15th May 2023, SIAM-DS23

Society for Industrial and Applied Mathematics

Timoteo Carletti

Global Topological Synchronisation on Simplicial and Cell Complexes

Namur Institute for Complex Systems

Department of mathematics

UNIVERSITÉ DE NAMUR

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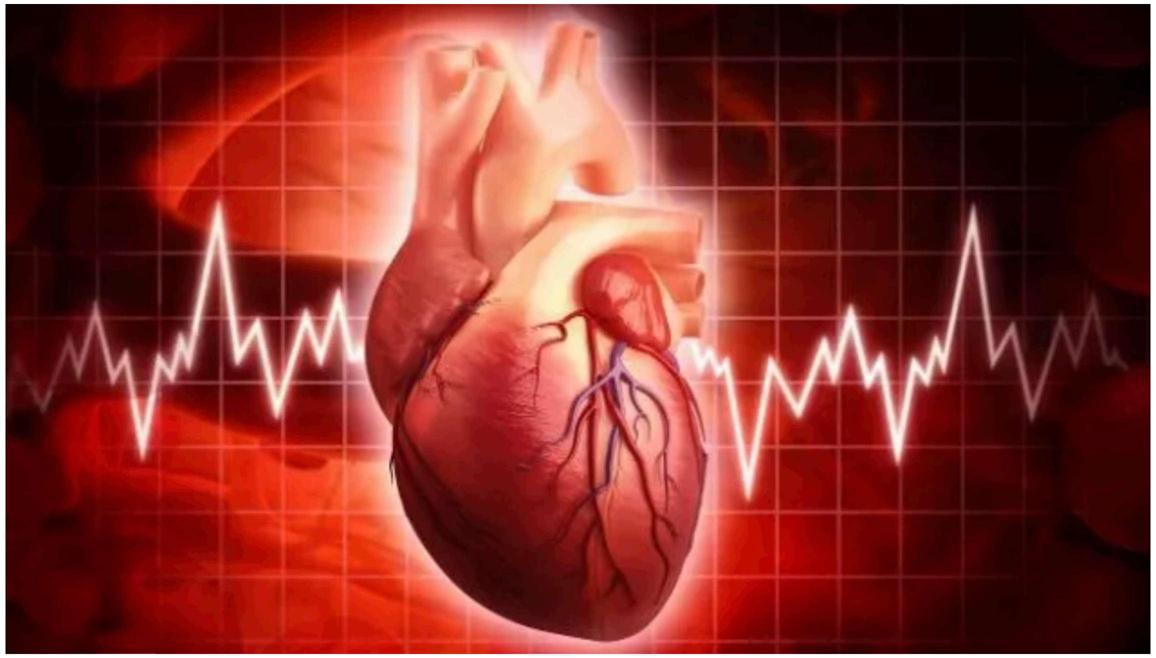
Synchronisation



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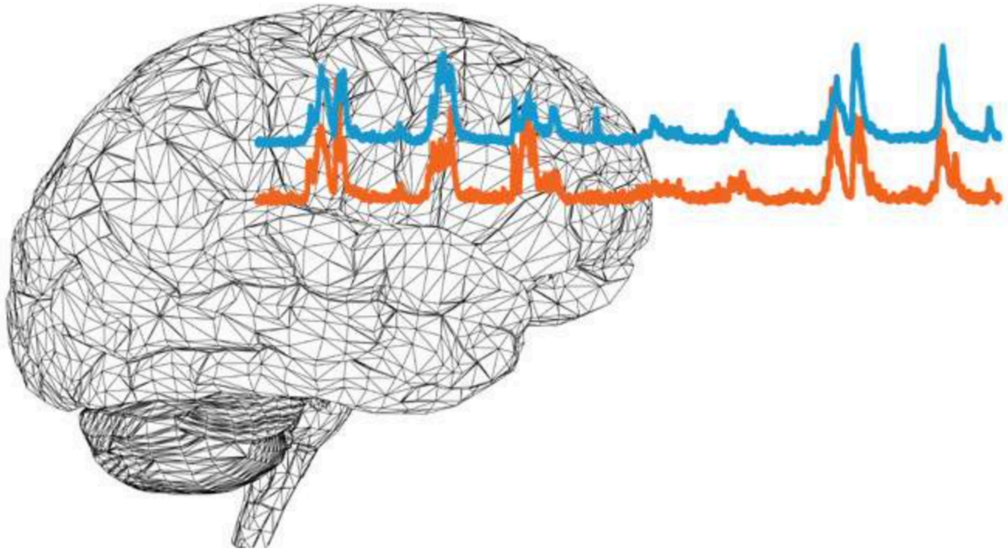
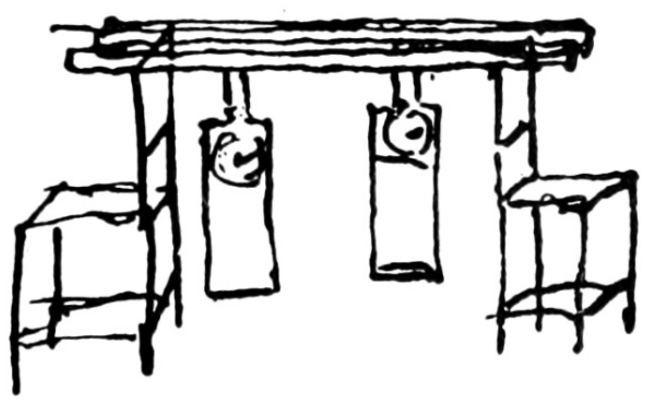
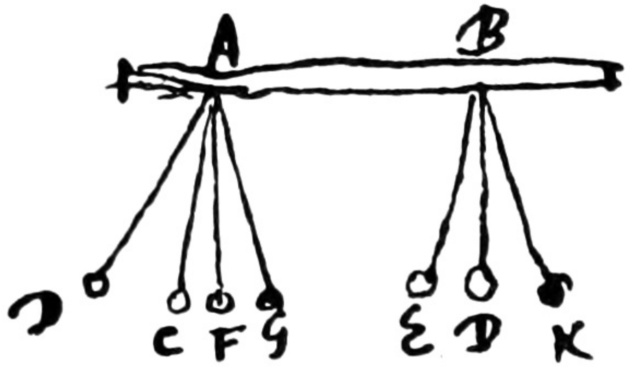
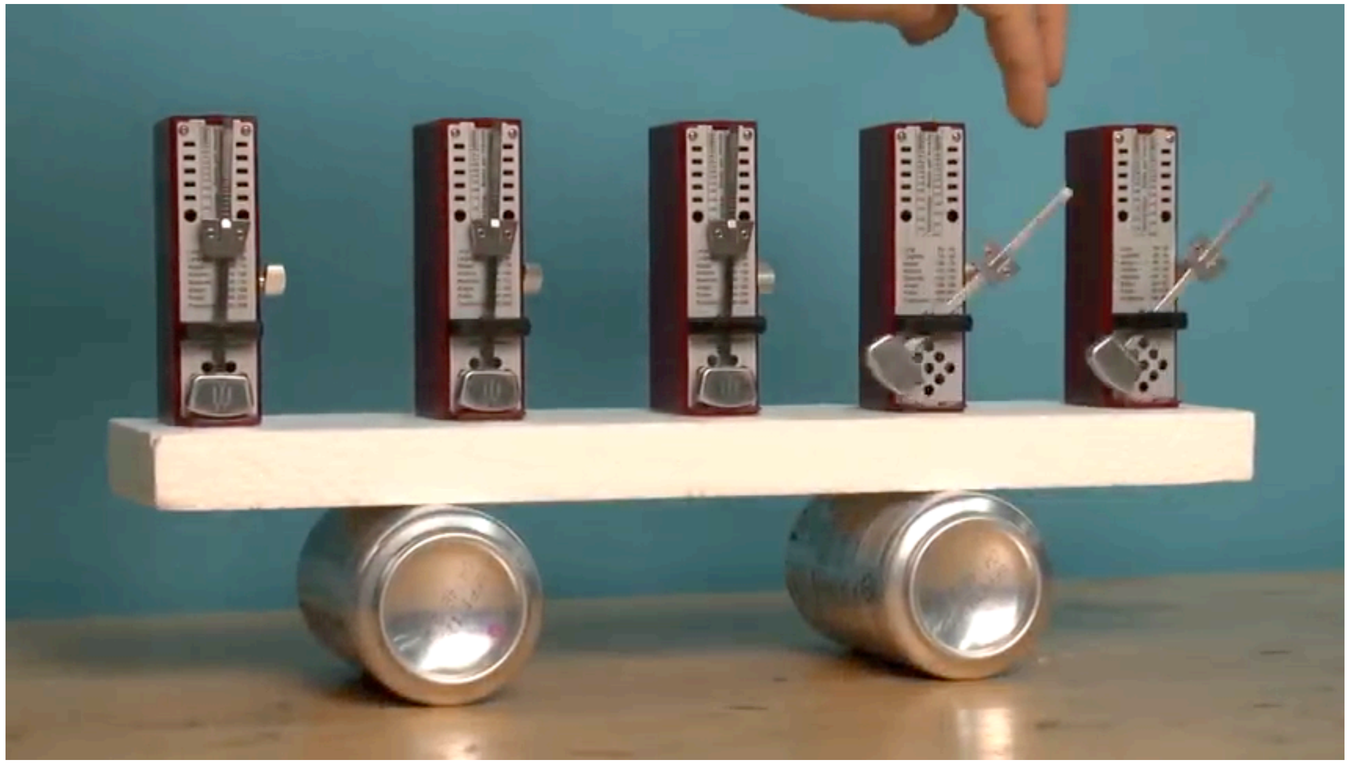


Global Synchronisation







www.youtube.com

Huygen

"An odd kind of sympathy"

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Global Synchronisation: Pecora et al.

