Higher-order networks

An introduction to simplicial complexes
Lesson III

Franqui Chair Lessons

18-19 April 2023

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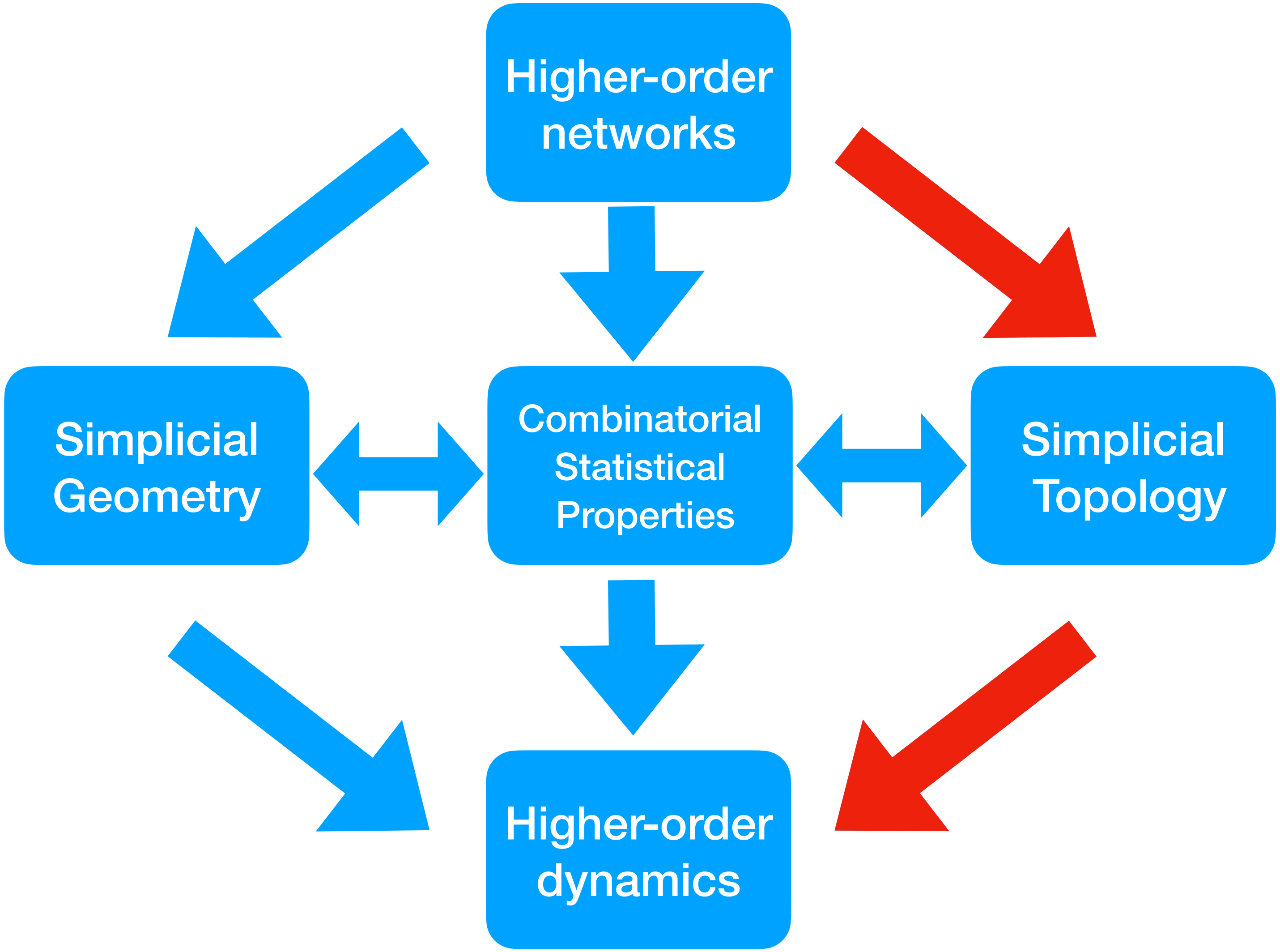
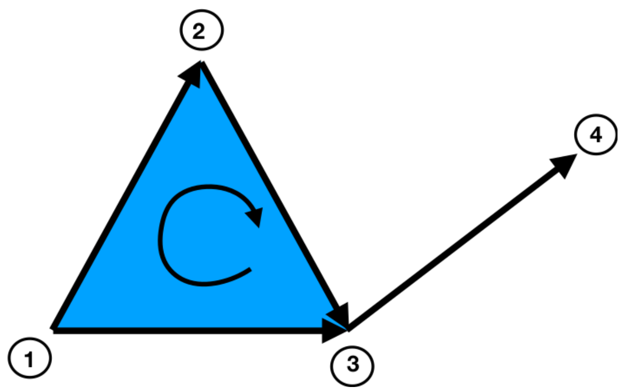
Higher-order structure and dynamics



Boundary Operators



Boundary operators

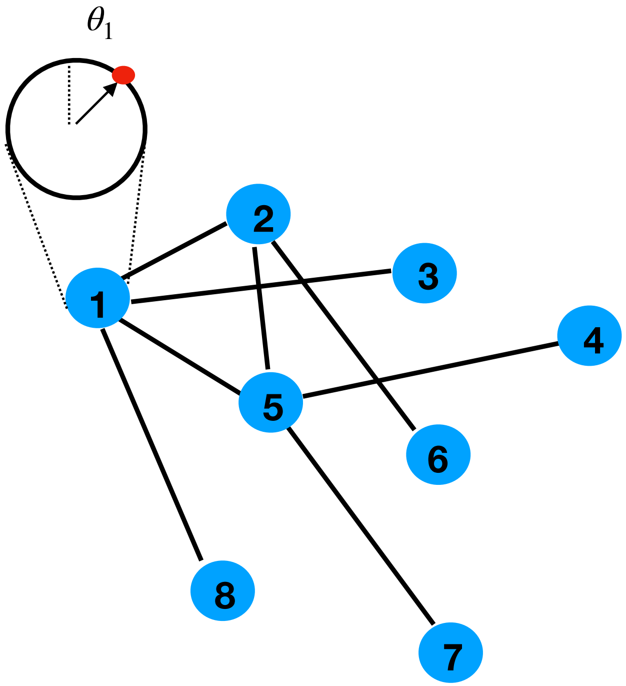
$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

