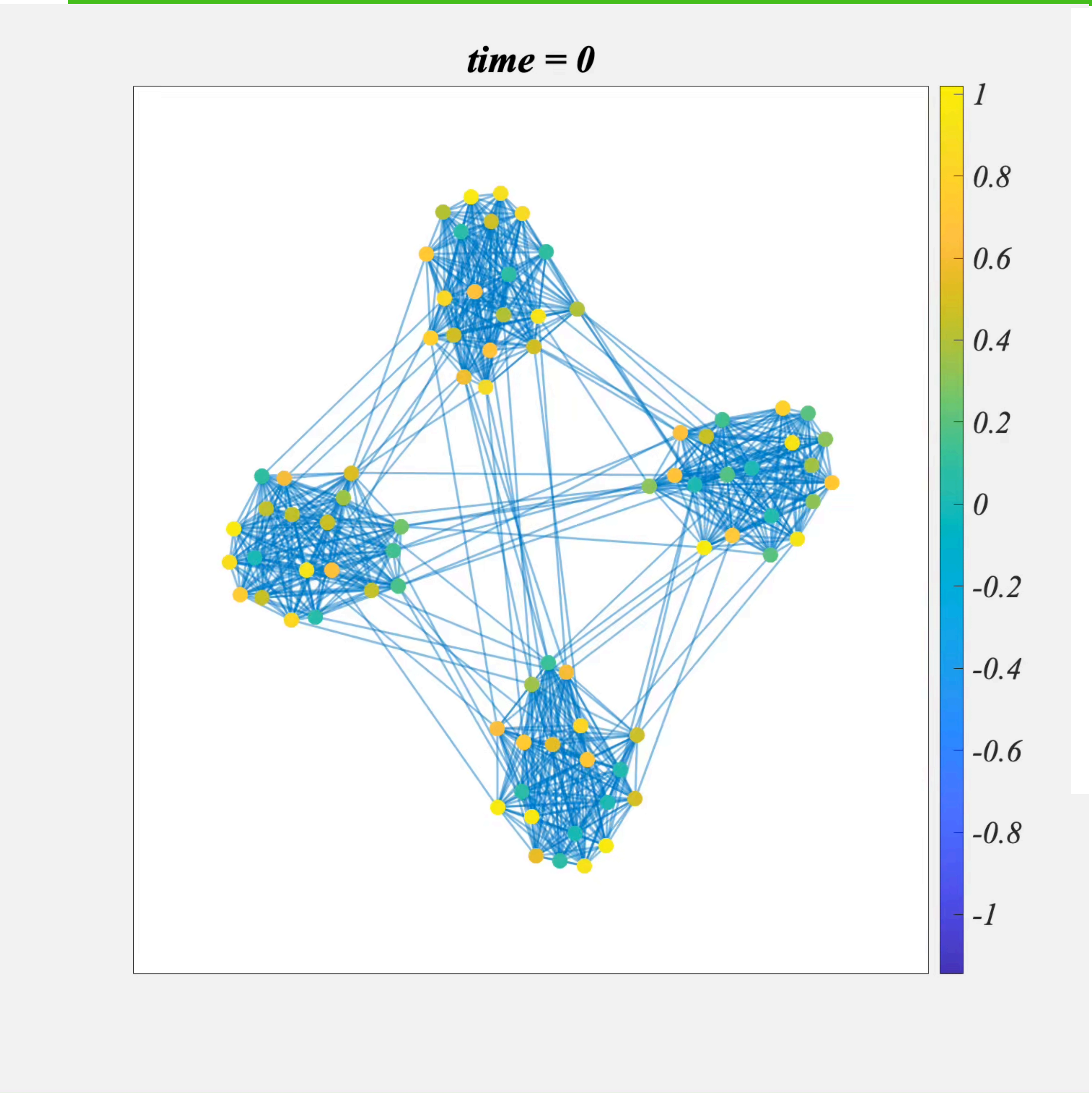


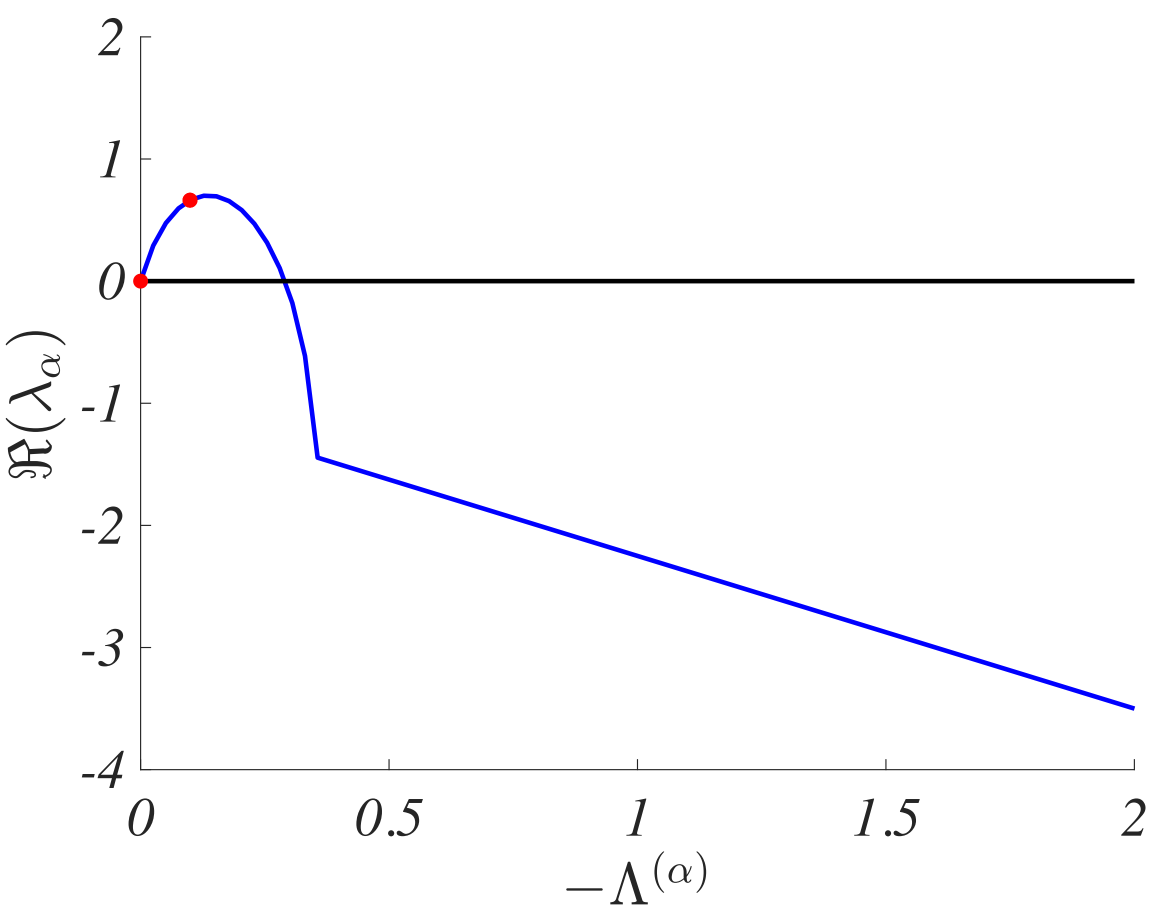


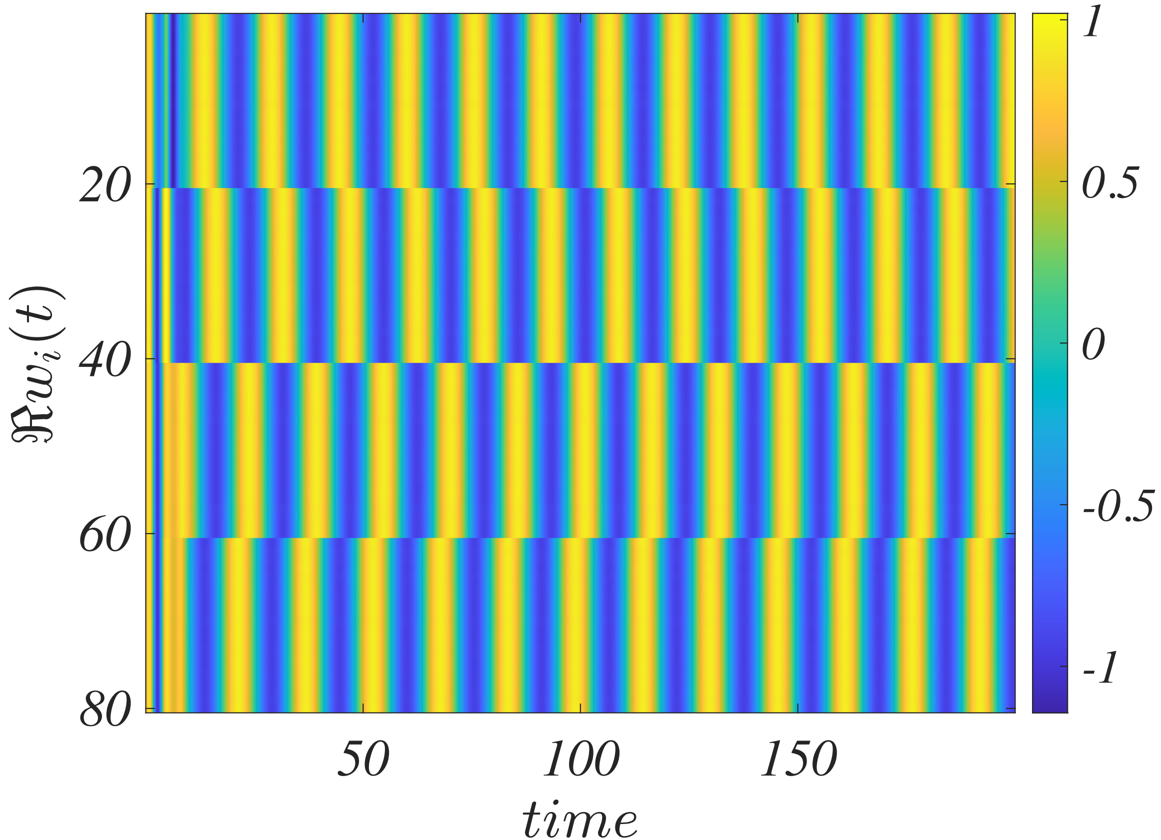
$a < 0$ $\quad$ $a = 0$ $\quad$ $a > 0$

$$\frac{dz^{(j)}}{dt} = z_j(a + ib - |z_j|^2) + \mu \sum_{j=1}^{n} A_{j\ell} \left[h(z^{(\ell)}) - h(z^{(j)}) \right]$$

