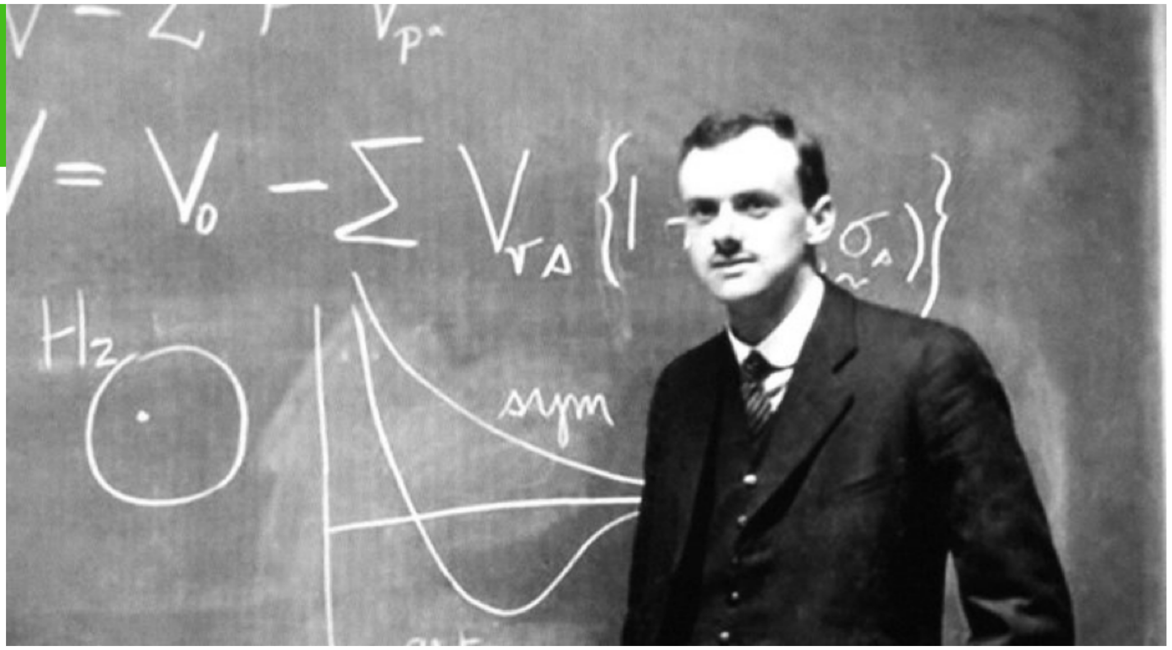
4th May 2023, MATDYNNET

Timoteo Carletti



Global Topological Synchronisation on Simplicial and Cell Complexes

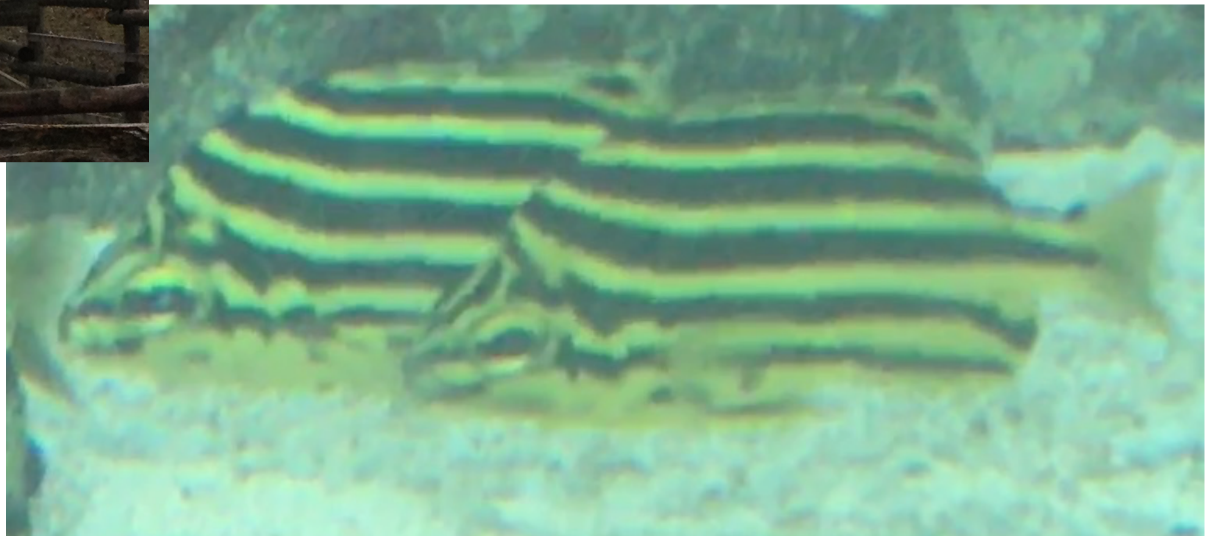
Namur Institute for Complex Systems

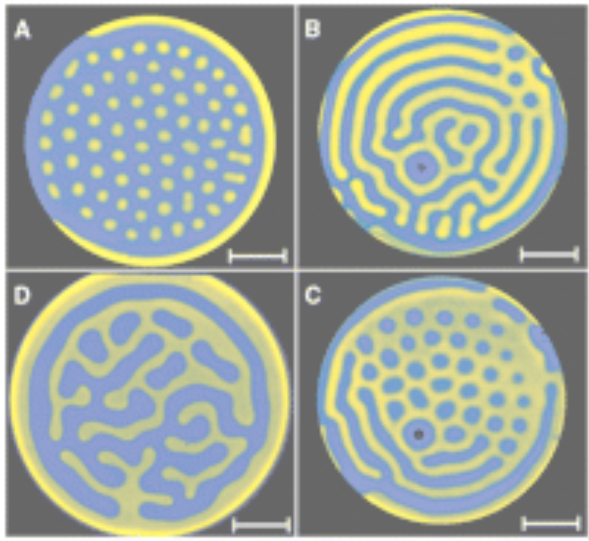
Department of mathematics

UNIVERSITÉ DE NAMUR

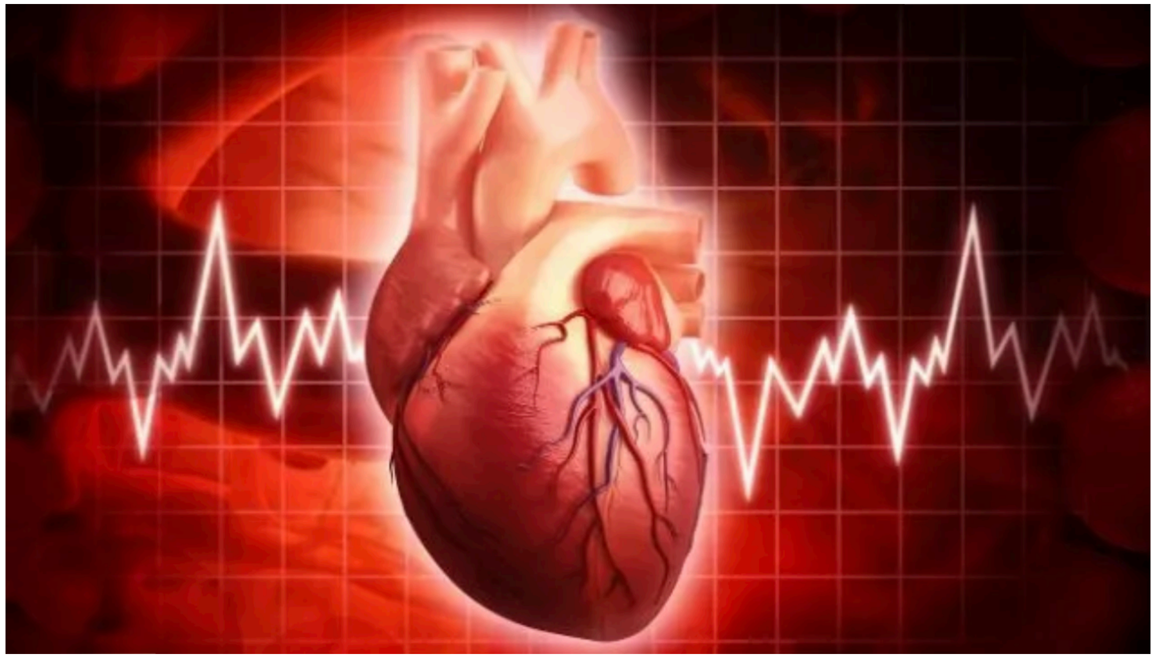
timoteo.carletti@unamur.be

Order from disorder …



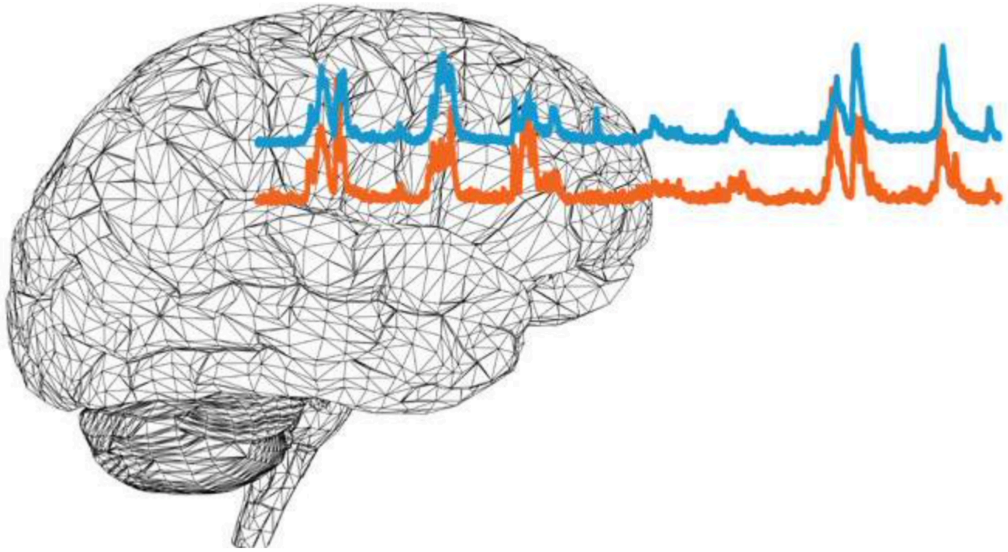


Synchronisation



www.youtube.com

www.quantamagazine.org



Kuramoto model

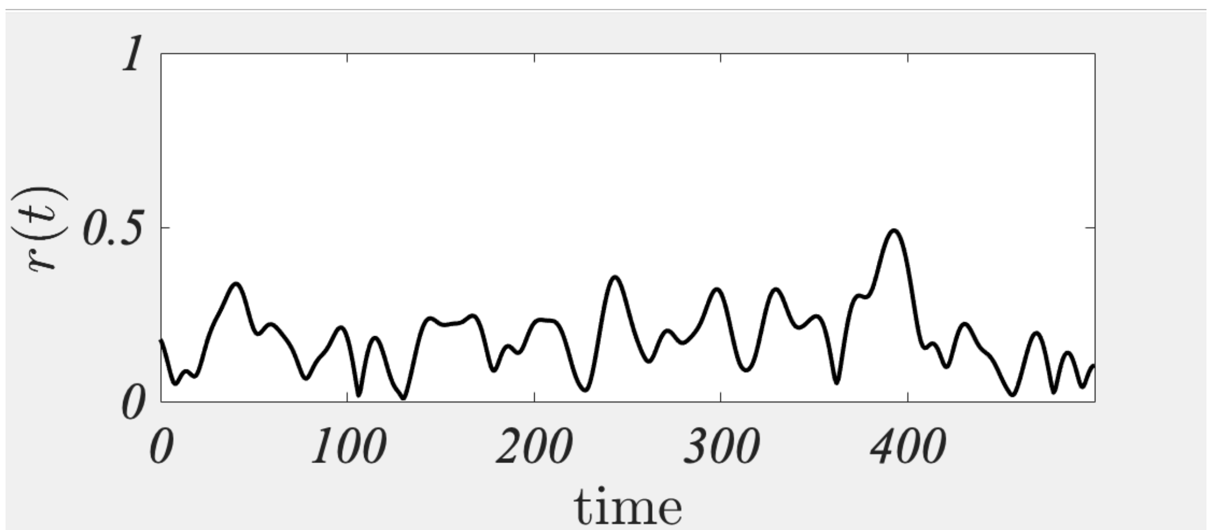
$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