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

PHYSICAL REVIEW E **80**, 036204 (2009)

Generic behavior of master-stability functions in coupled nonlinear dynamical systems

Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]
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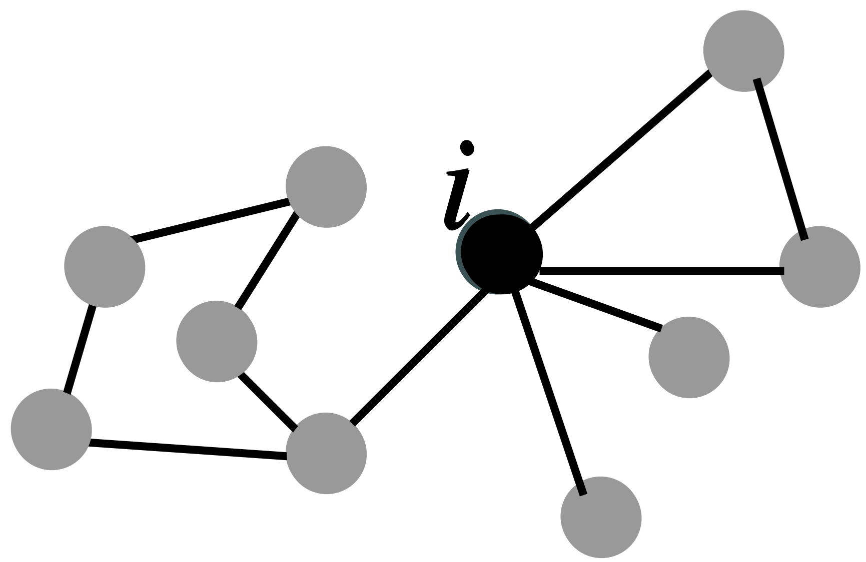
Global Synchronisation on networks

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} A_{ij} \mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

$$\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \mathbf{h}(\mathbf{x}^{(j)}) - \mathbf{h}(\mathbf{x}^{(i)})$$

$$= \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij} \mathbf{h}(\mathbf{x}^{(j)})$$

Diffusive-like coupling

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Global Synchronisation on networks

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

Synchronisation : $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

Does the whole system admit such (spatially) homogeneous solution?

$$\left.\frac{d\mathbf{x}^{(i)}}{dt}\right|_{\mathbf{x}^{(i)}=\mathbf{s}} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = 0$$

$$\mathbf{L}\mathbf{u} = 0 \qquad \mathbf{u} = (1, \dots, 1)^\top$$

Laplace matrix

Global Synchronisation on networks

Is $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \ldots, n$ stable?

- $\delta\mathbf{x}^{(i)}(t) = \mathbf{x}^{(i)}(t) - \mathbf{s}(t) \quad \forall i = 1, \ldots, n$
- $$\frac{d\delta\mathbf{x}^{(i)}}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}^{(i)} + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}^{(j)}$$

Time dependent linear system

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Global Synchronisation on networks

- $\mathbf{L}\phi^{(\alpha)} = \Lambda^{(\alpha)}\phi^{(\alpha)} \qquad \phi^{(\alpha)} \cdot \phi^{(\beta)} = \delta_{\alpha\beta} \qquad \Lambda^{(1)} = 0 \quad \Lambda^{(\alpha)} < 0$
- $\delta\mathbf{x}^{(i)} = \sum_{\alpha} \delta\mathbf{x}_\alpha \phi_i^{(\alpha)}$
- $\frac{d\delta\mathbf{x}_\alpha}{dt} = \mathbf{J}_\mathbf{f}(\mathbf{s}(t))\delta\mathbf{x}_\alpha + \sigma\Lambda^{(\alpha)}\mathbf{J}_\mathbf{h}(\mathbf{s}(t))\delta\mathbf{x}_\alpha := \mathbf{J}_\alpha(\mathbf{s}(t))\delta\mathbf{x}_\alpha$

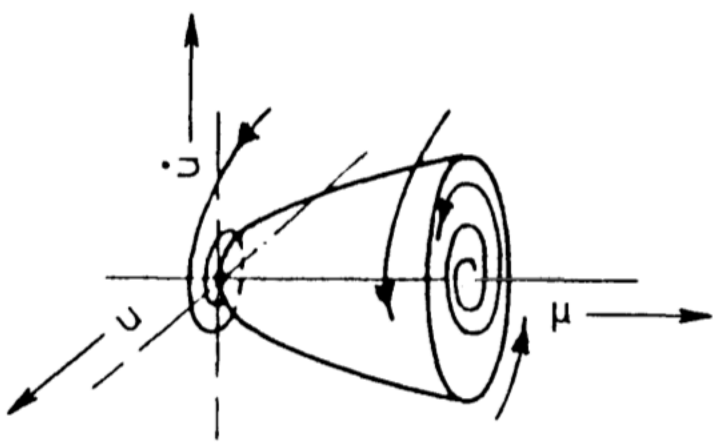
$\lambda(\Lambda^{(\alpha)})$ Master Stability Function = largest Lyapunov exponent of $\mathbf{J}_\alpha(\mathbf{s}(t))$

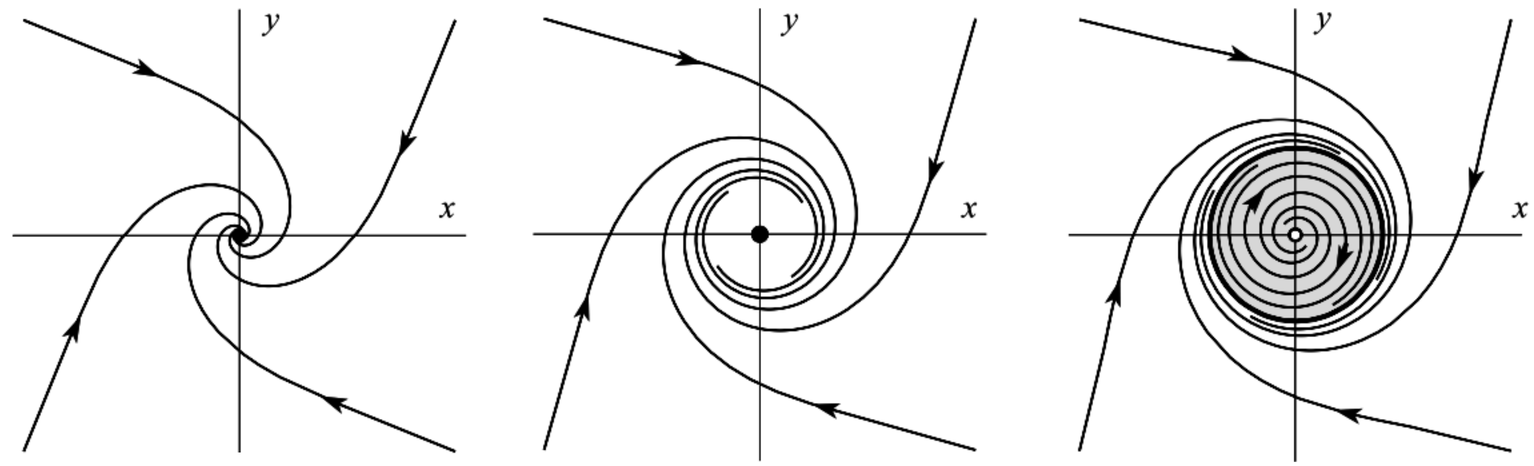
(function of $\Lambda^{(\alpha)}$)

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

Hopf Bifurcation





$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$

$$\frac{dz^{(j)}}{dt} = z_j(a + ib - |z_j|^2) + \mu \sum_{j=1}^{n} A_{j\ell} \left[h(z^{(\ell)}) - h(z^{(j)}) \right]$$