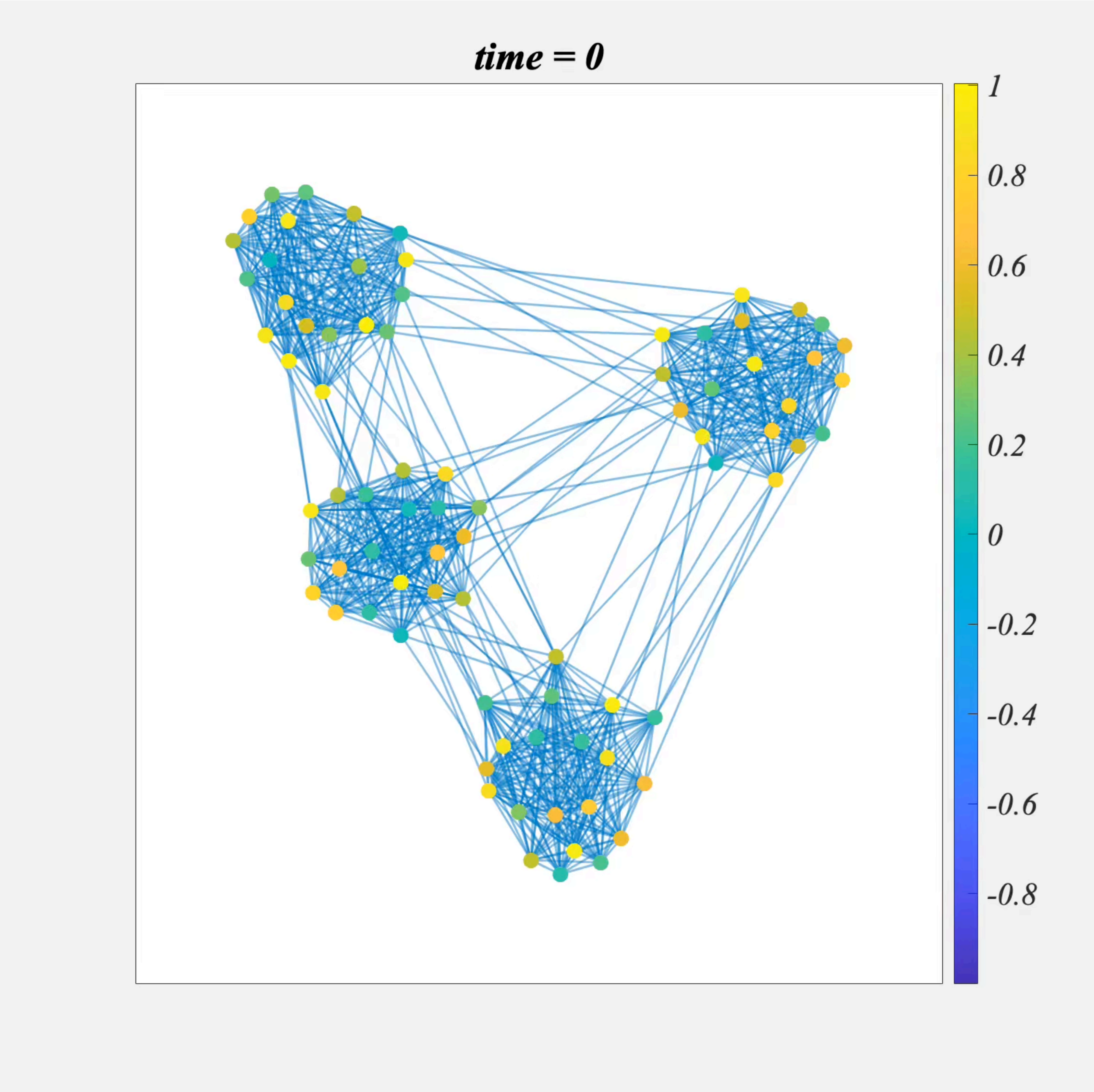
Stuart - Landau oscillator : no synch

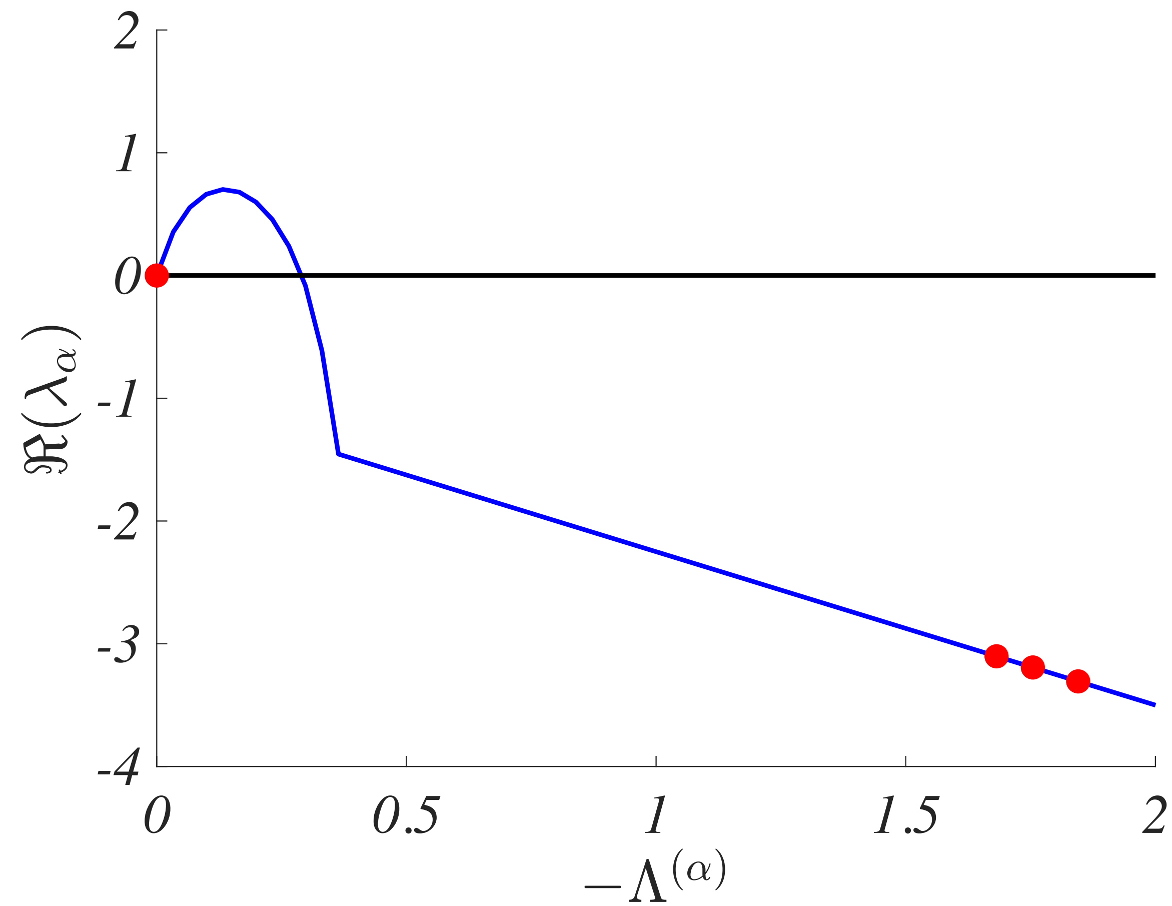


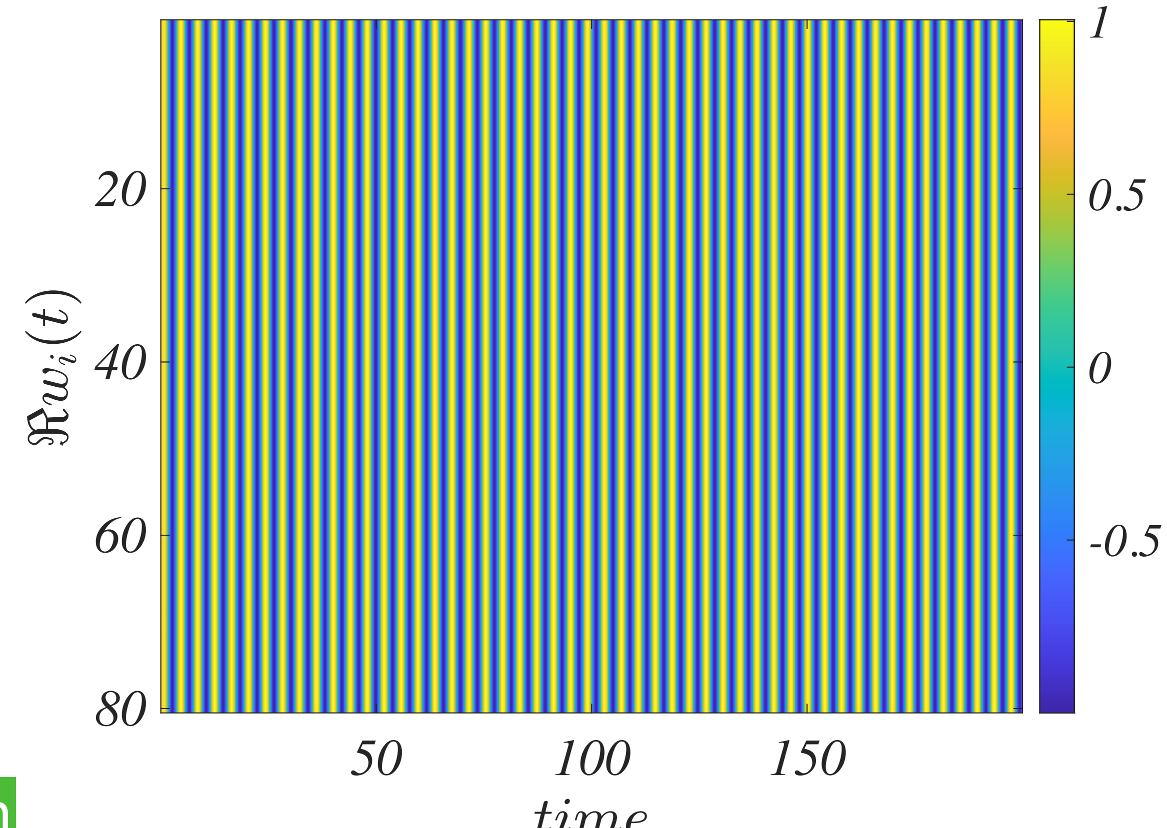




Stuart - Landau oscillator : synch







Global Synchronisation : beyond networks

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PAPER

Dynamical systems on hypergraphs

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Stability of synchronization in simplicial complexes