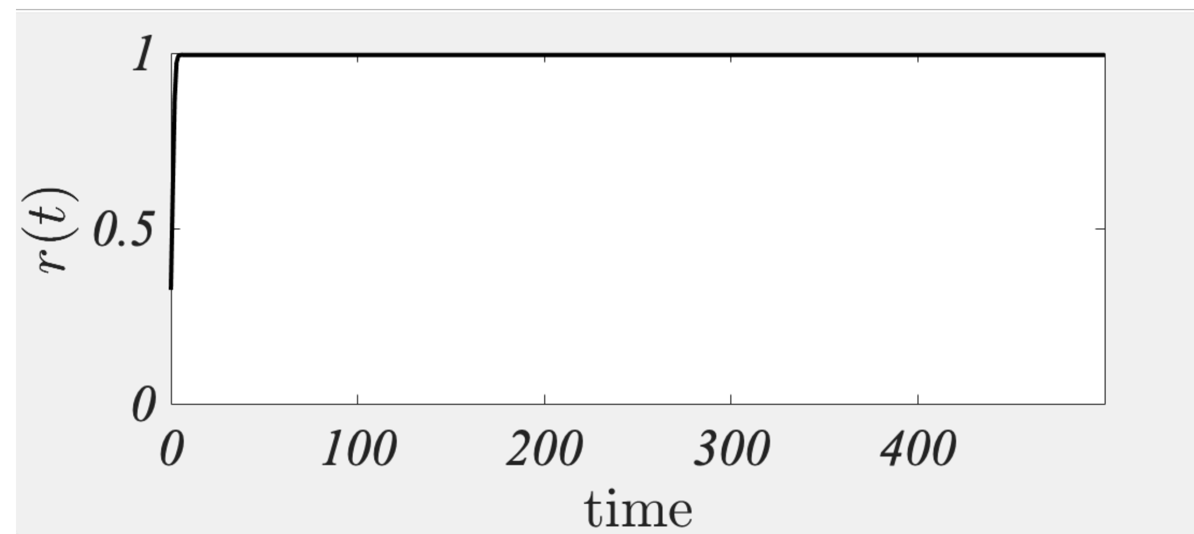
order parameter

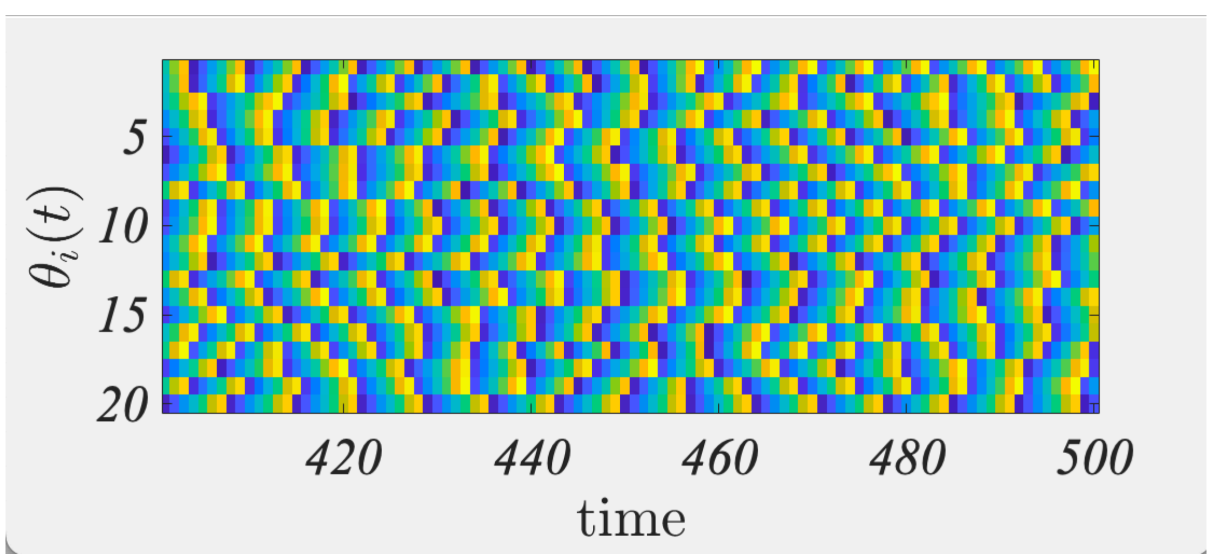
$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

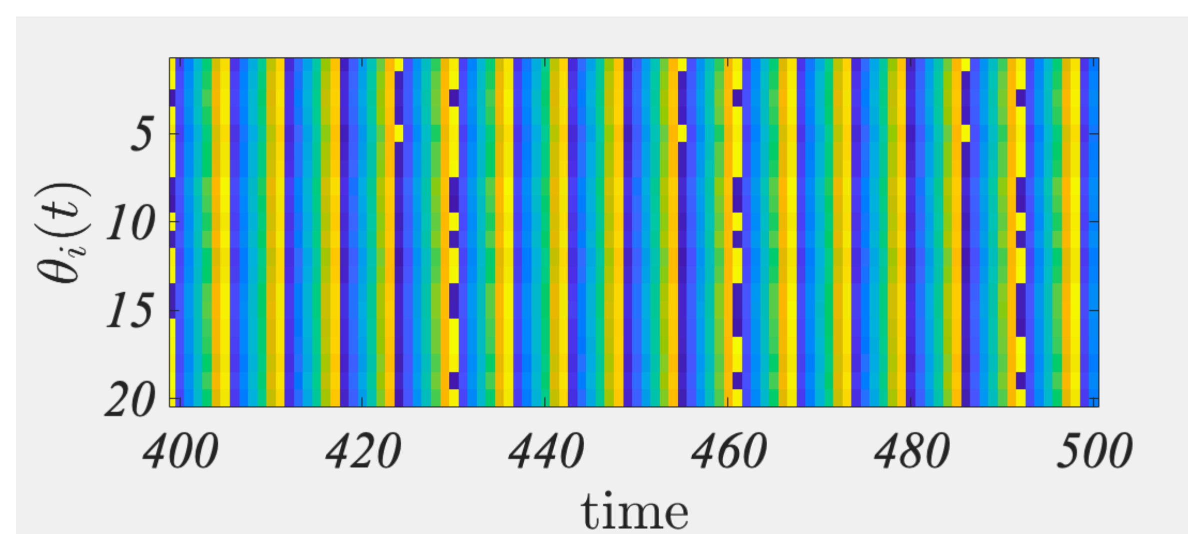
$K = 0.01$

$K = 1.0$

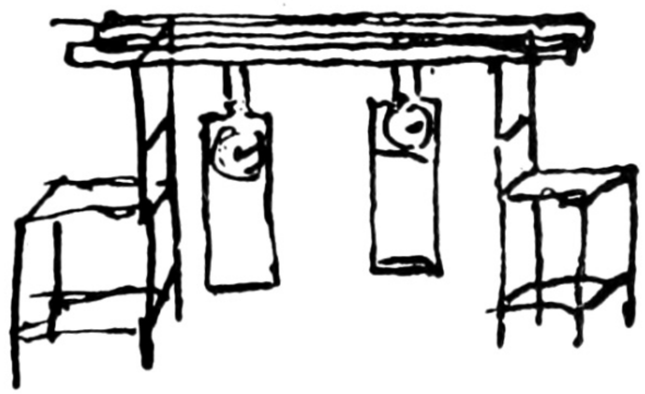


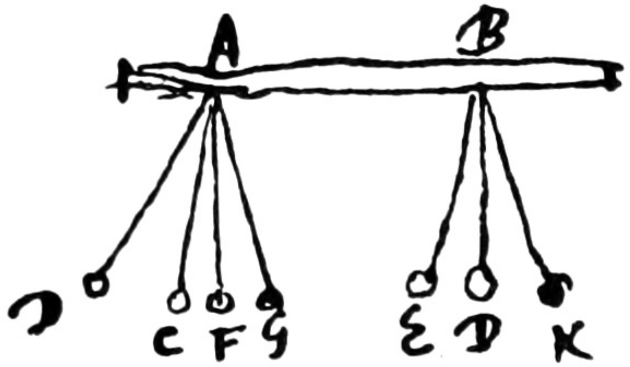






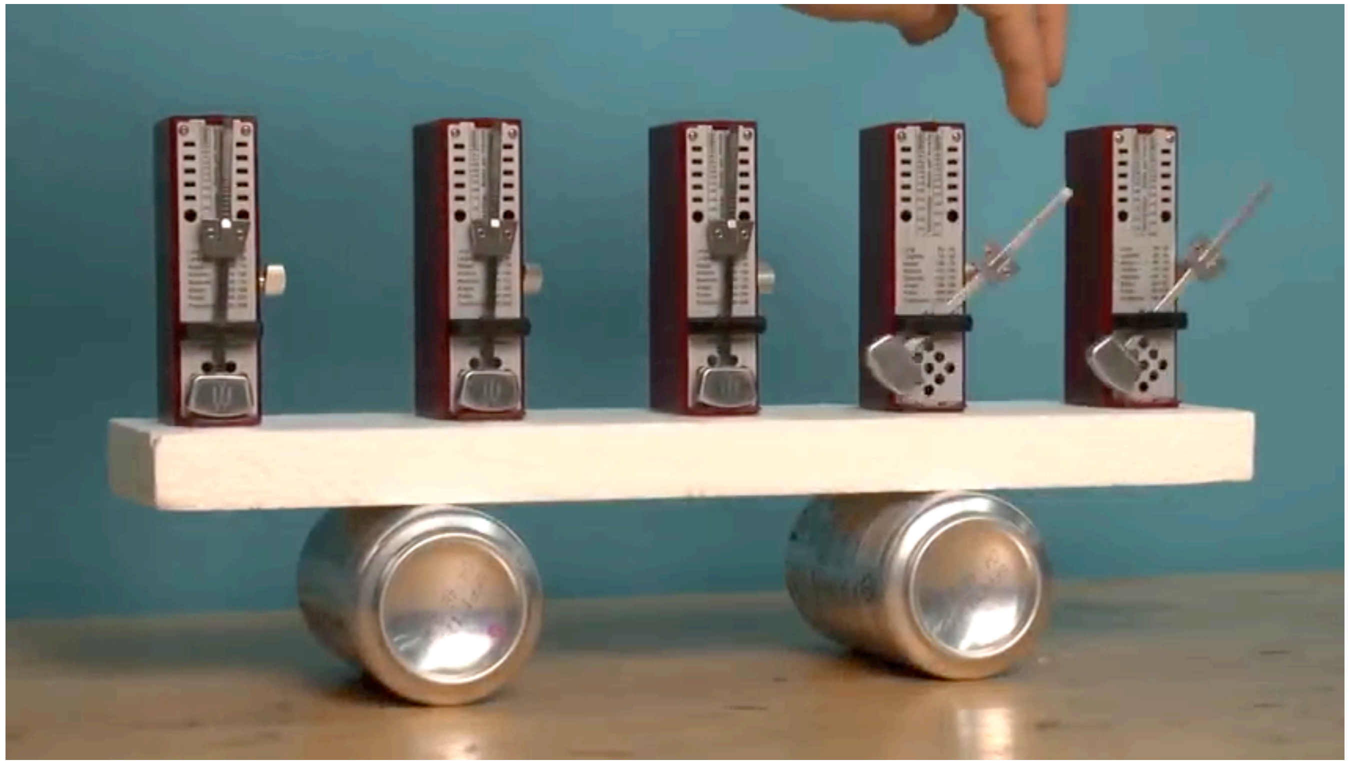
Global Synchronisation





Huygen

"An odd kind of sympathy"



www.youtube.com

Global Synchronisation: Pecora et al.