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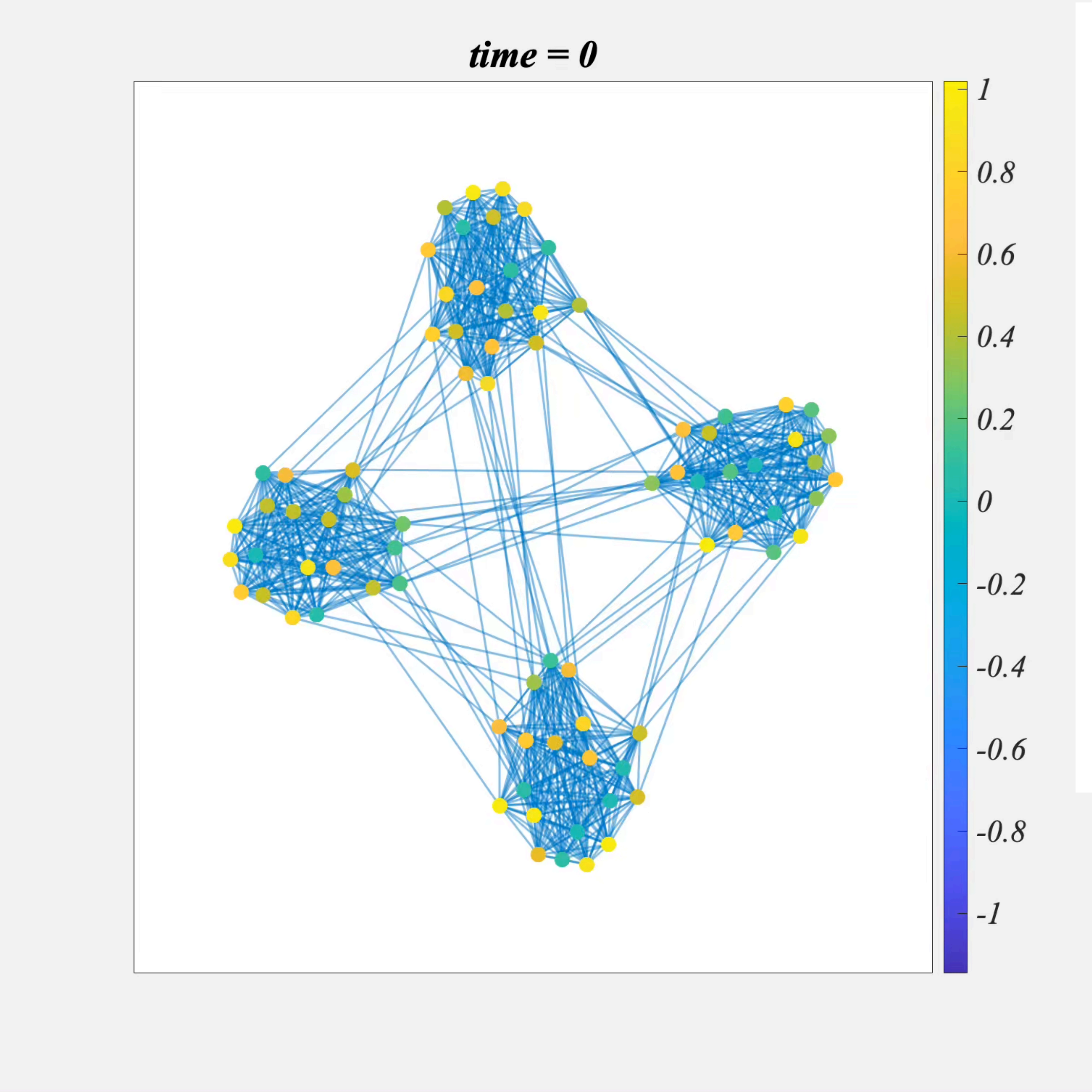
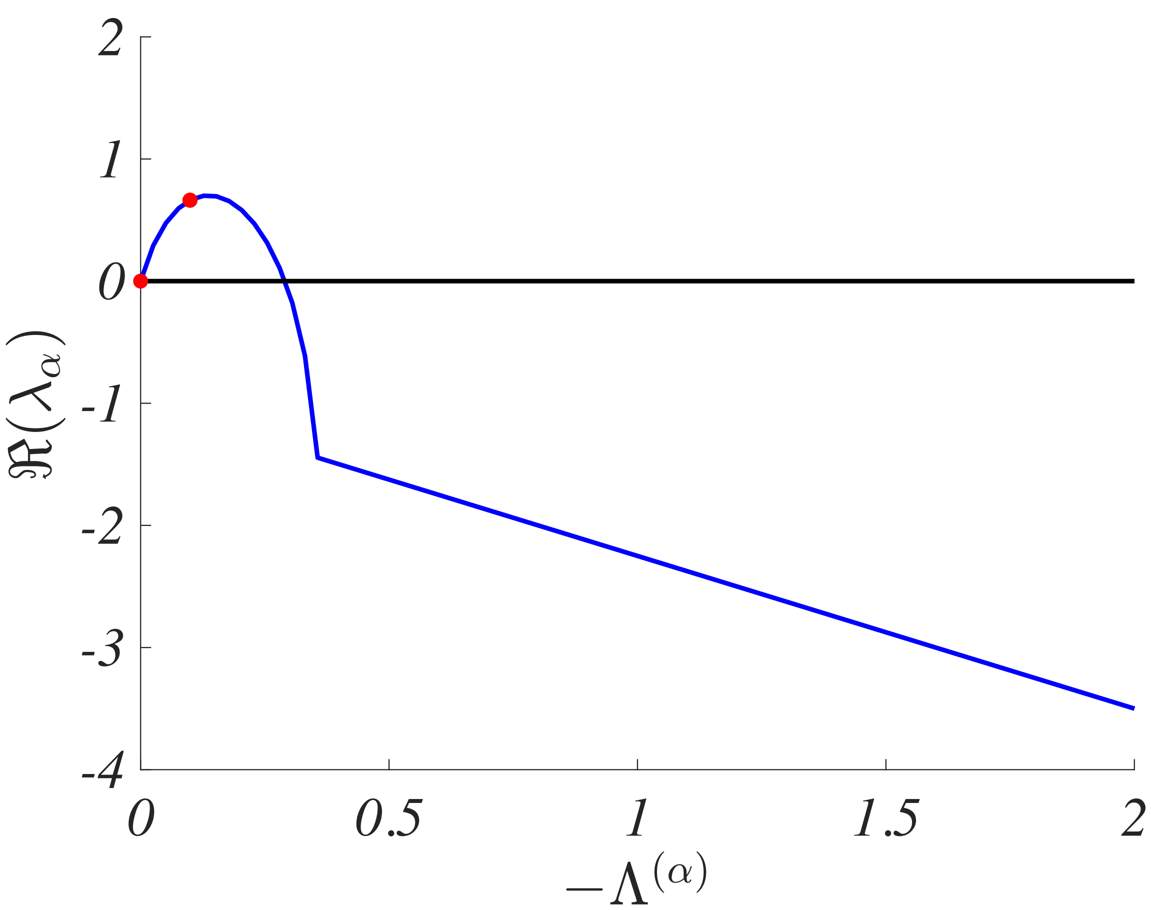