L. V. Gambuzza[1,12], F. Di Patti[2,12], L. Gallo[3,4,12], S. Lepri[2], M. Romance[5], R. Criado[5], M. Frasca[1,6,13]✉, V. Latora[3,4,7,8,13]✉ & S. Boccaletti[2,9,10,11,13]✉

Global Topological Synchronisation

PHYSICAL REVIEW LETTERS **130**, 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

Timoteo Carletti,[1] Lorenzo Giambagli,[1,2] and Ginestra Bianconi[3,4]

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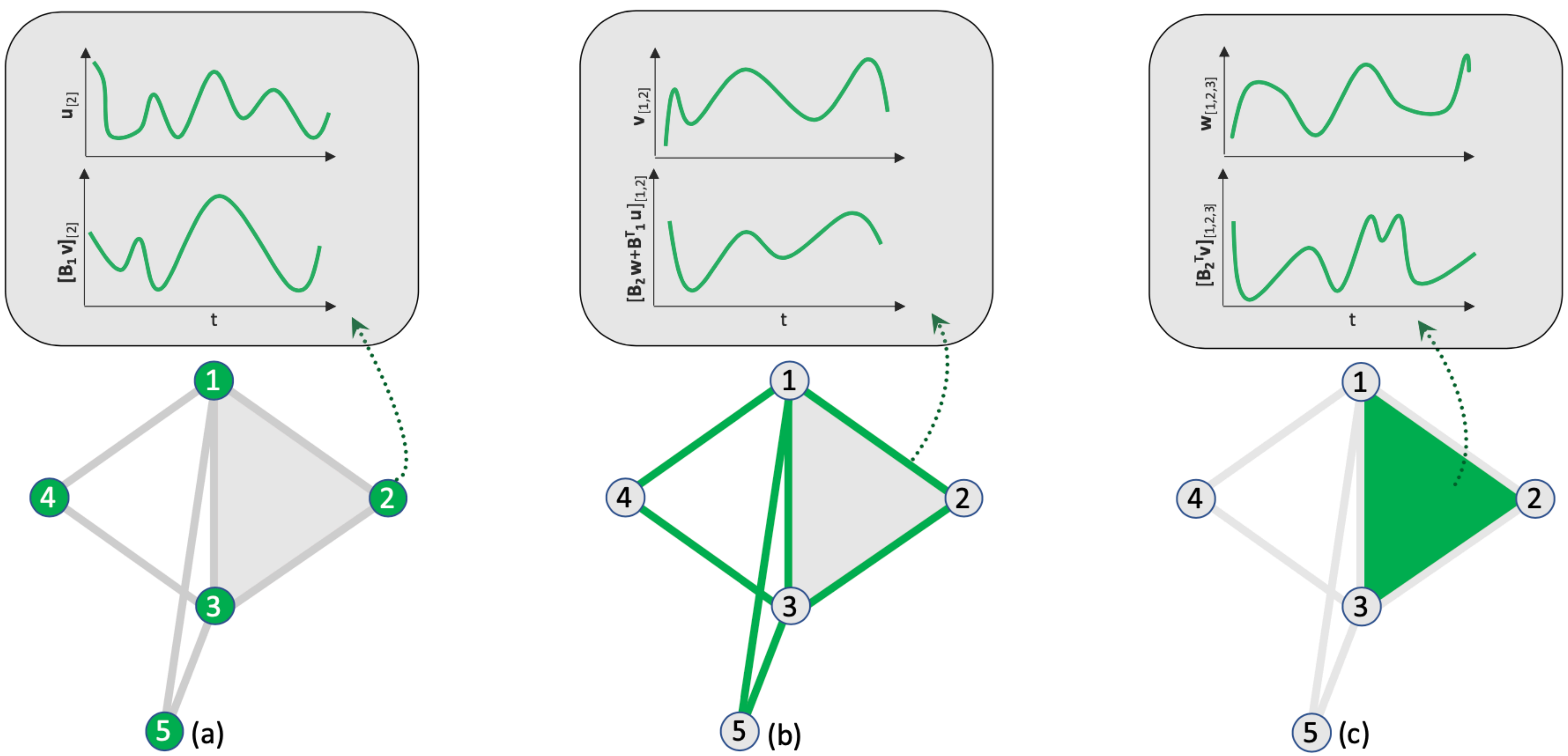


Simplicial complex

$\sigma_i^{(k)} = [i_0, \dots, i_k]$ k-simplex (it contains k+1 nodes)

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

Incidence matrix

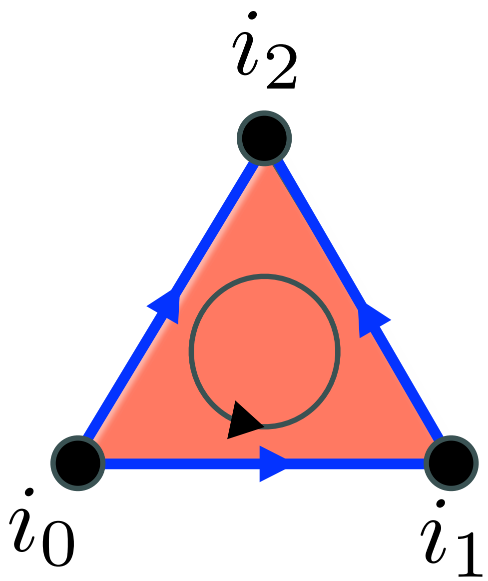
$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

Simplicial complex: an example

$k = 2$ Three nodes, hence a triangle $\sigma^{(2)} = [i_0, i_1, i_2]$



$\sigma_1^{(1)} = [i_0, i_1]$ $\sigma_2^{(1)} = [i_1, i_2]$ $\sigma_3^{(1)} = [i_0, i_2]$

Incidence matrices

$$\mathbf{B}_1 \in M^{N_0 \times N_1}$$

$$\mathbf{B}_1(\sigma_i^{(0)}, \sigma_j^{(1)}) = \begin{array}{c} \\ i_0 \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{ccc} [i_0, i_1] & [i_1, i_2] & [i_0, i_2] \end{array} \\ \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{B}_2 \in M^{N_1 \times N_2}$$

$$\mathbf{B}_2(\sigma_i^{(1)}, \sigma_j^{(2)}) = \begin{array}{c} \\ [i_0, i_1] \\ [i_1, i_2] \\ [i_0, i_2] \end{array} \begin{array}{c} [i_0, i_1, i_2] \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array}$$