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

PHYSICAL REVIEW E **80**, 036204 (2009)

Generic behavior of master-stability functions in coupled nonlinear dynamical systems

Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]
[1]*School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[2]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Code 6362, Naval Research Laboratory, Washington, DC 20375, USA*
(Received 9 June 2009; published 15 September 2009)

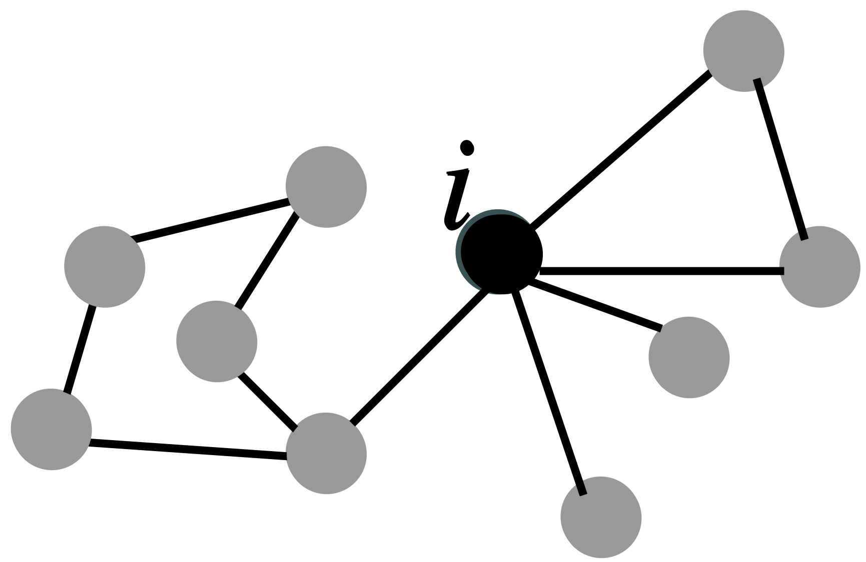
Global Synchronisation: rush course

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} A_{ij}\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

$$= \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})$$

$$\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \mathbf{h}(\mathbf{x}^{(j)}) - \mathbf{h}(\mathbf{x}^{(i)})$$

Diffusive-like coupling

Global Synchronisation: rush course

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

Synchronisation : $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

Does the whole system admit such (spatially) homogeneous solution?

$$\frac{d\mathbf{x}^{(i)}}{dt}\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = 0$$

$$\mathbf{L}\mathbf{u} = 0 \qquad \mathbf{u} = (1, \dots, 1)^{\top}$$

Laplace matrix

Global Synchronisation: rush course

Is $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \ldots, n$ stable?

- $\delta\mathbf{x}^{(i)}(t) = \mathbf{x}^{(i)}(t) - \mathbf{s}(t) \quad \forall i = 1, \ldots, n$

- $$\frac{d\delta\mathbf{x}^{(i)}}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}^{(i)} + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}^{(j)}$$

Time dependent linear system

Global Synchronisation: rush course

- $\mathbf{L}\phi^{(\alpha)} = \Lambda^{(\alpha)}\phi^{(\alpha)} \qquad \phi^{(\alpha)} \cdot \phi^{(\beta)} = \delta_{\alpha\beta} \qquad \Lambda^{(1)} = 0 \quad \Lambda^{(\alpha)} < 0$
- $\delta\mathbf{x}^{(i)} = \sum_{\alpha} \delta\mathbf{x}_\alpha \phi_i^{(\alpha)}$
- $\frac{d\delta\mathbf{x}_\alpha}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}_\alpha + \sigma\Lambda^{(\alpha)}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}_\alpha := \mathbf{J}_\alpha(\mathbf{s}(t))\delta\mathbf{x}_\alpha$

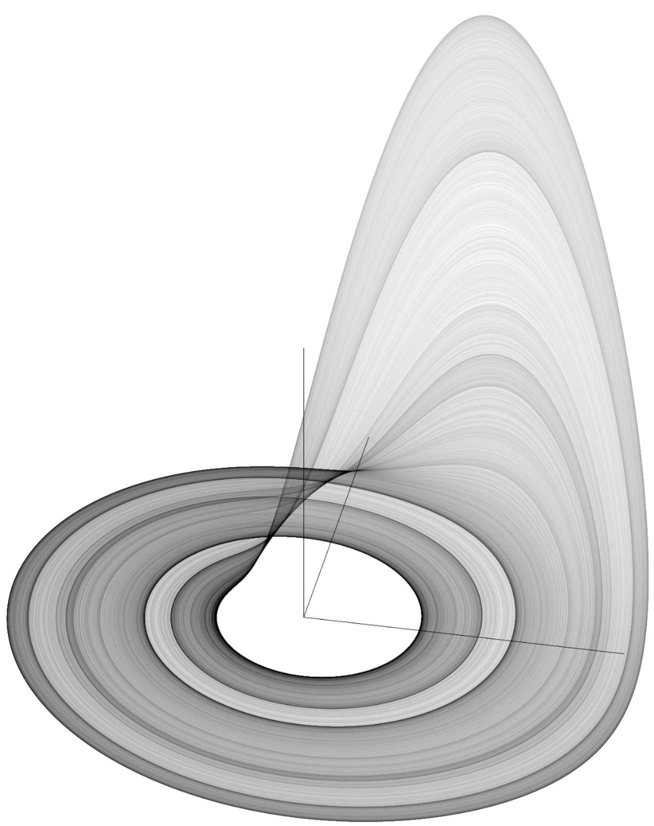
$\lambda(\Lambda^{(\alpha)})$ Master Stability Function = largest Lyapunov exponent of $\mathbf{J}_\alpha(\mathbf{s}(t))$

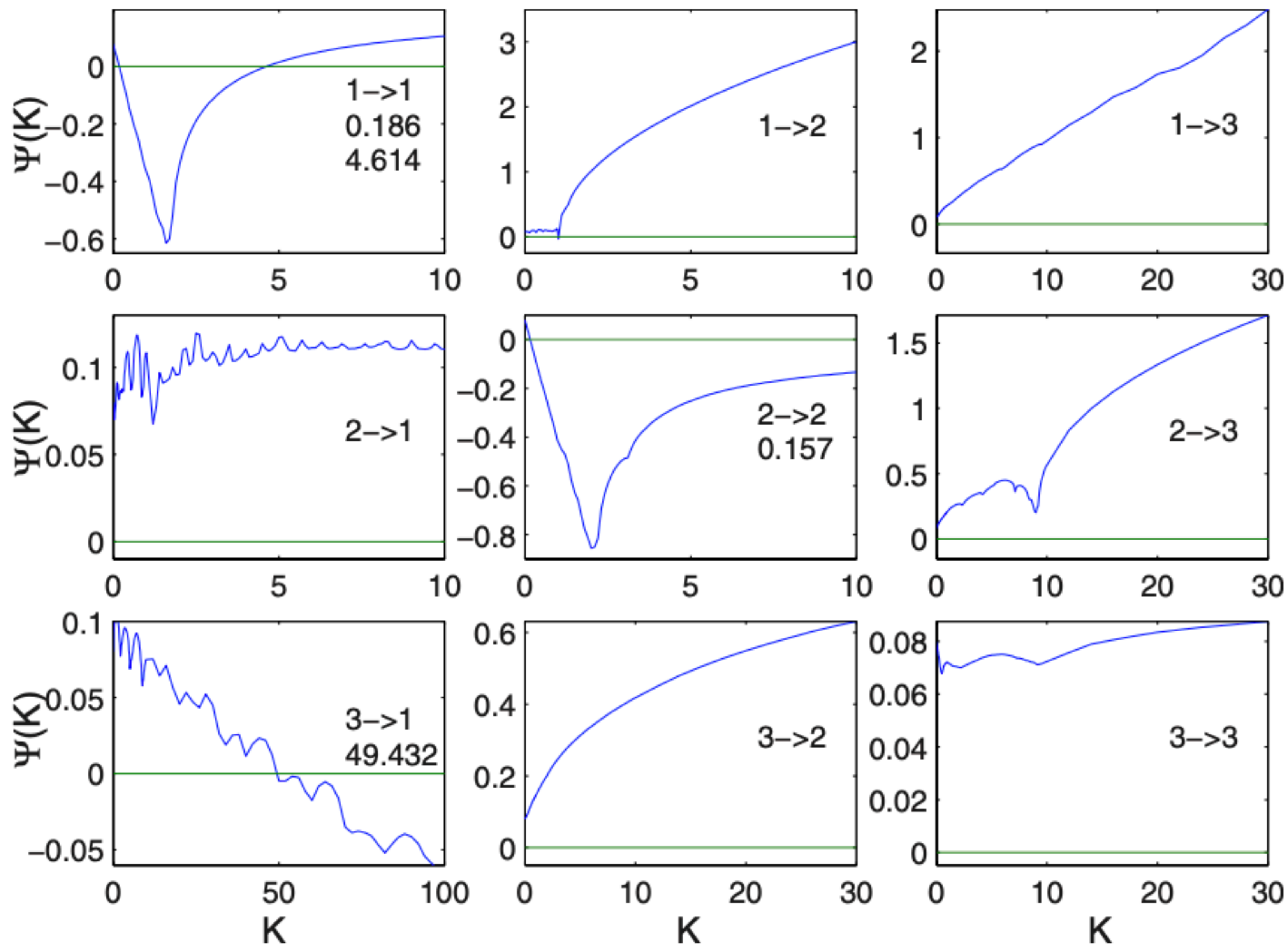
(function of $\Lambda^{(\alpha)}$)

timoteo.carletti@unamur.be

Global Synchronisation: Rössler oscillator

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + \alpha y, \\ \dot{z} = \beta + (x - \gamma) z, \end{cases}$$





$$K = -\sigma \Lambda^{(\alpha)}$$

PHYSICAL REVIEW E **80**, 036204 (2009)