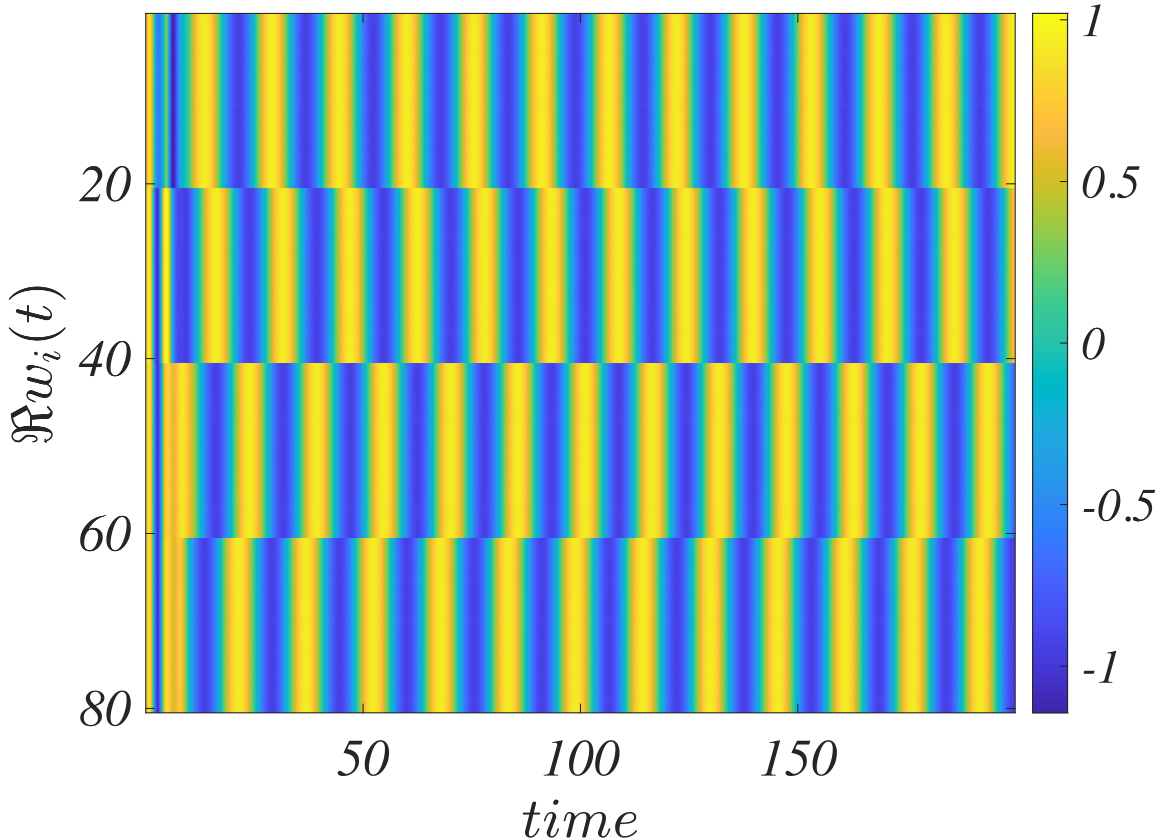
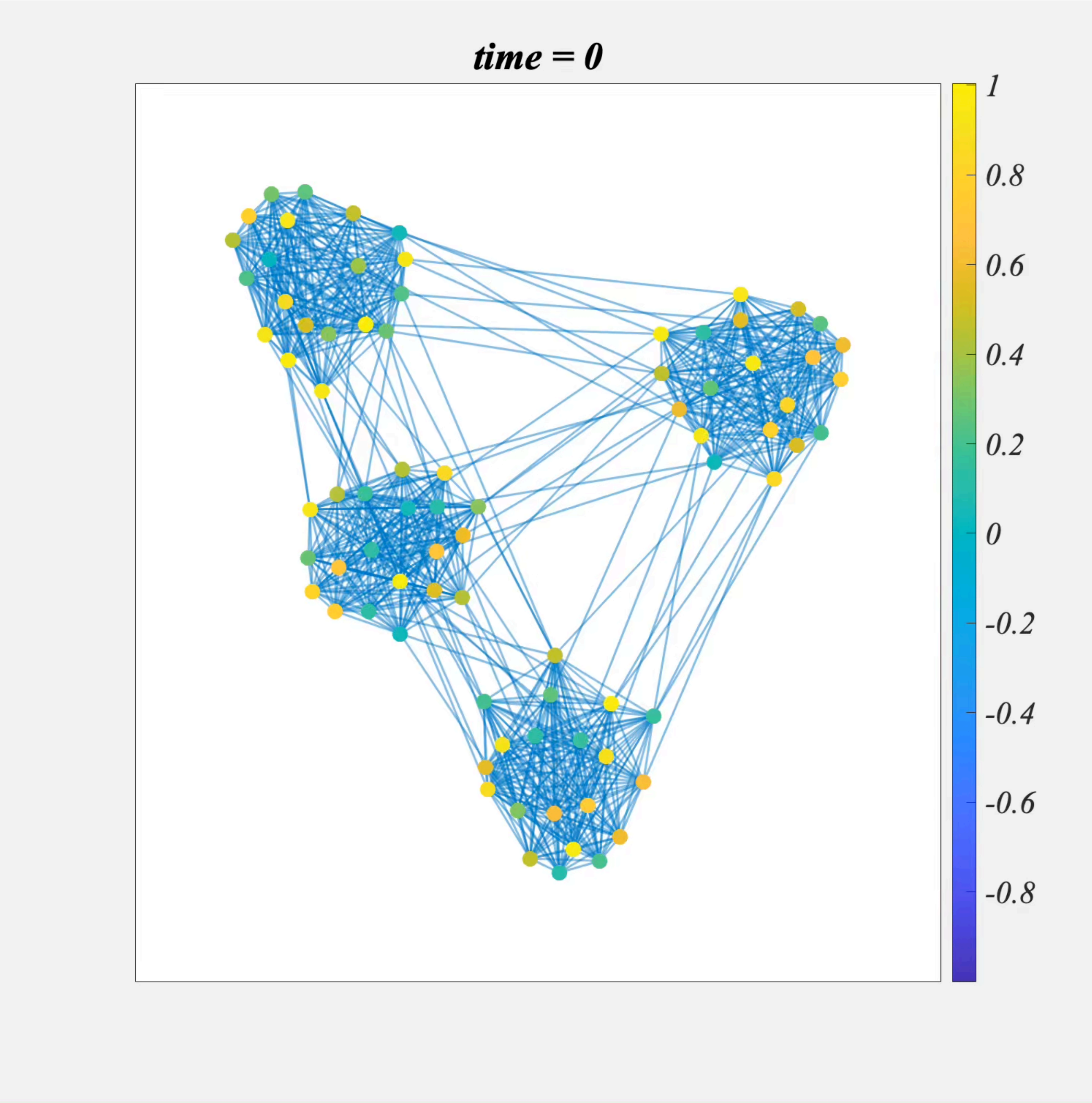
Stuart - Landau oscillator : no synch