Functions on simplicial complexes

$$\sigma_i^{(k)} = [i_0, \ldots, i_k]$$

$$\mathbf{x} : C_k \to \mathbb{R}^d$$ k-cochain

$$\mathbf{x}_i = \mathbf{x}(\sigma_i^{(k)}) = (x_i^1, \ldots, x_i^d)$$

$$\mathbf{x}(-\sigma_i^{(k)}) = -\mathbf{x}(\sigma_i^{(k)})$$

Dynamical system on a simplex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i)$$

$$\mathbf{f}(-\mathbf{x}_i) = -\mathbf{f}(\mathbf{x}_i)$$

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Global Topological Synchronisation

Dynamical system on a simplicial complex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N_k} L_k(i,j)\mathbf{h}(\mathbf{x}_j) \qquad \mathbf{h}(-\mathbf{x}_i) = -\mathbf{h}(\mathbf{x}_i)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

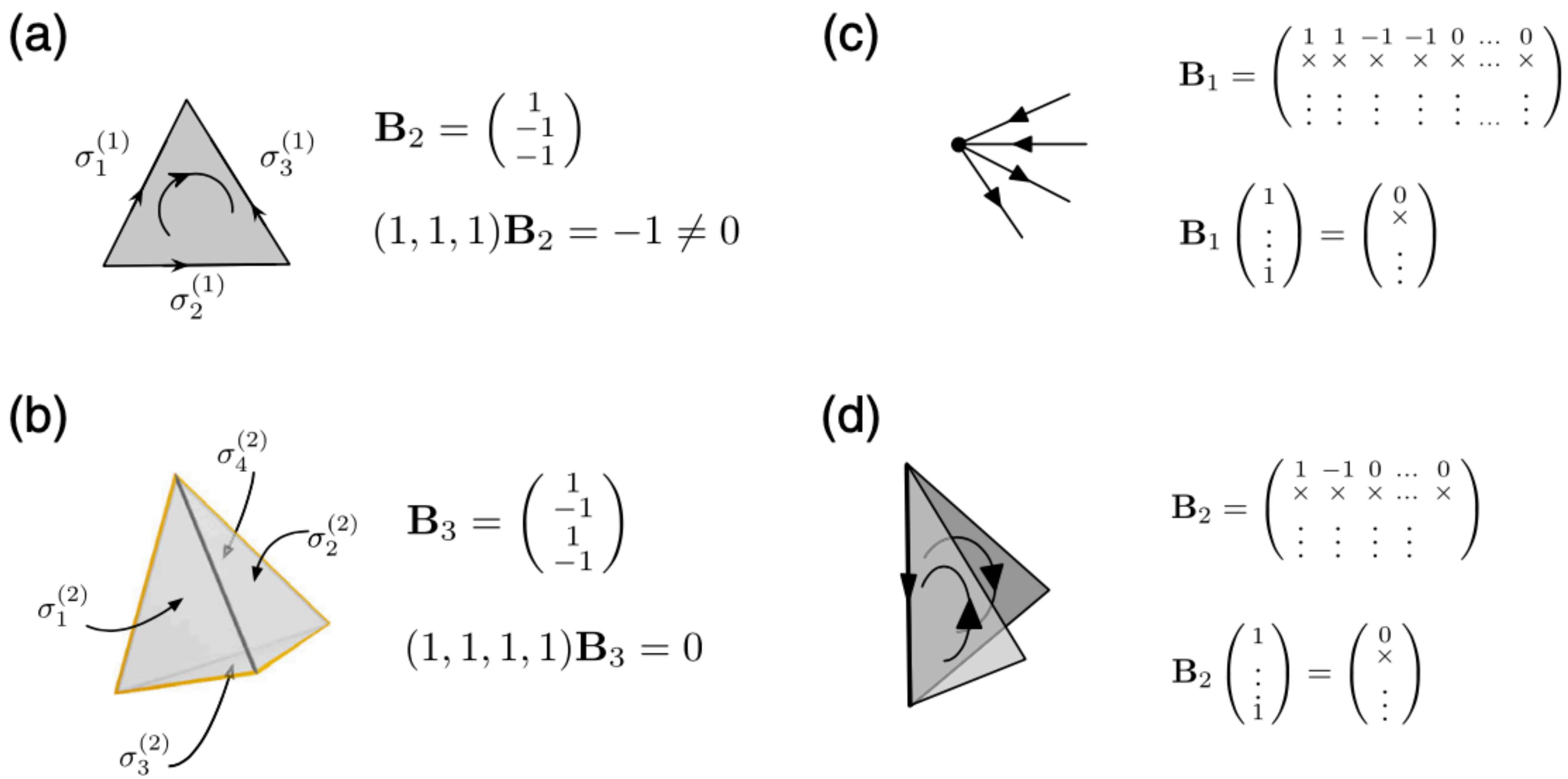
Synchronisation : $\mathbf{x}_i(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, N_k$

$$\left.\frac{d\mathbf{x}_i}{dt}\right|_{\mathbf{x}_i=\mathbf{s}} = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}_j)\Big|_{\mathbf{x}_i=\mathbf{s}} \overset{?}{=} 0$$

Global Topological Synchronisation

Necessary condition $\mathbf{L}_k u = 0 \quad \Leftrightarrow \quad \mathbf{B}_k u = 0$ and $\mathbf{B}_{k+1}^\top u = 0$