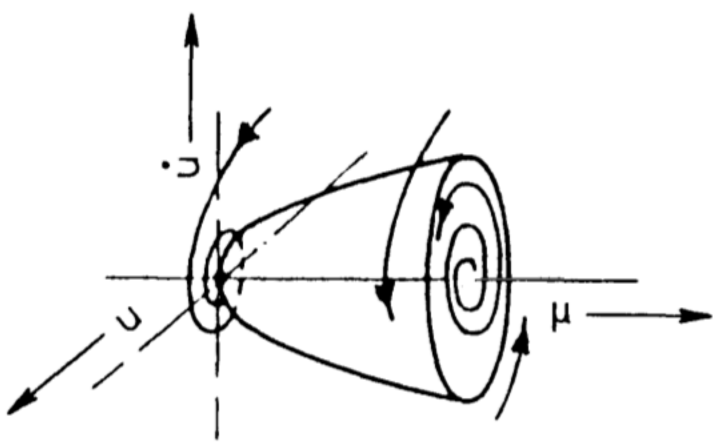
Generic behavior of master-stability functions in coupled nonlinear dynamical systems

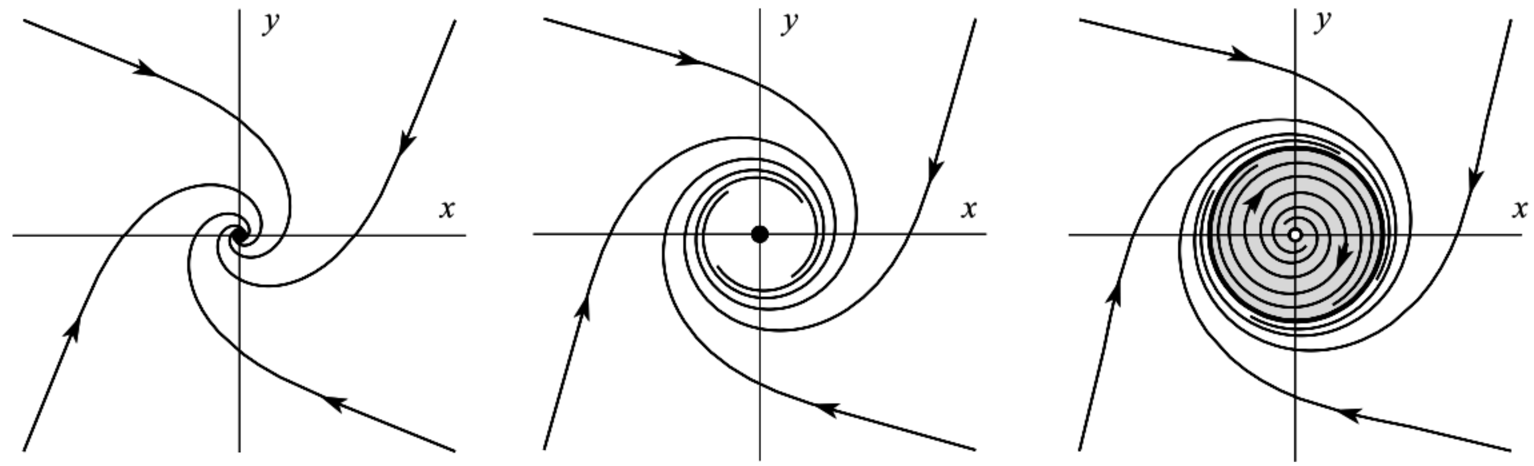
Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