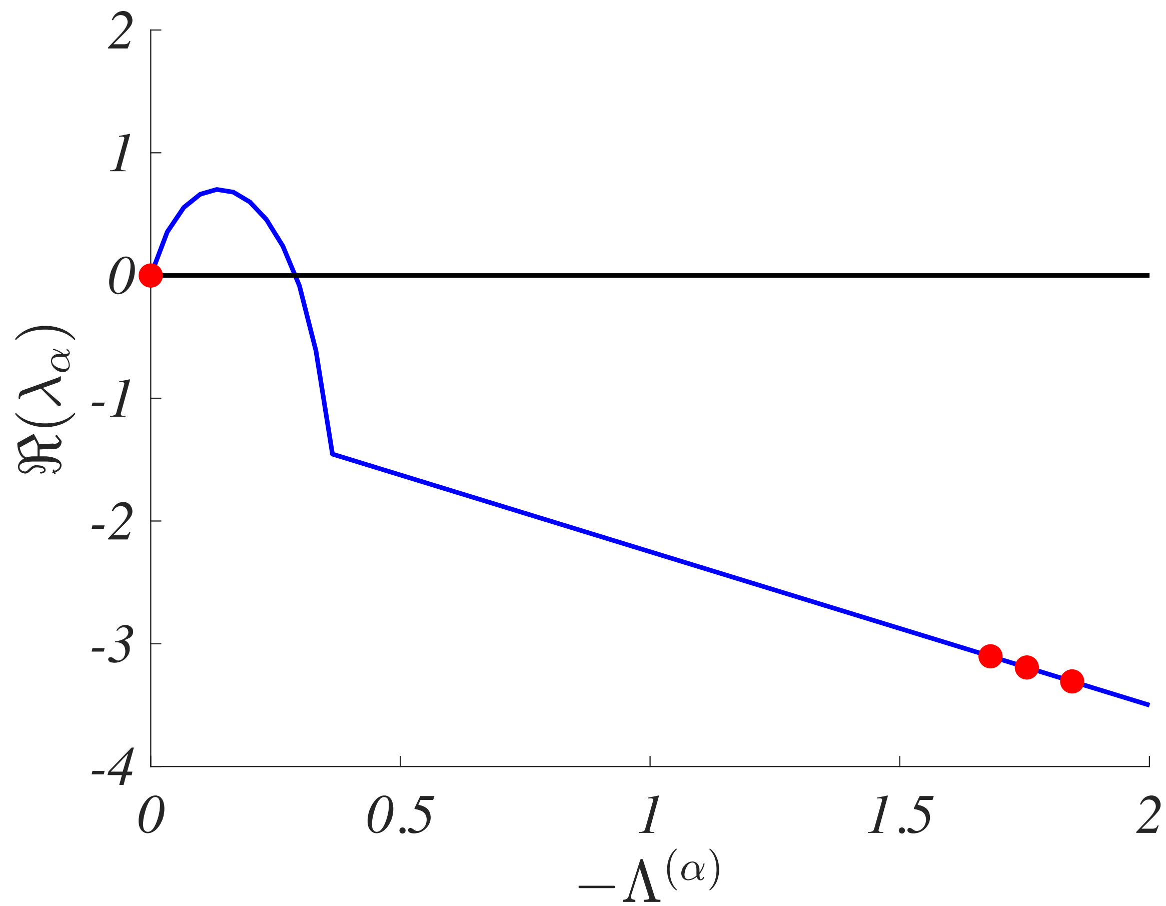


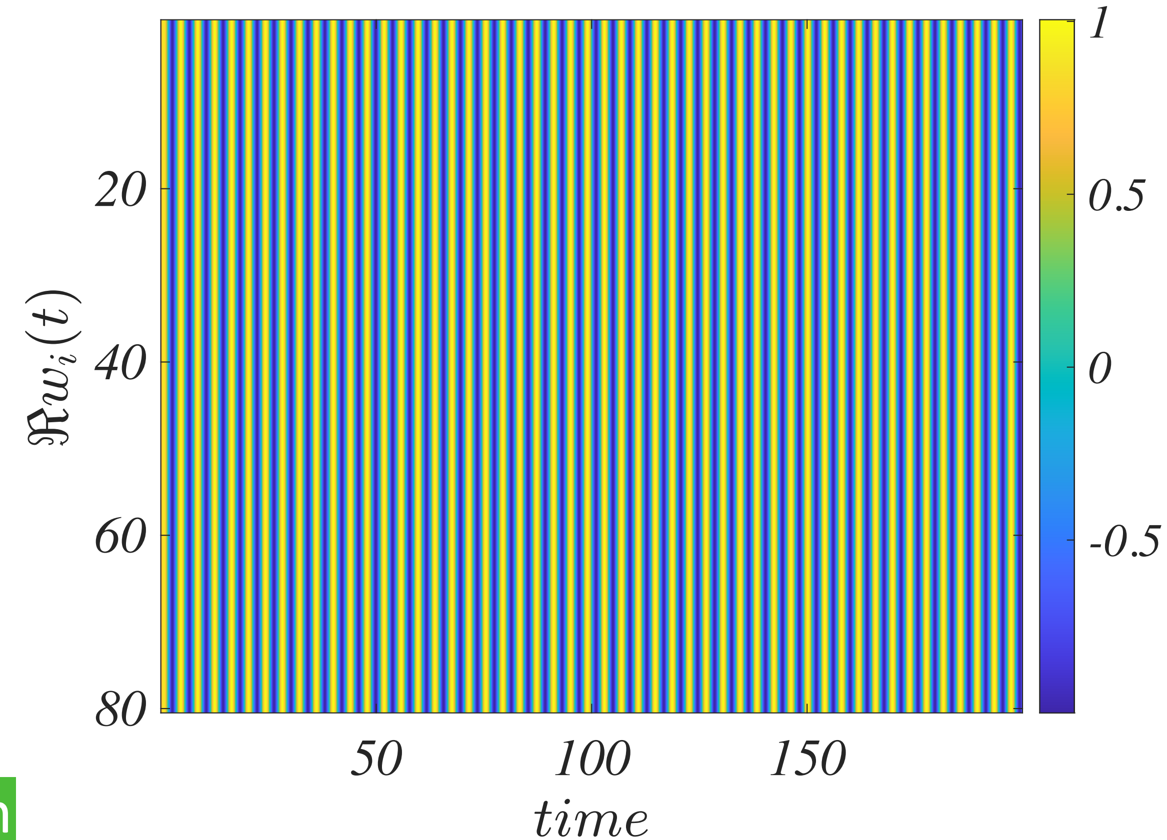




Stuart - Landau oscillator : synch







Global Synchronisation : beyond networks

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PAPER

Dynamical systems on hypergraphs

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ARTICLE

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Stability of synchronization in simplicial complexes

L. V. Gambuzza[1,12], F. Di Patti[2,12], L. Gallo[3,4,12], S. Lepri[2], M. Romance[5], R. Criado[5], M. Frasca[1,6,13]✉, V. Latora[3,4,7,8,13]✉ & S. Boccaletti[2,9,10,11,13]✉

Global Topological Synchronisation

PHYSICAL REVIEW LETTERS **130,** 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

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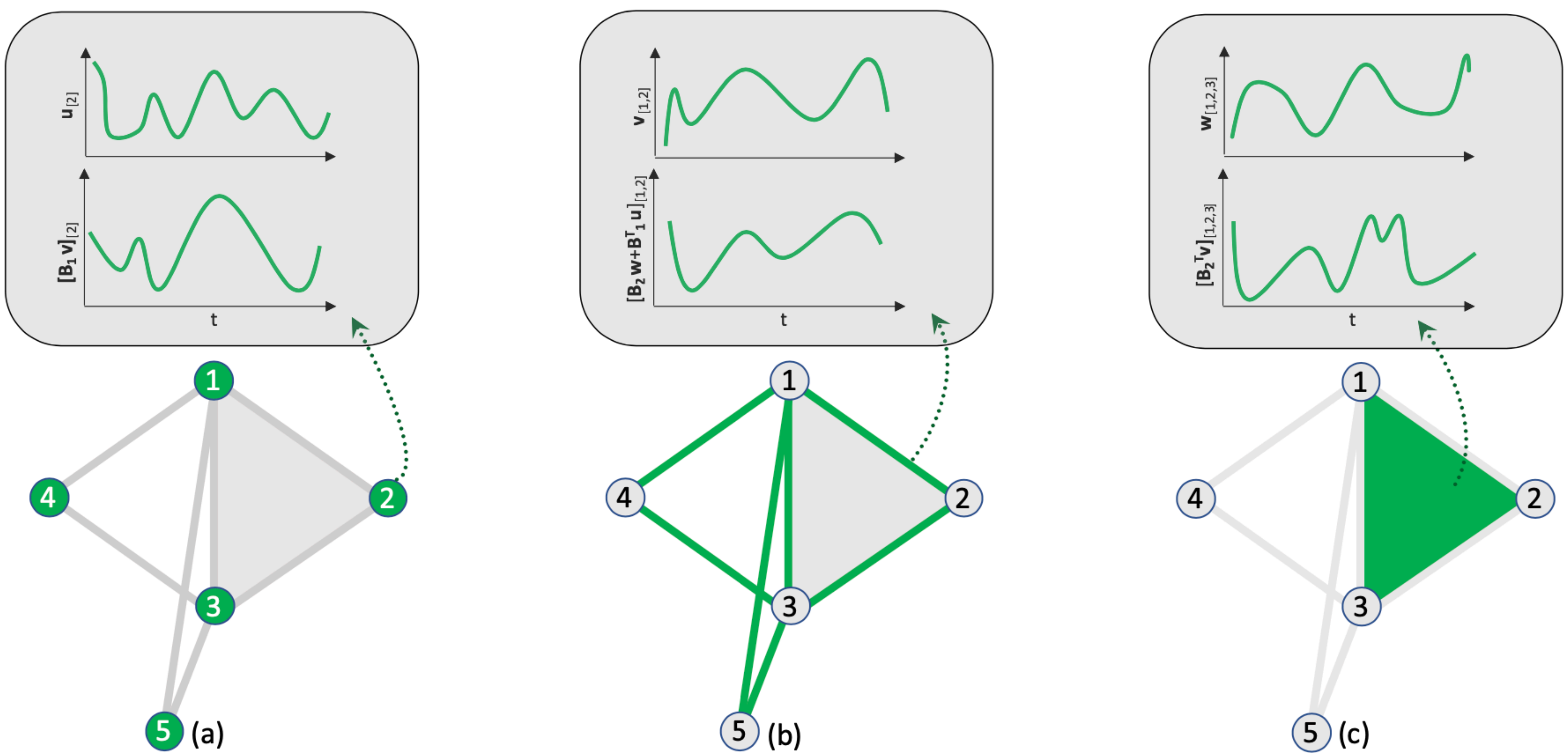

Global Topological Synchronisation

$$\sigma_i^{(k)} = [i_0, \ldots, i_k]$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

Incidence matrix

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

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Global Topological Synchronisation

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{x} : C_k \to \mathbb{R}^d$$ k-cochain

$$\mathbf{x}(-\sigma_i^{(k)}) = -\mathbf{x}(\sigma_i^{(k)})$$

$$\mathbf{x}_i = \mathbf{x}(\sigma_i^{(k)}) = (x_i^1, \dots, x_i^d)$$

Dynamical system on a simplex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i)$$

$$\mathbf{f}(-\mathbf{x}_i) = -\mathbf{f}(\mathbf{x}_i)$$

Global Topological Synchronisation

Dynamical system on a simplicial complex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N_k} L_k(i,j)\mathbf{h}(\mathbf{x}_j) \qquad \mathbf{h}(-\mathbf{x}_i) = -\mathbf{h}(\mathbf{x}_i)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