odd dim = non global synch



even dim = global synch if <u>balanced</u>

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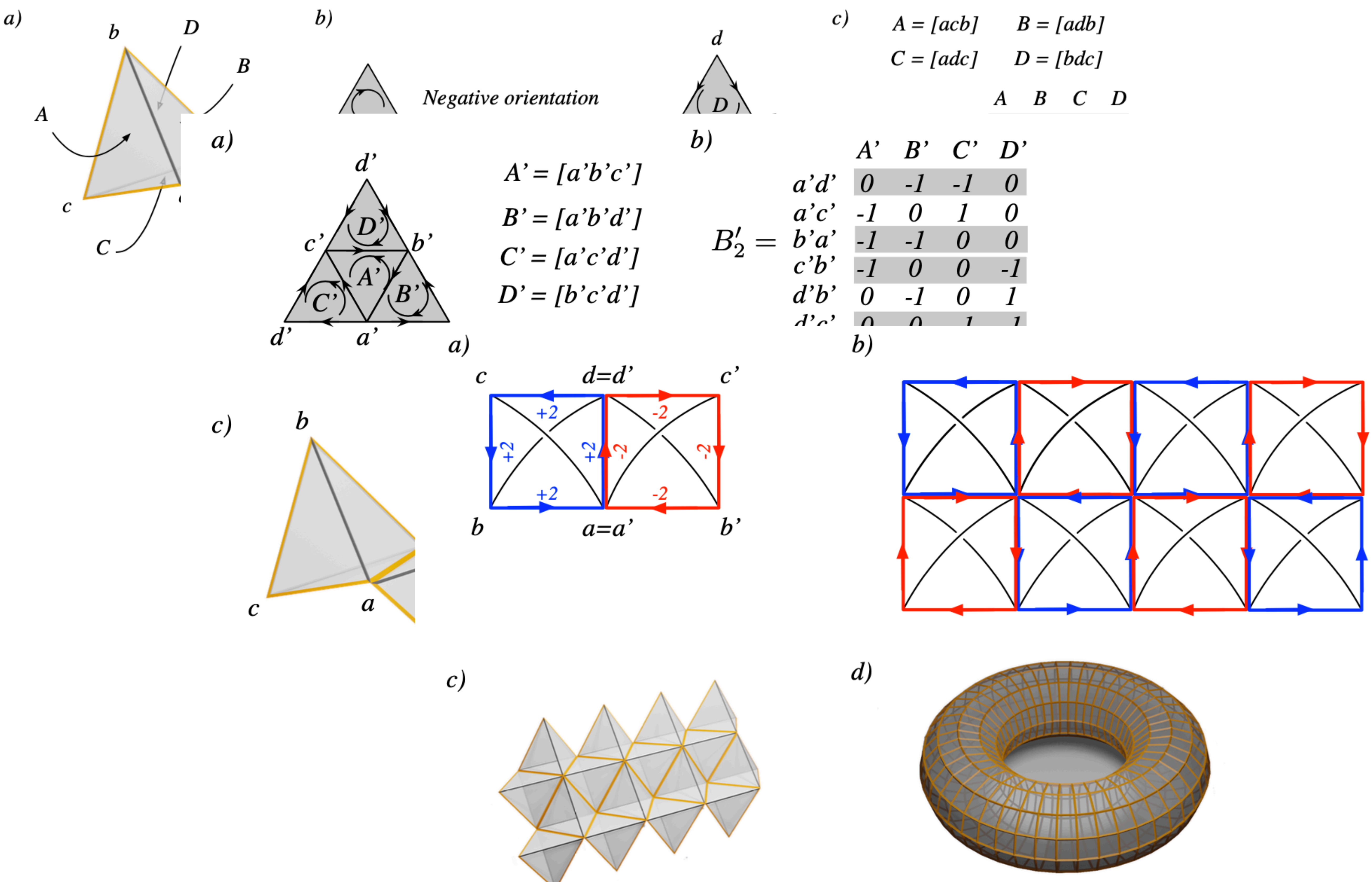
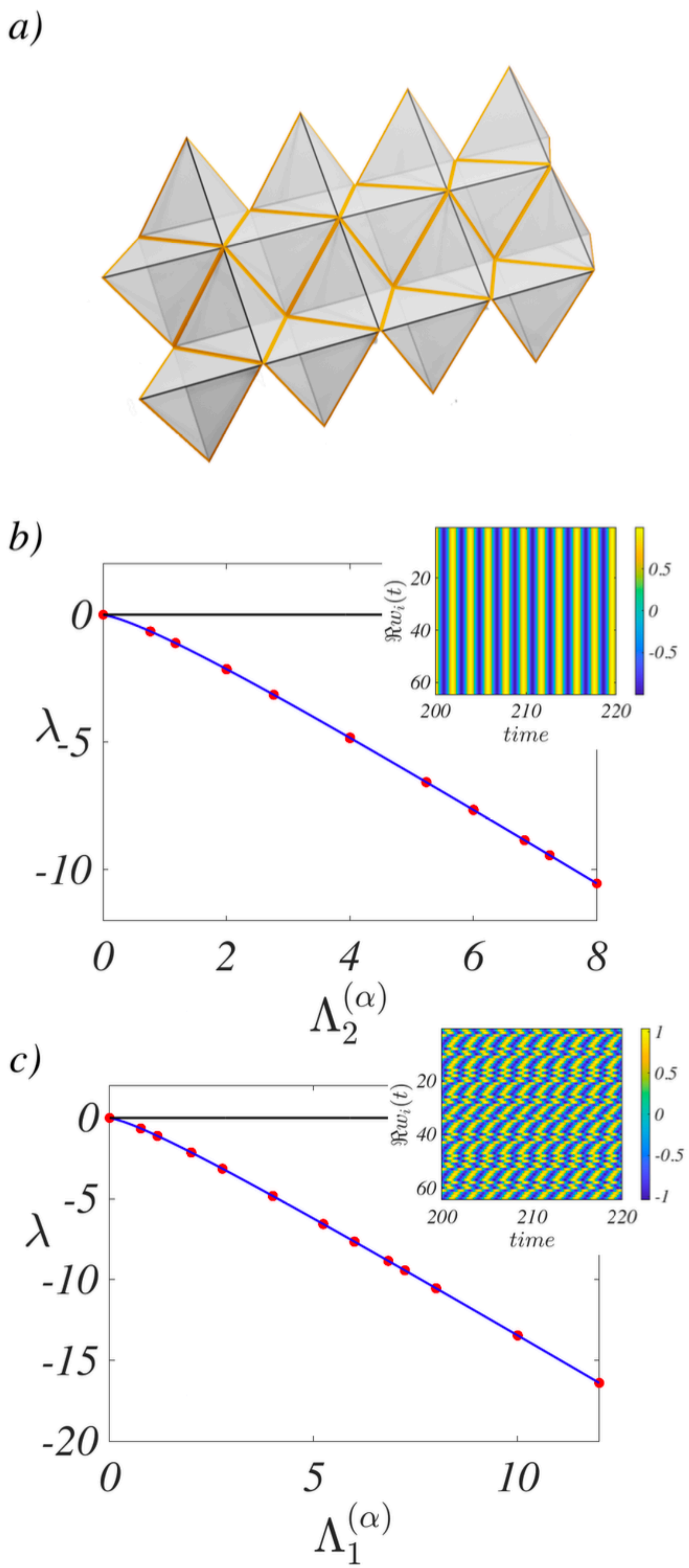
The "waffle" 3-simplicial complex



Global Topological Synchronisation : Stuart-Landau



global synch
for faces (k=2)