Hopf Bifurcation



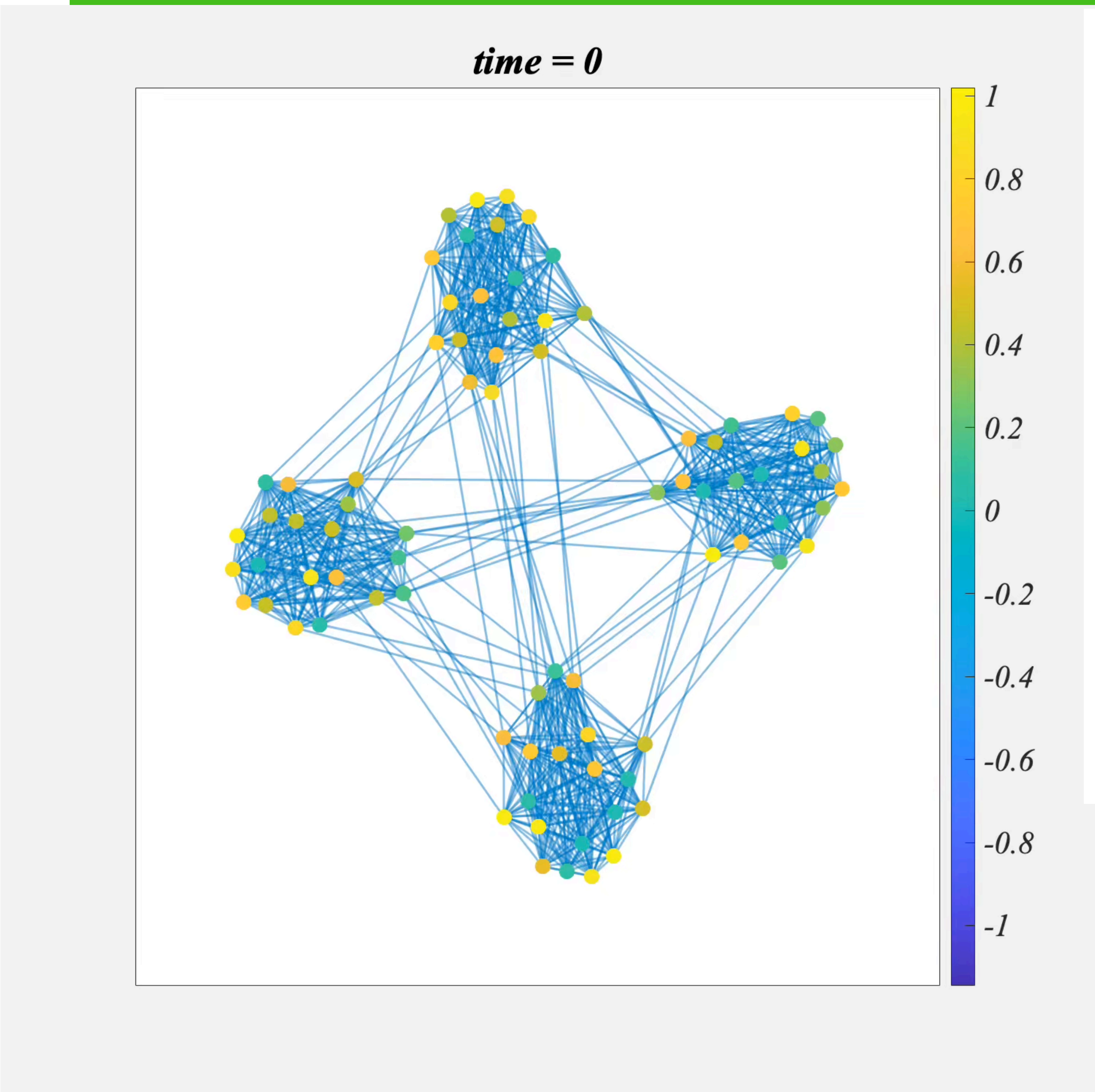


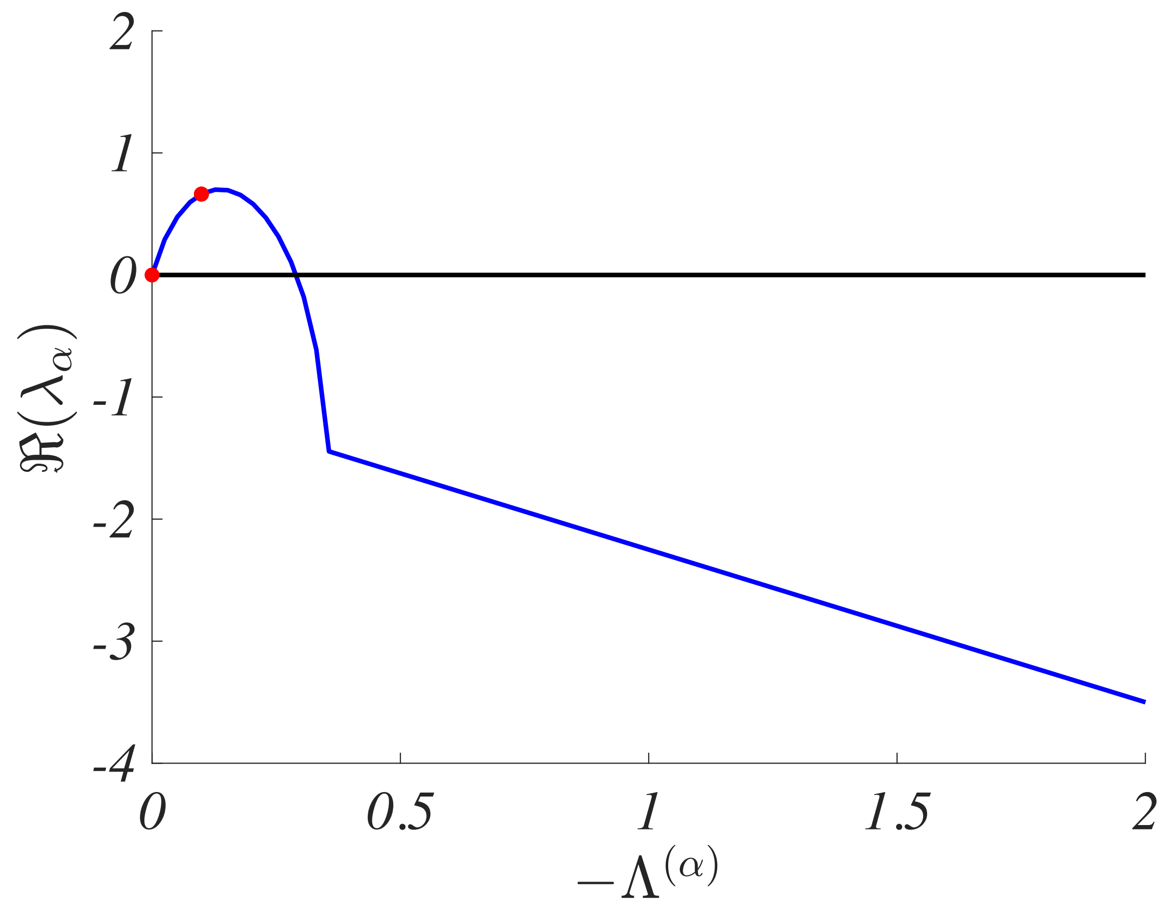
$a < 0$ $\qquad a = 0$ $\qquad a > 0$

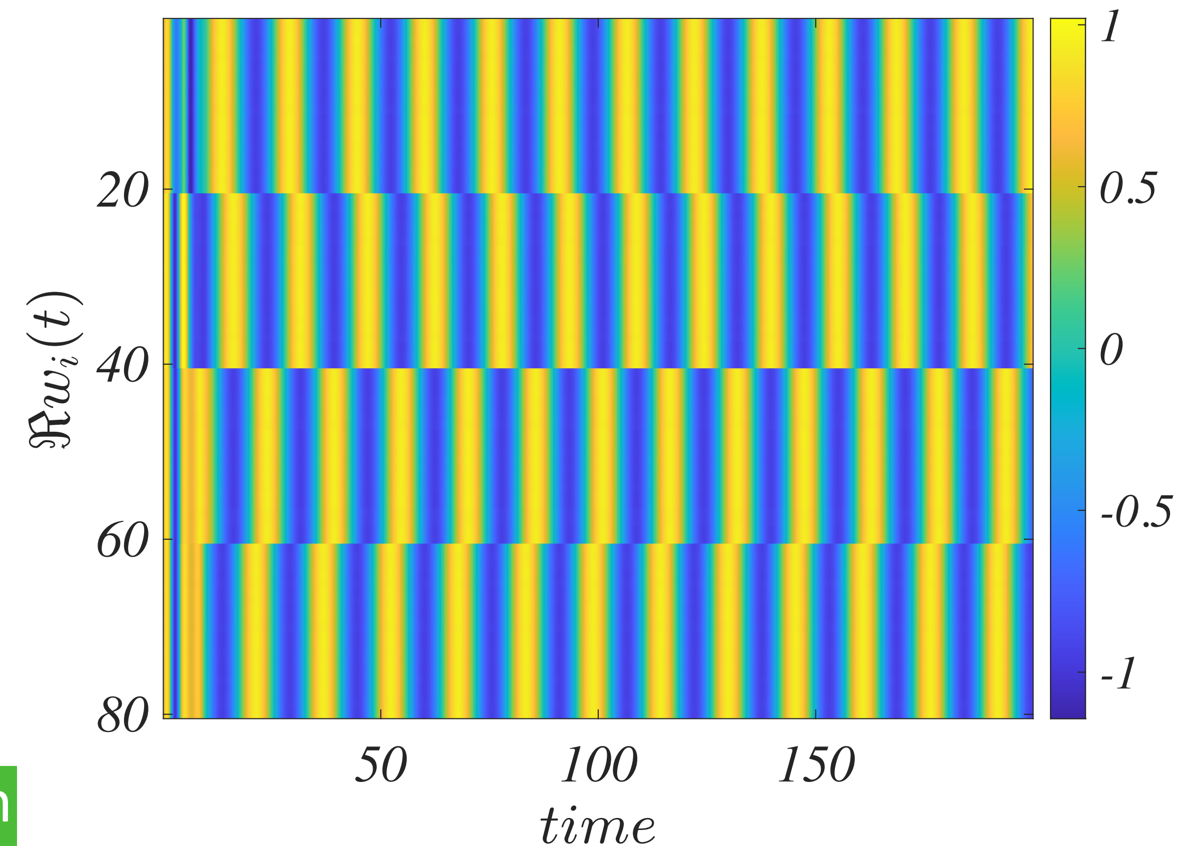
$$\frac{dz^{(j)}}{dt} = z_j(a + ib - |z_j|^2) + \mu \sum_{j=1}^{n} A_{j\ell} \left[h(z^{(\ell)}) - h(z^{(j)}) \right]$$