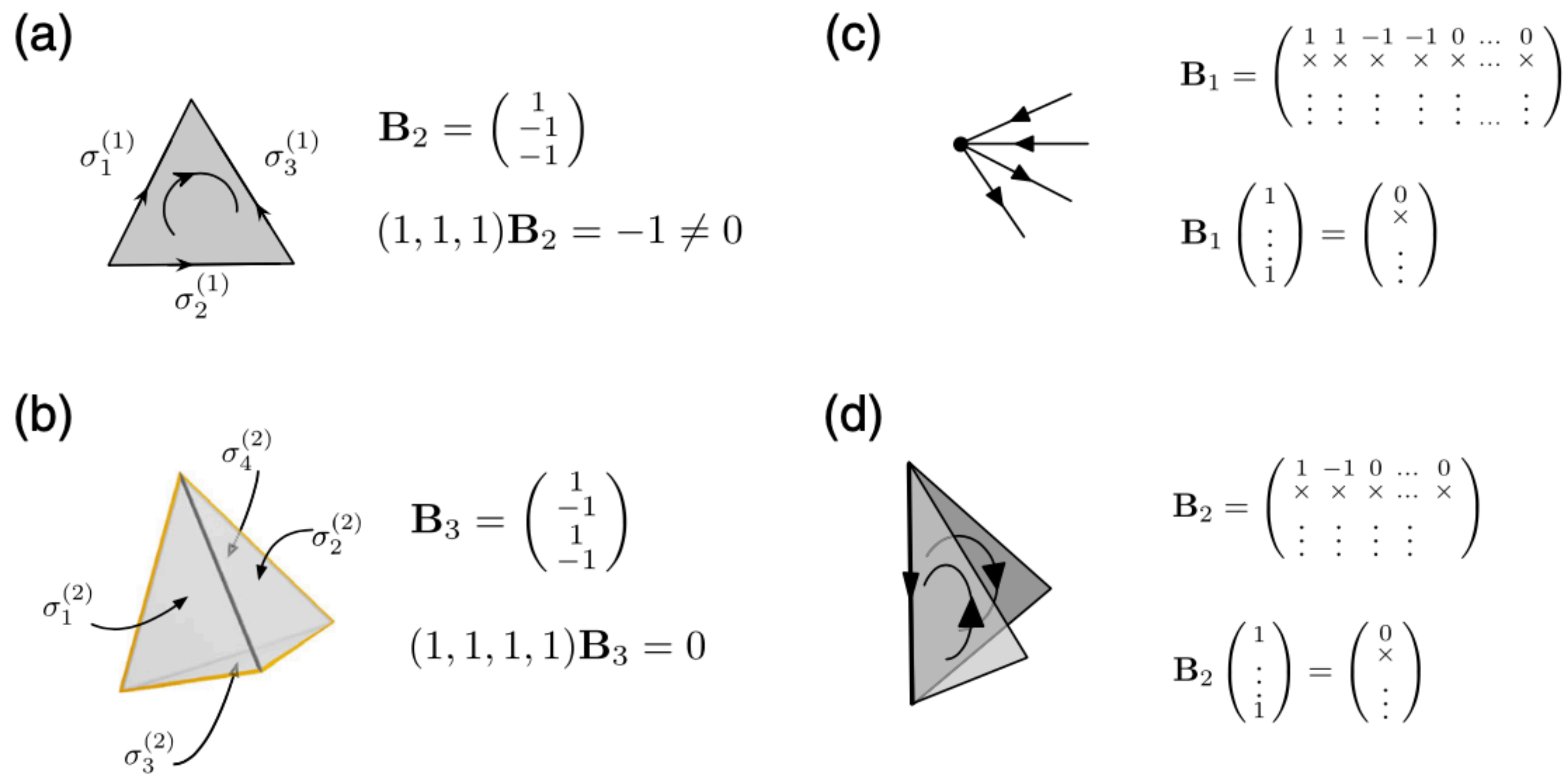
Synchronisation : $\mathbf{x}_i(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, N_k$

$$\left.\frac{d\mathbf{x}_i}{dt}\right|_{\mathbf{x}_i=\mathbf{s}} = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}_j)\Big|_{\mathbf{x}_i=\mathbf{s}} \overset{?}{=} 0$$

Global Topological Synchronisation

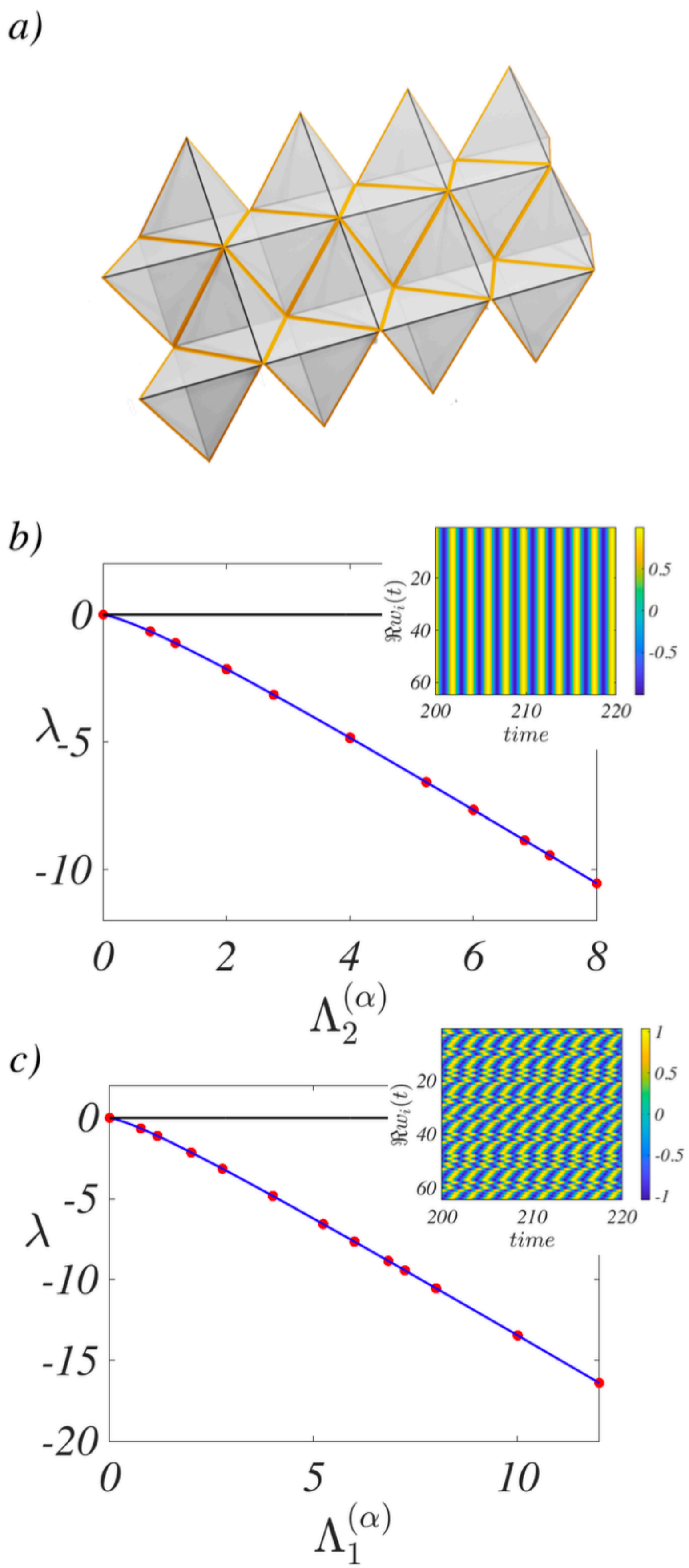
Necessary condition $\mathbf{L}_k u = 0 \quad \Leftrightarrow \quad \mathbf{B}_k u = 0$ and $\mathbf{B}_{k+1}^\top u = 0$

odd dim = non global synch



even dim = global synch if balanced

Global Topological Synchronisation : Stuart-Landau



global synch
for faces (k=2)