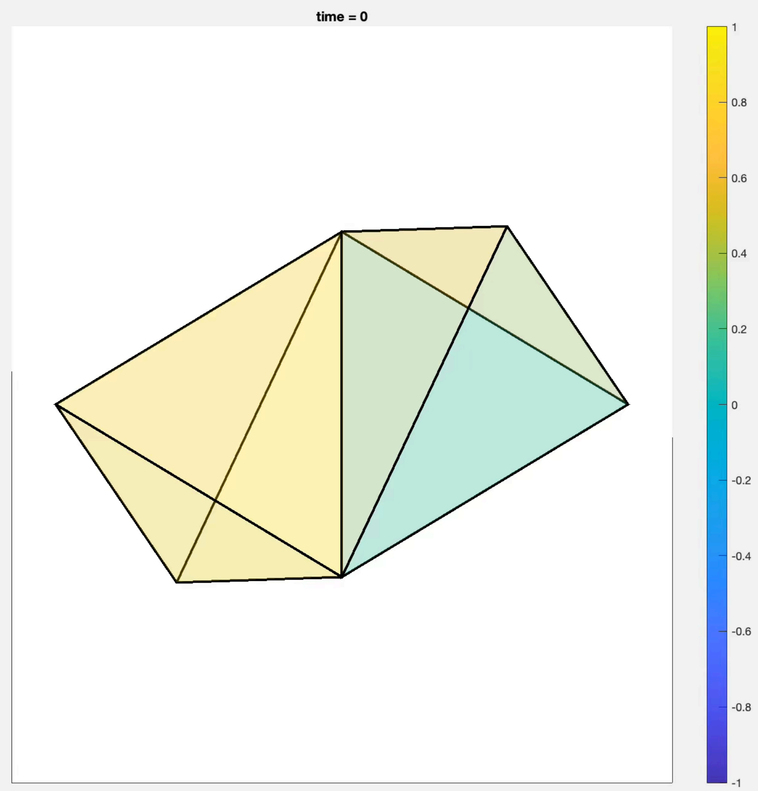
$$\mathbf{B}_2(1,\ldots,1)^\top = 0$$

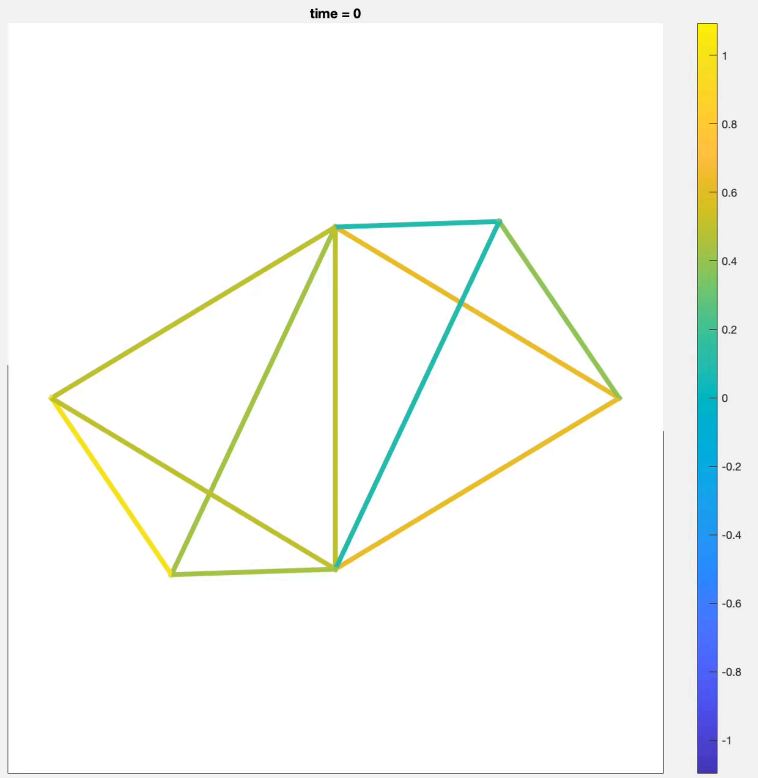
$$\mathbf{B}_3^\top(1,\ldots,1)^\top = 0$$

no global synch
for links (k=1)

$$\mathbf{B}_1(1,\ldots,1)^\top = 0$$

$$\mathbf{B}_2^\top(1,\ldots,1)^\top \neq 0$$





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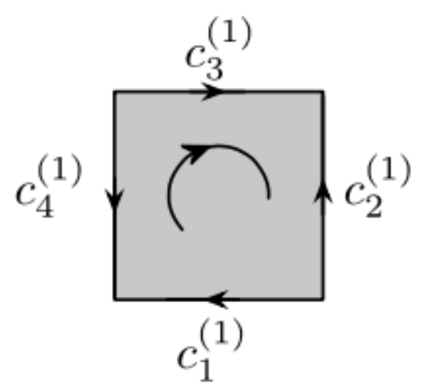
Global Topological Synchronisation

The topological obstruction
does not exist for cell complexes

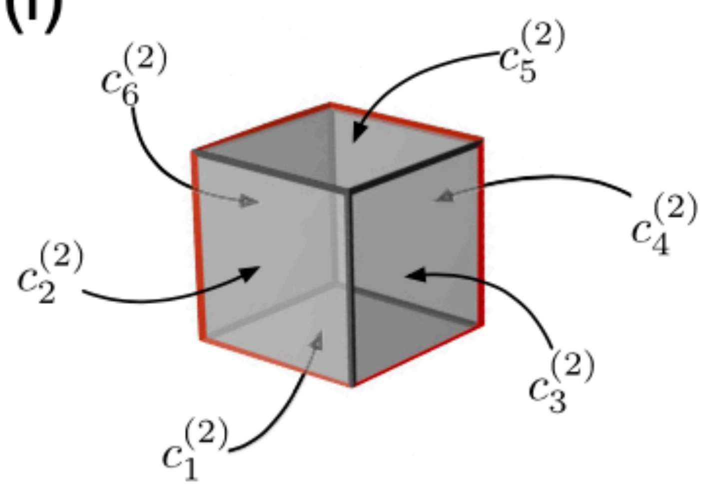
(e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_2 = 0$$

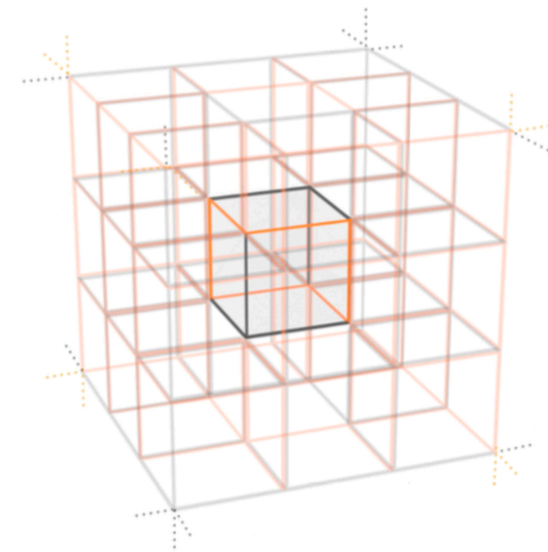
(f)