naxys Namur Center for Complex Systems

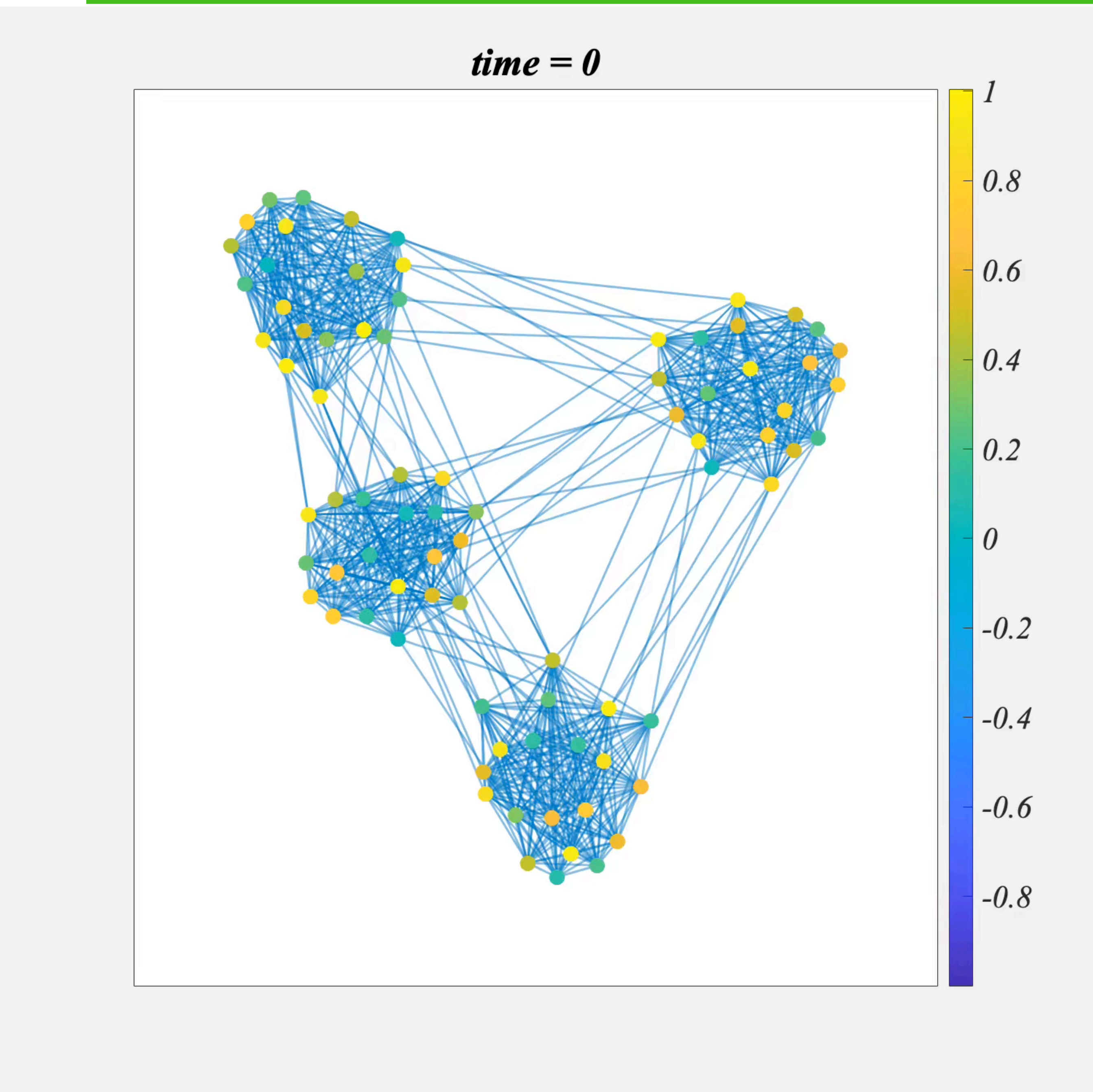
Stuart - Landau oscillator : no synch

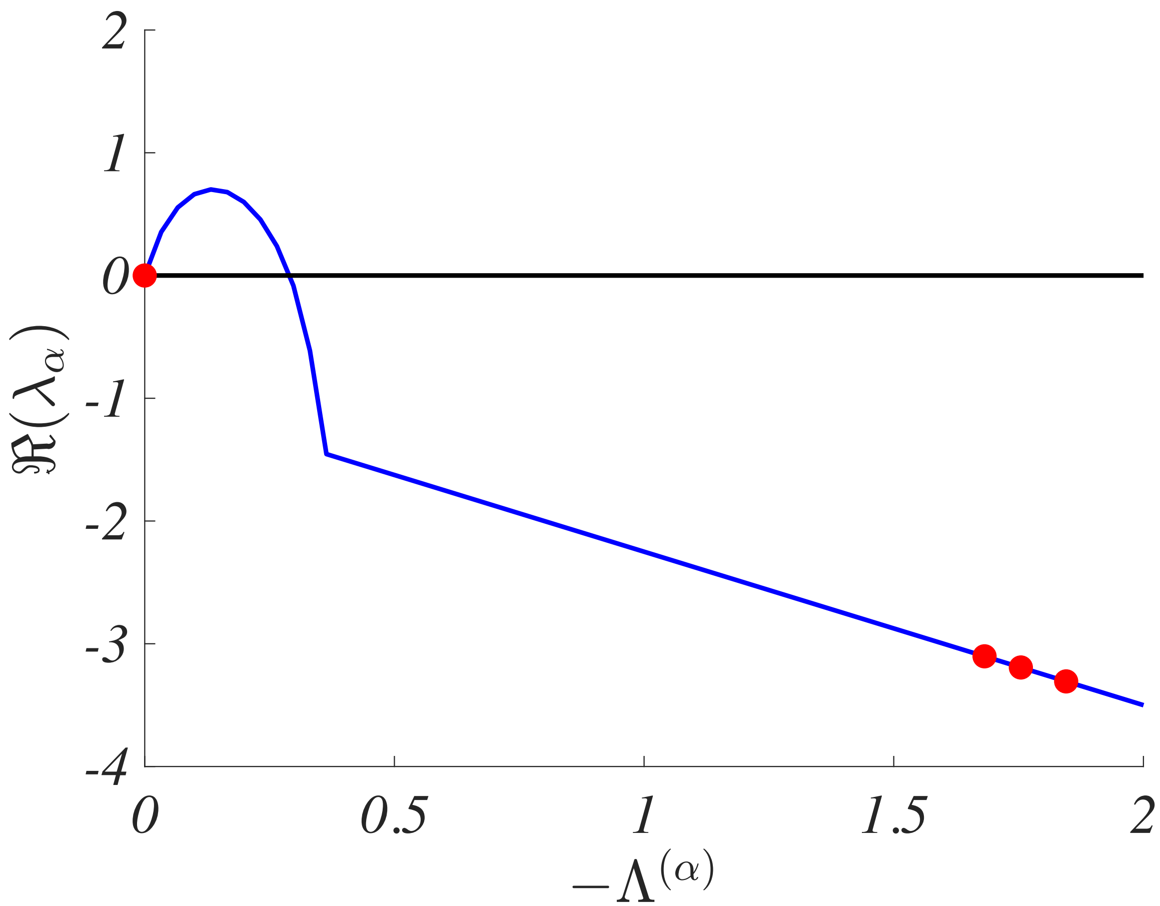


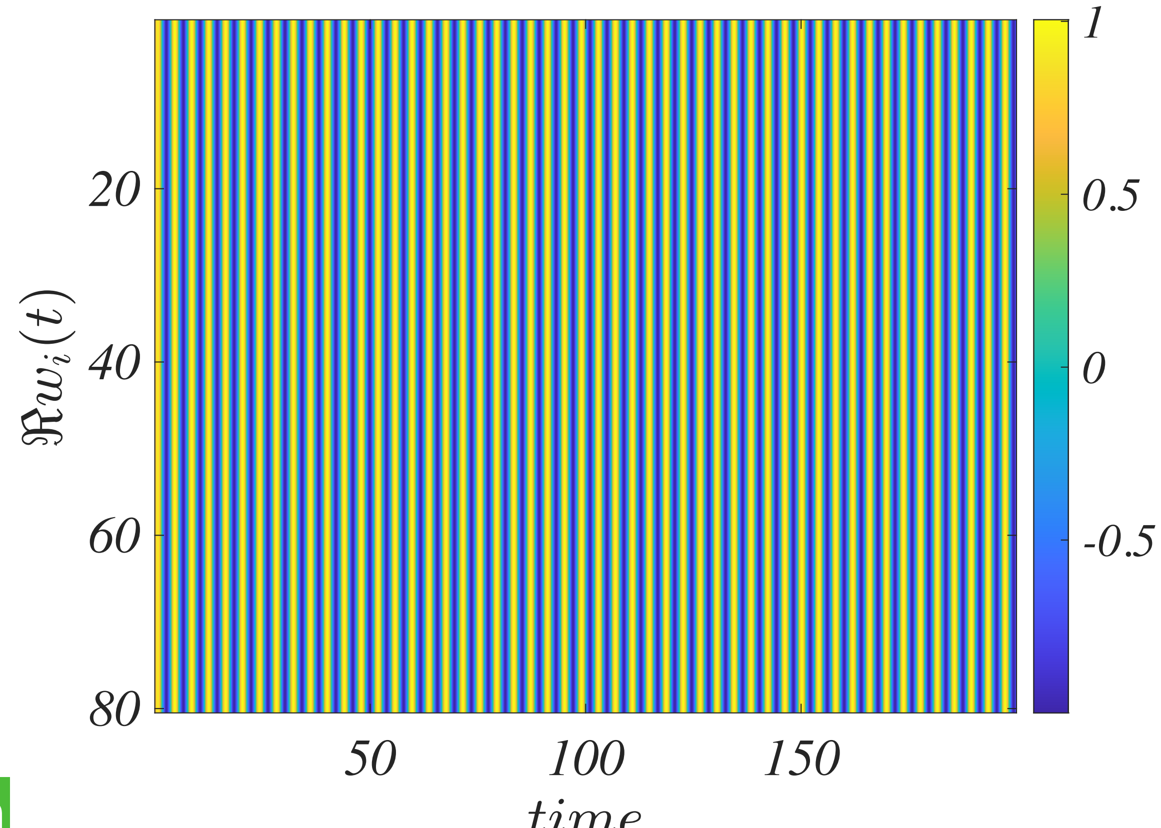




Stuart - Landau oscillator : synch







Global Synchronisation : beyond networks

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PAPER

Dynamical systems on hypergraphs

CrossMark

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Timoteo Carletti[1,4], Duccio Fanelli[2] and Sara Nicoletti[2,3]

1 naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge, 8 B5000 Namur, Belgium
2 Università degli Studi di Firenze, Dipartimento di Fisica e Astronomia, CSDC and INFN, via G. Sansone 1, 50019 Sesto Fiorentino, Italy
3 Dipartimento di Ingegneria dell'Informazione, Università di Firenze, Via S. Marta 3, 50139 Florence, Italy
4 Author to whom any correspondence should be addressed.

E-mail: timoteo.carletti@unamur.be

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ARTICLE

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Stability of synchronization in simplicial complexes

L. V. Gambuzza[1,12], F. Di Patti[2,12], L. Gallo[3,4,12], S. Lepri[2], M. Romance[5], R. Criado[5], M. Frasca[1,6,13]✉, V. Latora[3,4,7,8,13]✉ & S. Boccaletti[2,9,10,11,13]✉

Global Topological Synchronisation

PHYSICAL REVIEW LETTERS **130,** 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

Timoteo Carletti,[1] Lorenzo Giambagli,[1,2] and Ginestra Bianconi[3,4]

[1]*Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, Rue Grafé 2, B5000 Namur, Belgium*

[2]*Department of Physics and Astronomy, University of Florence, INFN and CSDC, 50019 Sesto Fiorentino, Italy*

[3]*School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, United Kingdom*

[4]*The Alan Turing Institute, 96 Euston Road, London, NW1 2DB, United Kingdom*

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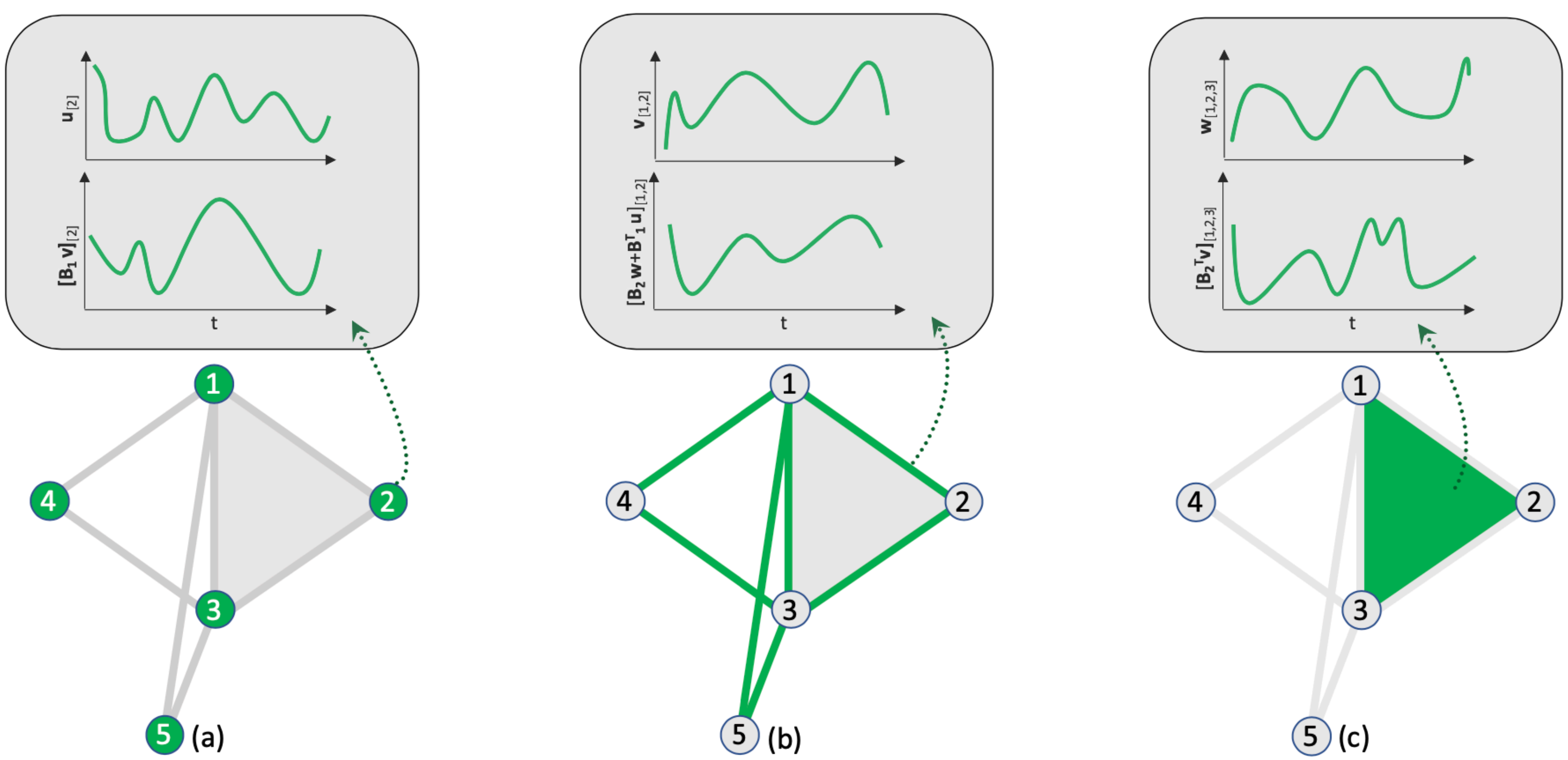


Global Topological Synchronisation

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

$$\mathbf{x} : C_k \to \mathbb{R}^d$$

k-cochain

$$\mathbf{x}(-\sigma_i^{(k)}) = -\mathbf{x}(\sigma_i^{(k)})$$

$$\mathbf{x}_i = \mathbf{x}(\sigma_i^{(k)}) = (x_i^1, \dots, x_i^d)$$

Global Topological Synchronisation

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) \qquad \mathbf{f}(-\mathbf{x}_i) = -\mathbf{f}(\mathbf{x}_i)$$

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N_k} L_k(i,j)\mathbf{h}(\mathbf{x}_j) \qquad \mathbf{h}(-\mathbf{x}_i) = -\mathbf{h}(\mathbf{x}_i)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

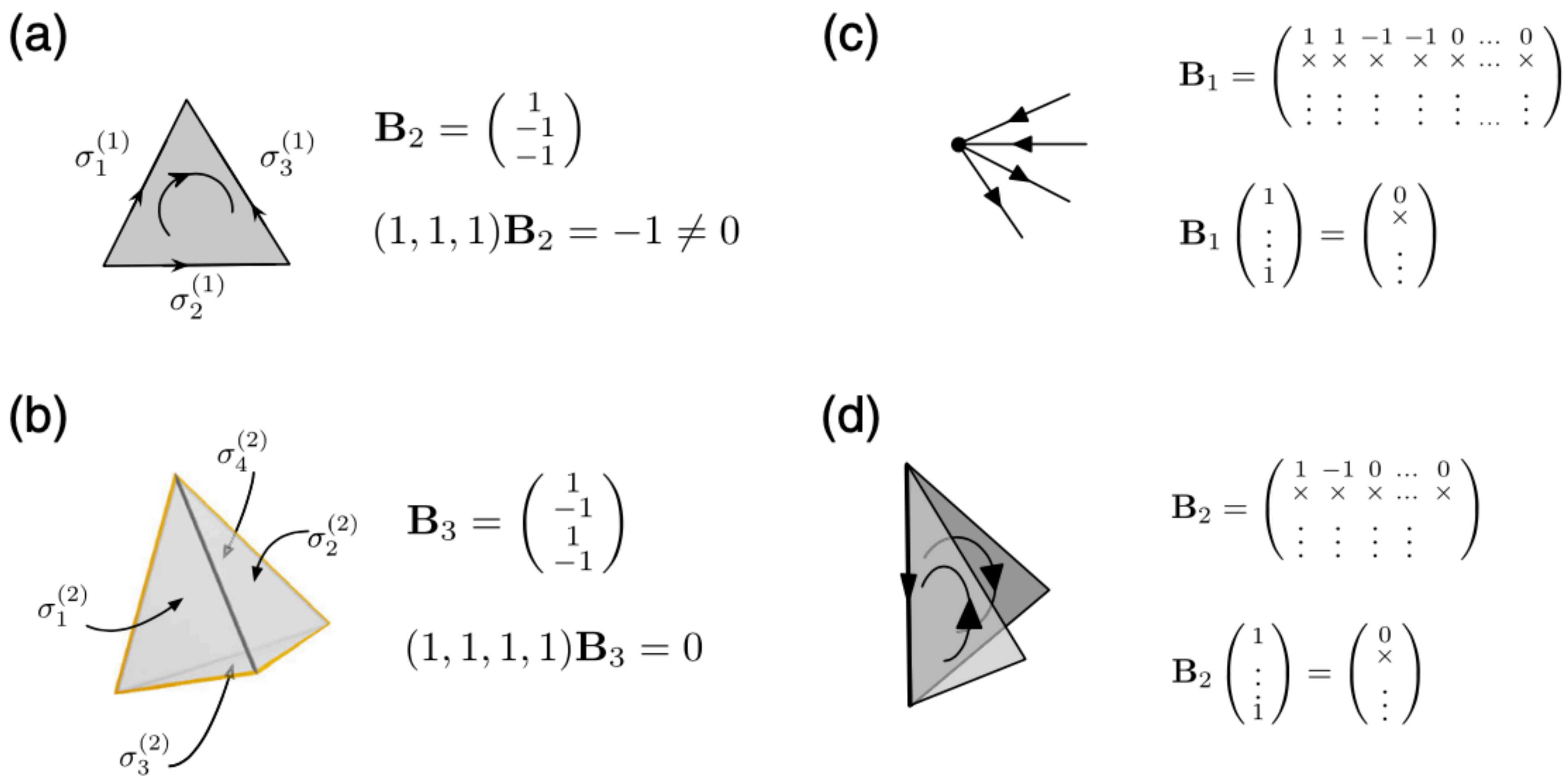
Synchronisation : $\mathbf{x}_i(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, N_k$

$$\left.\frac{d\mathbf{x}_i}{dt}\right|_{\mathbf{x}_i=\mathbf{s}} = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}_j)\Big|_{\mathbf{x}_i=\mathbf{s}} \overset{?}{=} 0$$