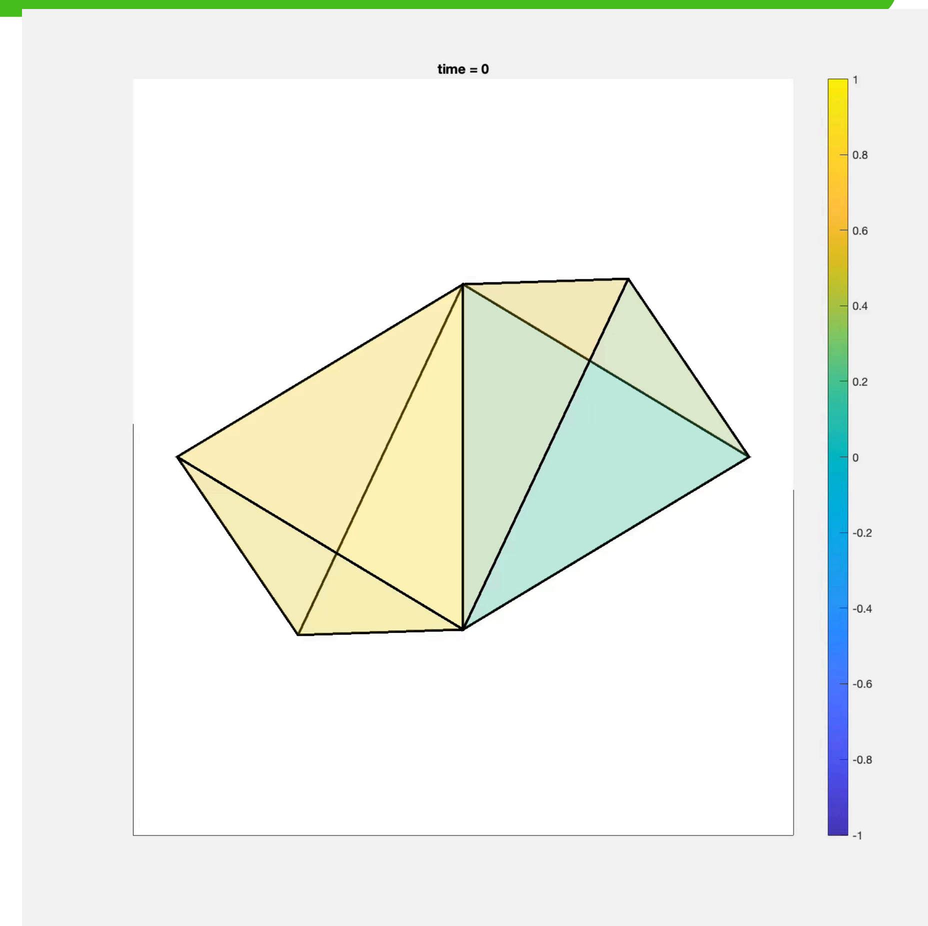
$$\mathbf{B}_2(1,\ldots,1)^\top = 0$$

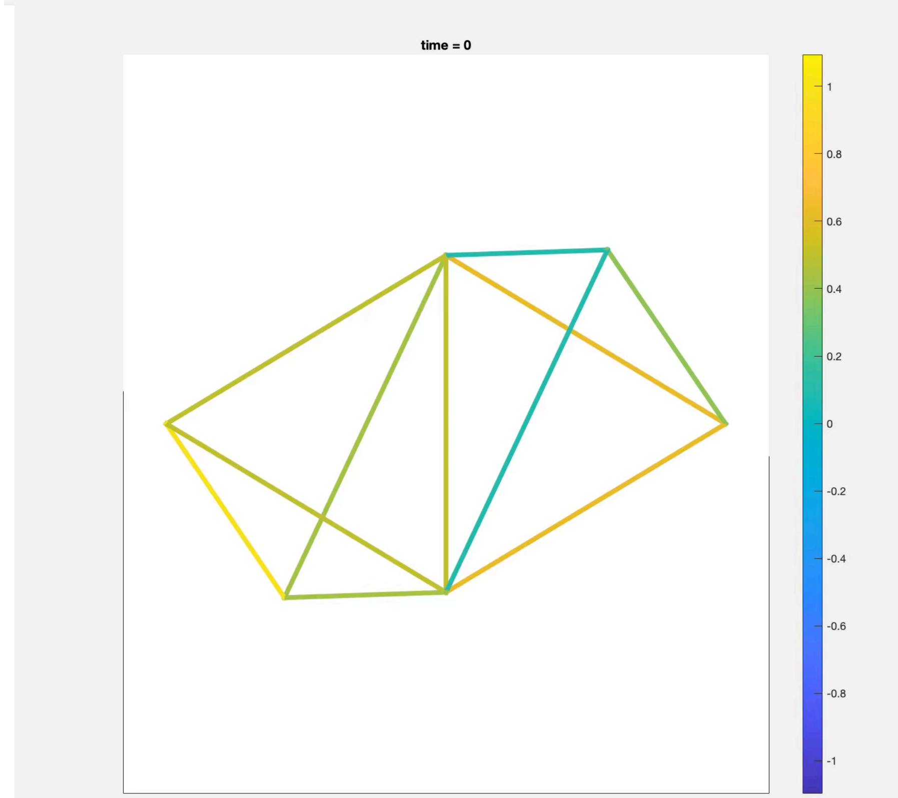
$$\mathbf{B}_3^\top(1,\ldots,1)^\top = 0$$

no global synch
for links (k=1)

$$\mathbf{B}_1(1,\ldots,1)^\top = 0$$

$$\mathbf{B}_2^\top(1,\ldots,1)^\top \neq 0$$





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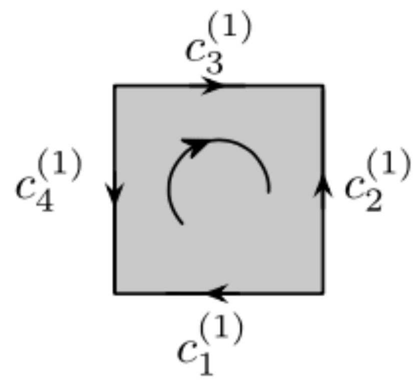
Global Topological Synchronisation

The topological obstruction
<u>does not exist</u> for cell complexes

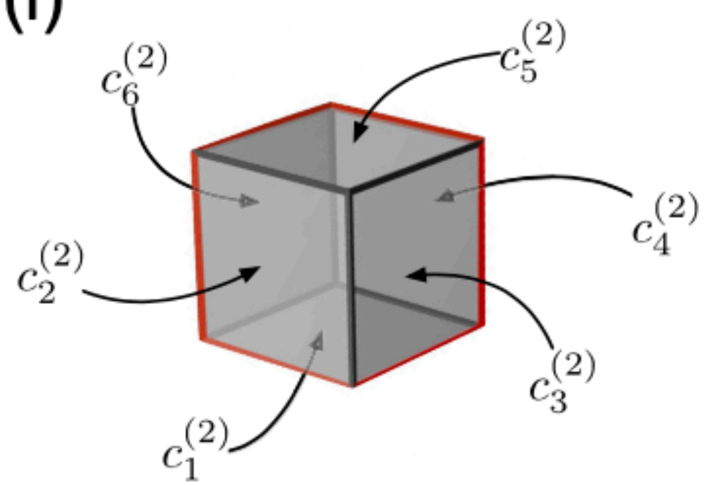
(e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

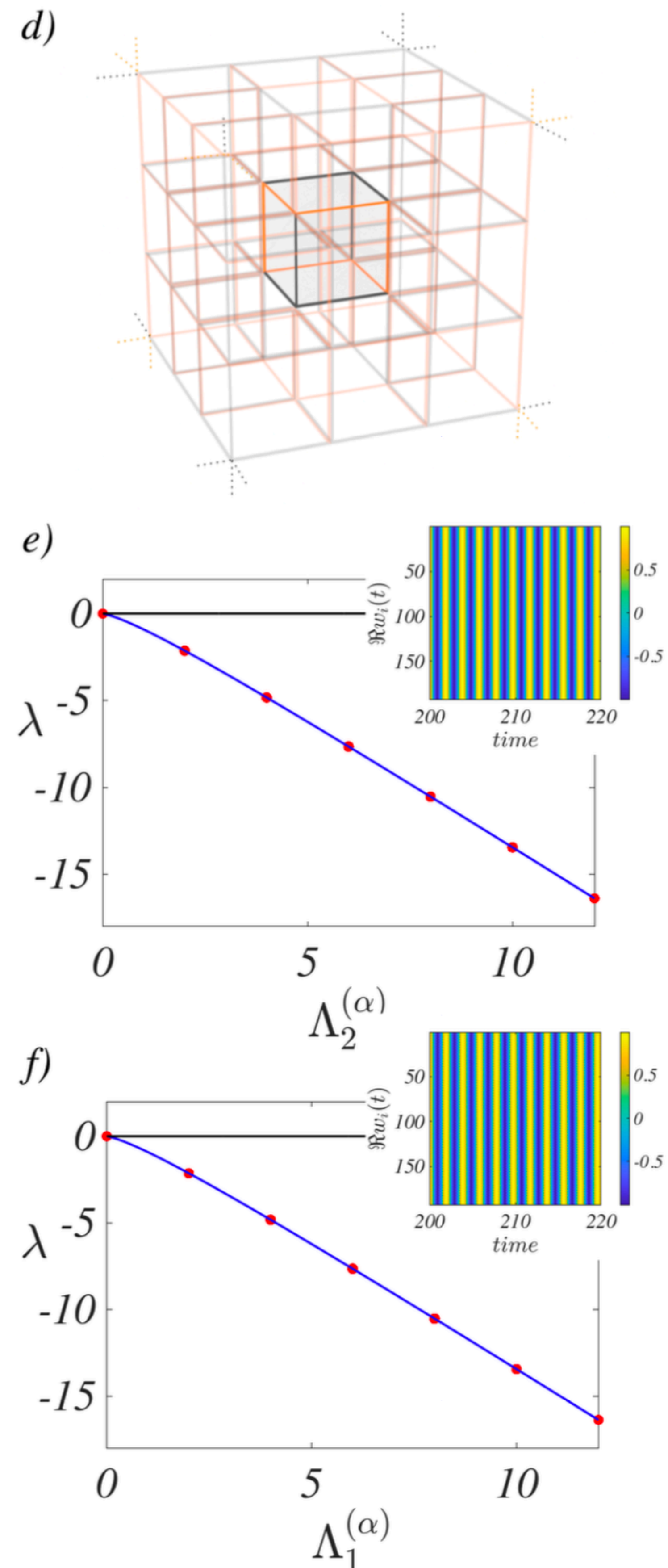
$$(1,1,1,1)\mathbf{B}_2 = 0$$

(f)