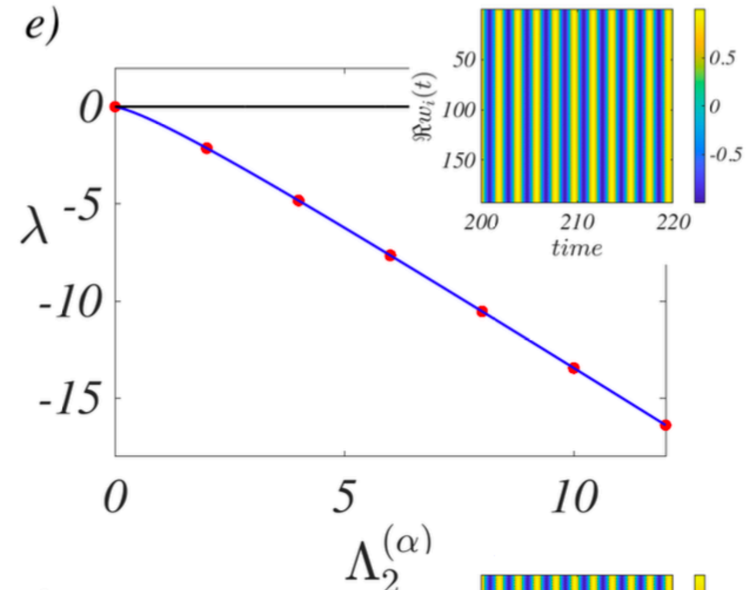
$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$

d)

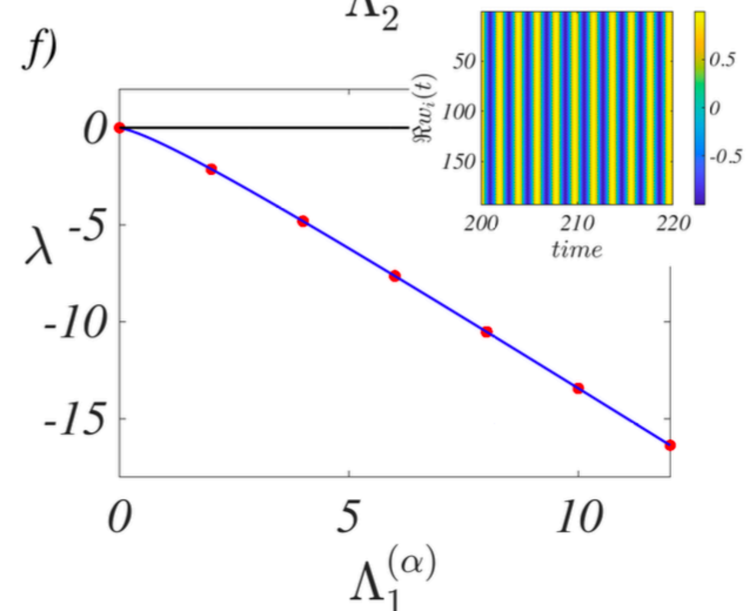


e)



global synch for faces

f)

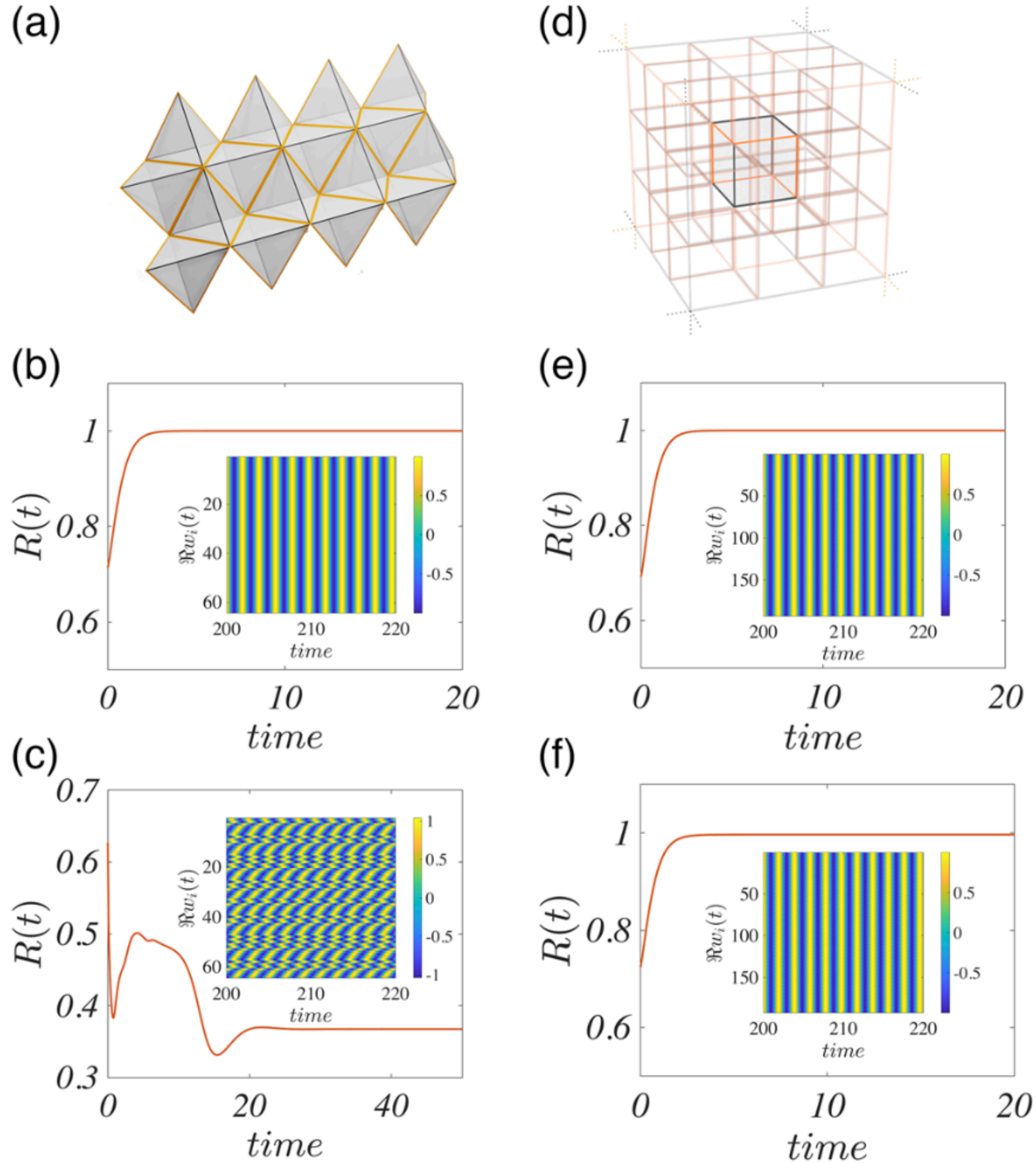


global synch for links

Global Topological Synchronisation : Stuart-Landau

global sync
for faces

no global sync
for links



global sync
for faces

global sync
for links

27th September 2023

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25-28 Sept,

Campus ... the University of Dschang, Cameroon

University of Dschang

Timoteo Carletti

Thank you

Any questions?

Global Topological Synchronisation on Simplicial Complexes

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