Global Topological Synchronisation

Necessary condition $\mathbf{L}_k u = 0 \quad \Leftrightarrow \quad \mathbf{B}_k u = 0$ and $\mathbf{B}_{k+1}^\top u = 0$

odd dim = non global synch

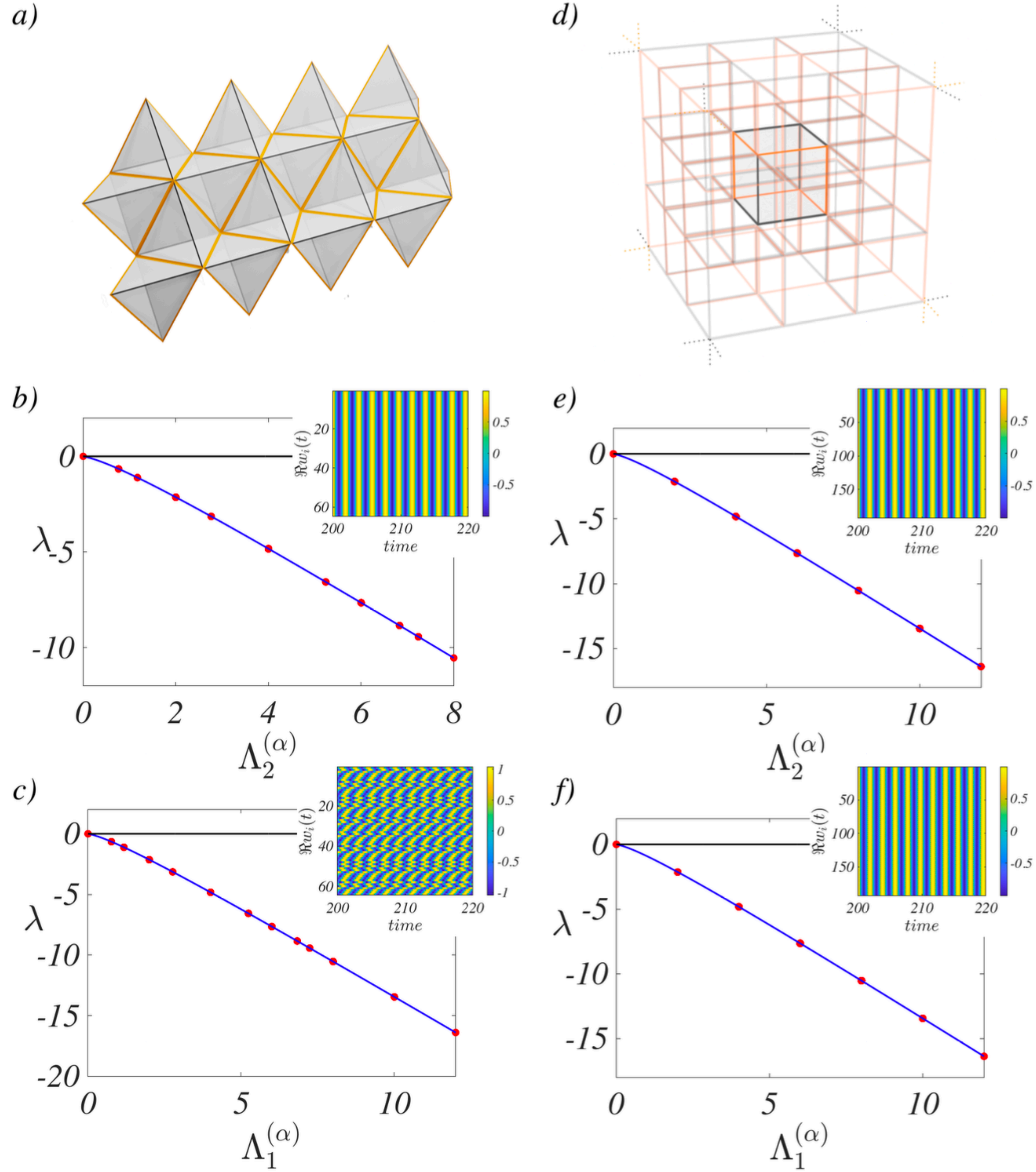


even dim = global synch if balanced

Global Topological Synchronisation : Stuart-Landau

global synch for faces

no global synch for links



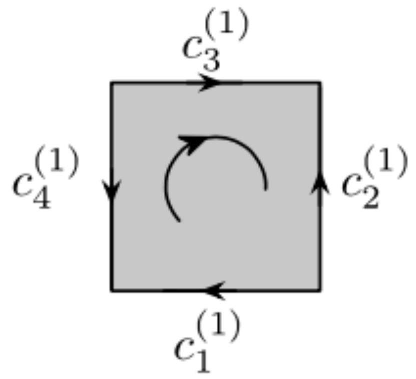
global synch for faces

global synch for links

Global Topological Synchronisation

The topological obstruction does not exist for cell complexes

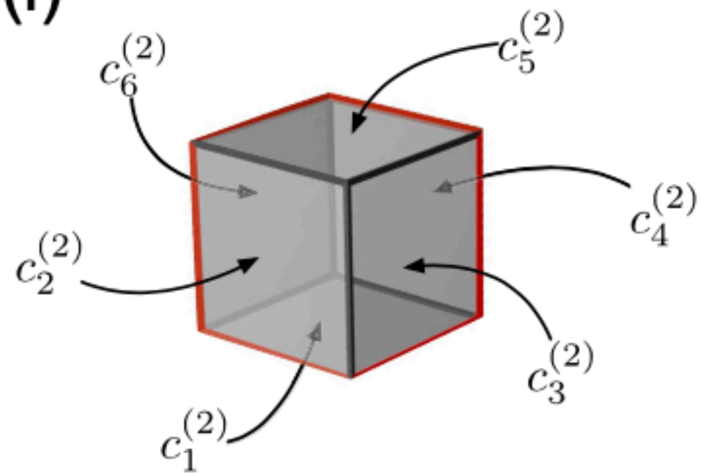
(e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_2 = 0$$

(f)

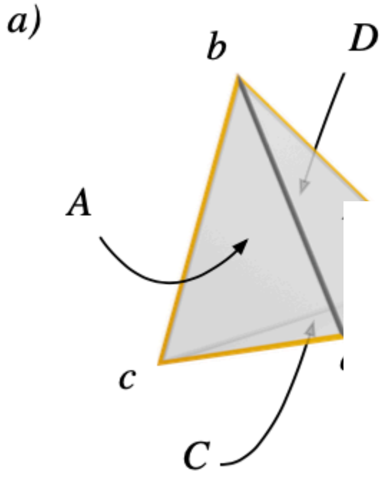


$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$

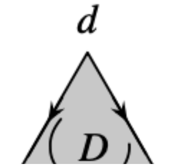
The "waffle" 3-simplicial complex

a)



b)

Negative orientation



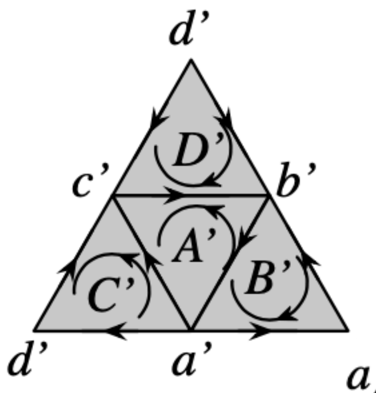
c)

A = [acb] B = [adb]

C = [adc] D = [bdc]

A B C D

a)



A' = [a'b'c']

B' = [a'b'd']

C' = [a'c'd']

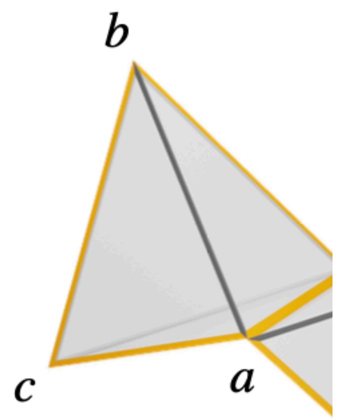
D' = [b'c'd']

b)

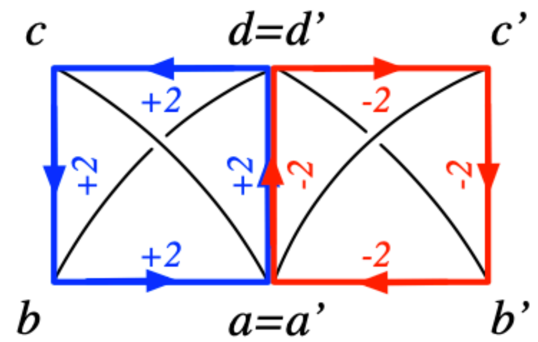
$B'_2 =$

	A'	B'	C'	D'
a'd'	0	-1	-1	0
a'c'	-1	0	1	0
b'a'	-1	-1	0	0
c'b'	-1	0	0	-1
d'b'	0	-1	0	1
d'c'	0	0	1	1

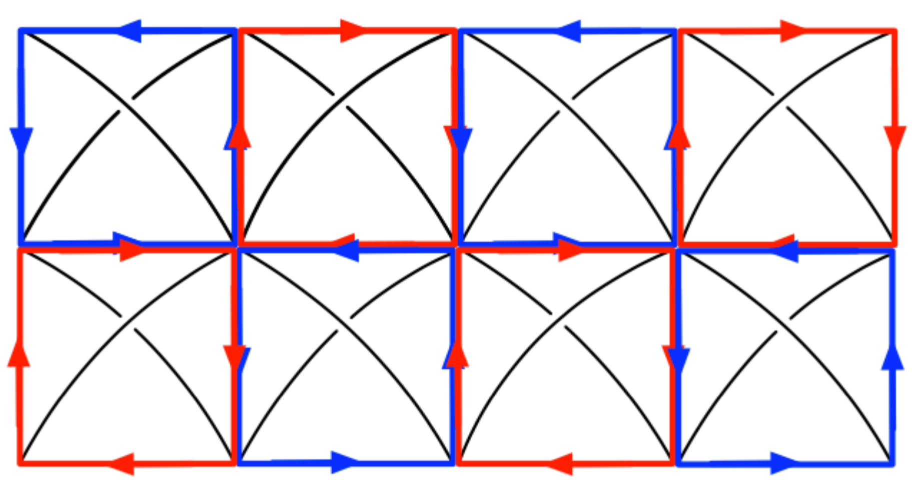
c)