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

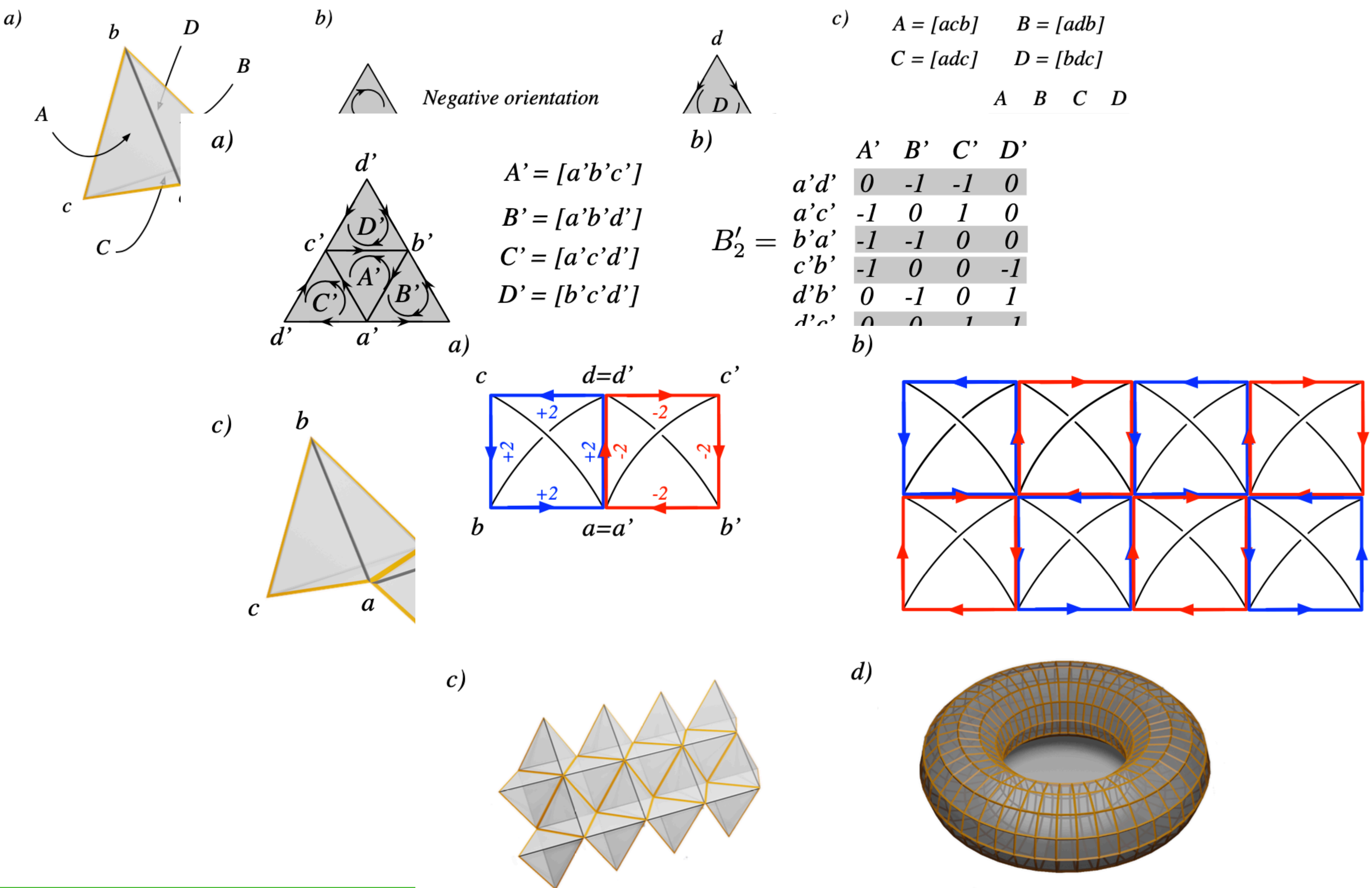
$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$



global synch
for faces

global synch
for links

The "waffle" 3-simplicial complex



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Thank you

Any questions?

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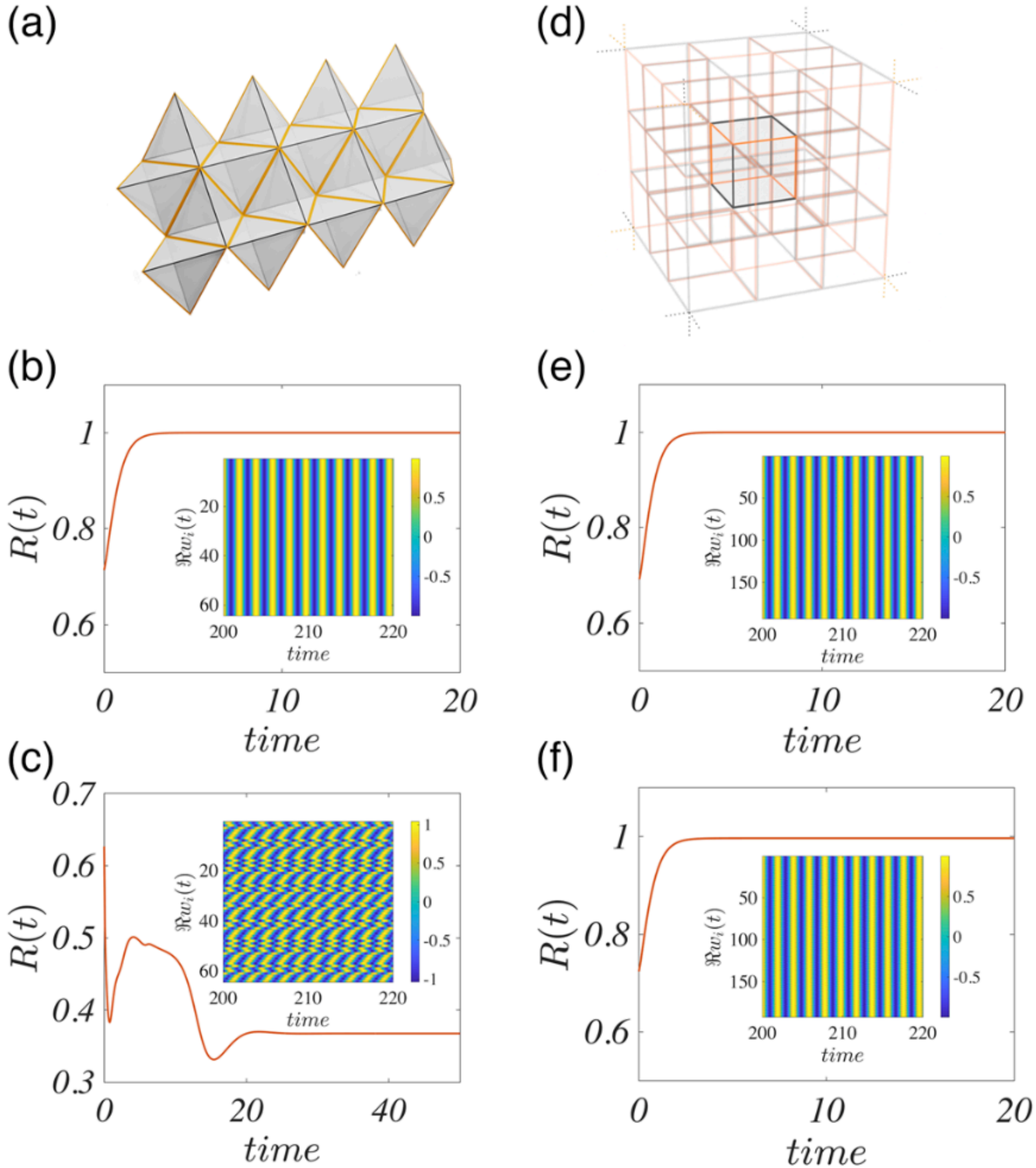
Global Topological Synchronisation : Stuart-Landau

global sync
for faces

no global sync
for links



global sync
for faces

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