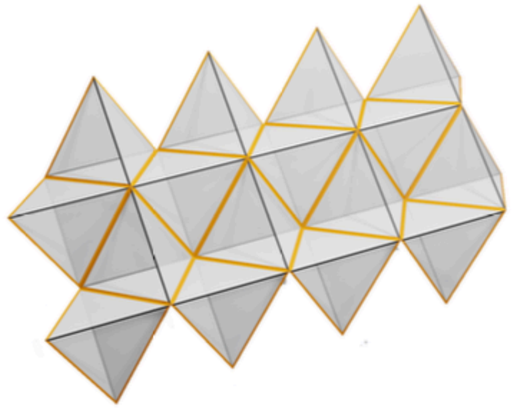
a)



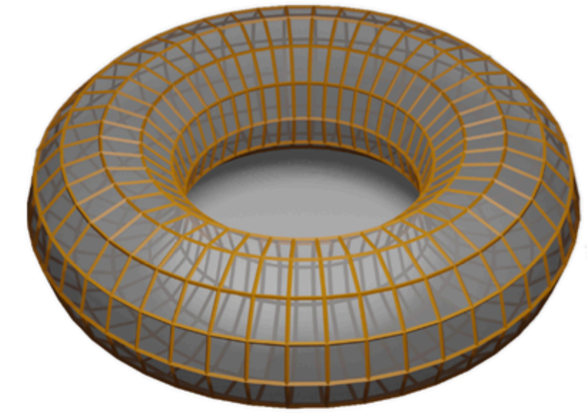
b)



c)



d)



Global Topological Synchronisation : Dirac coupling

$$\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_{N_0})^\top \qquad \mathbf{u}_i \in \mathbb{R}^d$$ 0-cochain (nodes)

$$\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_{N_1})^\top \qquad \mathbf{v}_i \in \mathbb{R}^d$$ 1-cochain (links)

$$\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{N_2})^\top \qquad \mathbf{w}_i \in \mathbb{R}^d$$ 2-cochain (faces)

$$\frac{d\mathbf{u}_i}{dt} = \mathbf{f}_0(\mathbf{u}_i) + \sum_{j=1}^{N_1} B_1(i,j)\mathbf{h}_1(\mathbf{v}_j)$$

$$\frac{d\mathbf{v}_i}{dt} = \mathbf{f}_1(\mathbf{v}_i) + \sum_{j=1}^{N_2} B_2(i,j)\mathbf{h}_2(\mathbf{w}_j) + \sum_{j=1}^{N_0} B_1^\top(i,j)\mathbf{h}_0(\mathbf{u}_j)$$

$$\frac{d\mathbf{w}_i}{dt} = \mathbf{f}_2(\mathbf{w}_i) + \sum_{j=1}^{N_2} B_2^\top(i,j)\mathbf{h}_1(\mathbf{v}_j)$$

Global Topological Synchronisation : Dirac coupling

$$\mathcal{D} = \begin{pmatrix} 0 & \mathbf{B}_1 \otimes \mathbf{1}_d & 0 \\ \mathbf{B}_1^\top \otimes \mathbf{1}_d & 0 & \mathbf{B}_2 \otimes \mathbf{1}_d \\ 0 & \mathbf{B}_2^\top \otimes \mathbf{1}_d & 0 \end{pmatrix}$$

The reference orbits are solution of

$$\frac{d\mathbf{s}_0}{dt} = \mathbf{f}_0(\mathbf{s}_0) \qquad \frac{d\mathbf{s}_1}{dt} = \mathbf{f}_1(\mathbf{s}_1) \qquad \frac{d\mathbf{s}_2}{dt} = \mathbf{f}_2(\mathbf{s}_2)$$

Global synchronisation

$$\mathbf{u}_i(t) = \mathbf{s}_0(t) \qquad \mathbf{v}_i(t) = \mathbf{s}_1(t) \qquad \mathbf{w}_i(t) = \mathbf{s}_2(t)$$

timoteo.carletti@unamur.be

Global Topological Synchronisation : Dirac coupling

$\mathbf{B}_1(1,\dots,1)^\top = 0$ Balanced condition

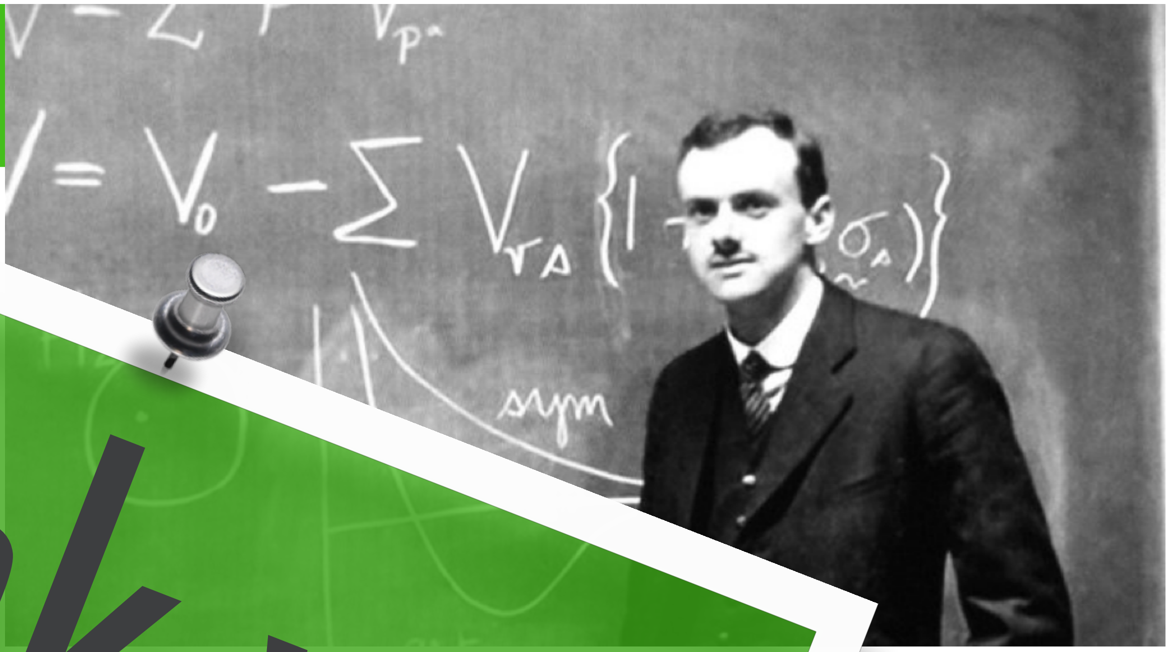
$\mathbf{B}_1^\top(1,\dots,1)^\top = 0$ Ok

$\mathbf{B}_2(1,\dots,1)^\top = 0$ Balanced condition

$\mathbf{B}_2^\top(1,\dots,1)^\top = 0$ Impossible on simplicial complexes

- No global synchronisation with Dirac coupling on generic simplicial complexes
- Global synchronisation with Dirac coupling is possible on cell complexes
- Global synchronisation with Dirac coupling is possible on networks

4th May 2023, MATDYNN



Thank you

Timoteo Carletti

Global Topological Synchronisation on Simplicial and Cell Complexes

Any questions?

Namur Institute for Complex Systems

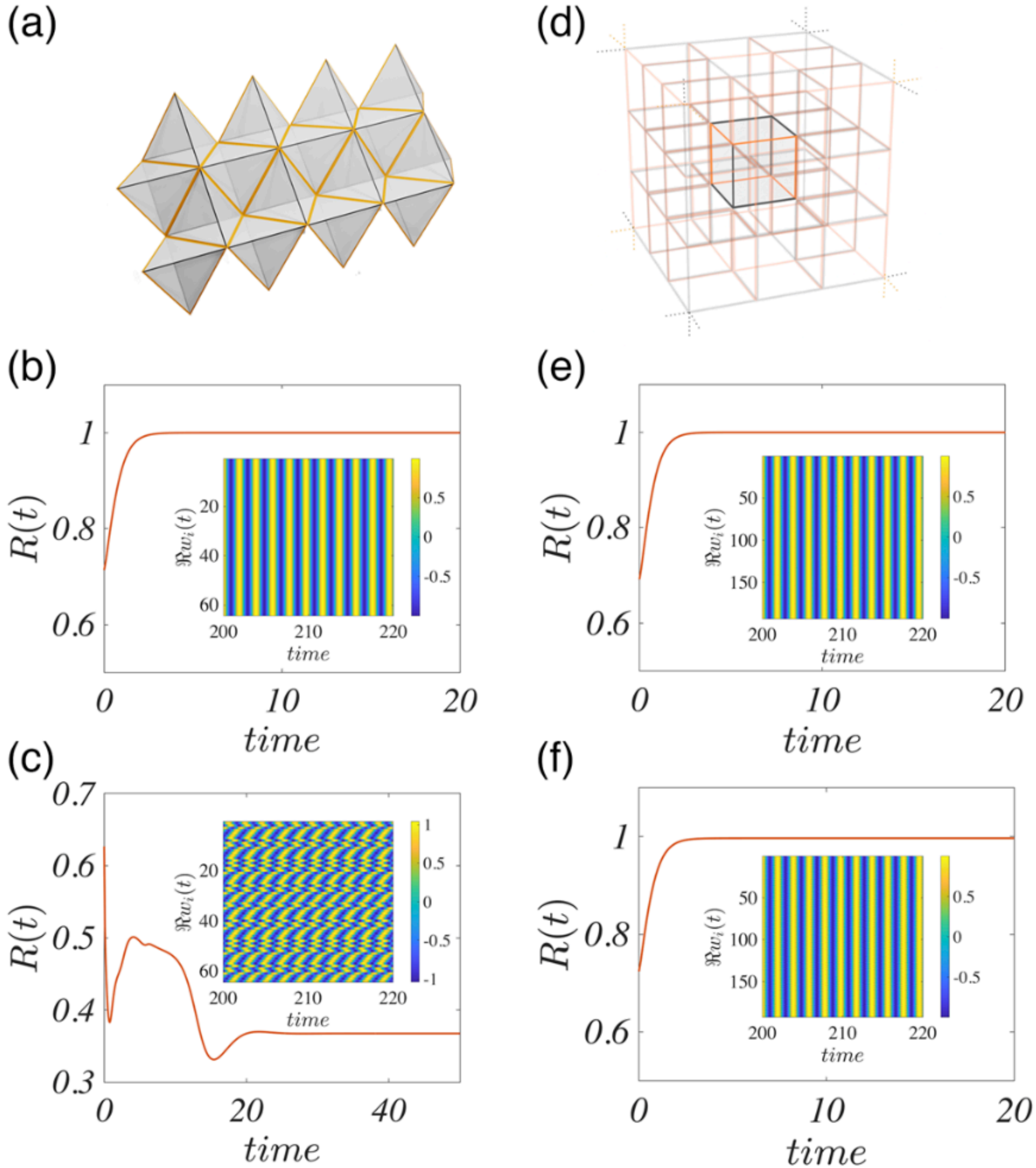
Department of mathematics

UNIVERSITÉ DE NAMUR

Global Topological Synchronisation : Stuart-Landau

global sync
for faces

global sync
for faces



no global sync
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