$\mathbf{B}_{[1]}$	**Discrete divergence**
$\mathbf{B}_{[1]}^{\top}$	**Discrete gradient**
$\mathbf{B}_{[2]}^{\top}$	**Discrete Curl**

The boundary of the boundary is null

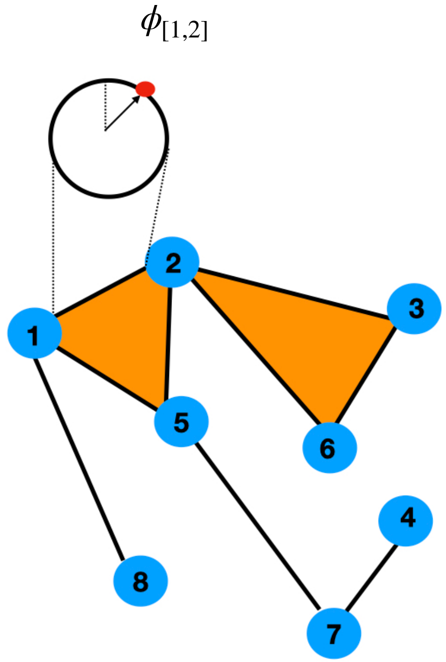
$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

Topological Kuramoto model



Standard Kuramoto model

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$



Topological Higher-order Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},$$

A. P. Millan, J.J. Torres and G. Bianconi PRL (2020)

Hodge Laplacians

The Hodge Laplacians describe diffusion

from n-simplices to m-simplices through (m-1) and (m+1)

simplices

$$\mathbf{L}_{[m]} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]} + \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

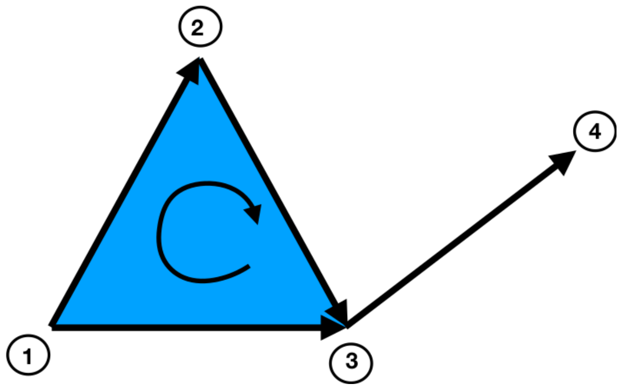
The higher order Hodge Laplacian can be decomposed as

$$\mathbf{L}_{[m]} = \mathbf{L}_{[m]}^{down} + \mathbf{L}_{[m]}^{up},$$

with

$$\mathbf{L}_{[m]}^{down} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]},$$
$$\mathbf{L}_{[m]}^{up} = \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

Simplicial complexes and Hodge Laplacians



Hodge Laplacians

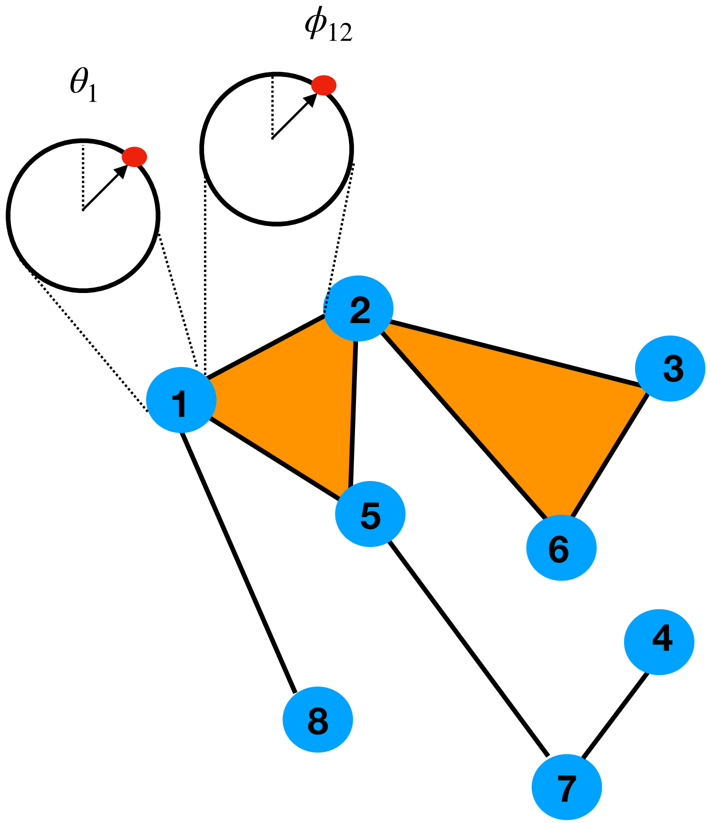
The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices

For a 2-dimensional simplicial complex we have

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^{\top} \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^{\top}\mathbf{B}_{[2]}$$

Coupling topological signals of different dimension



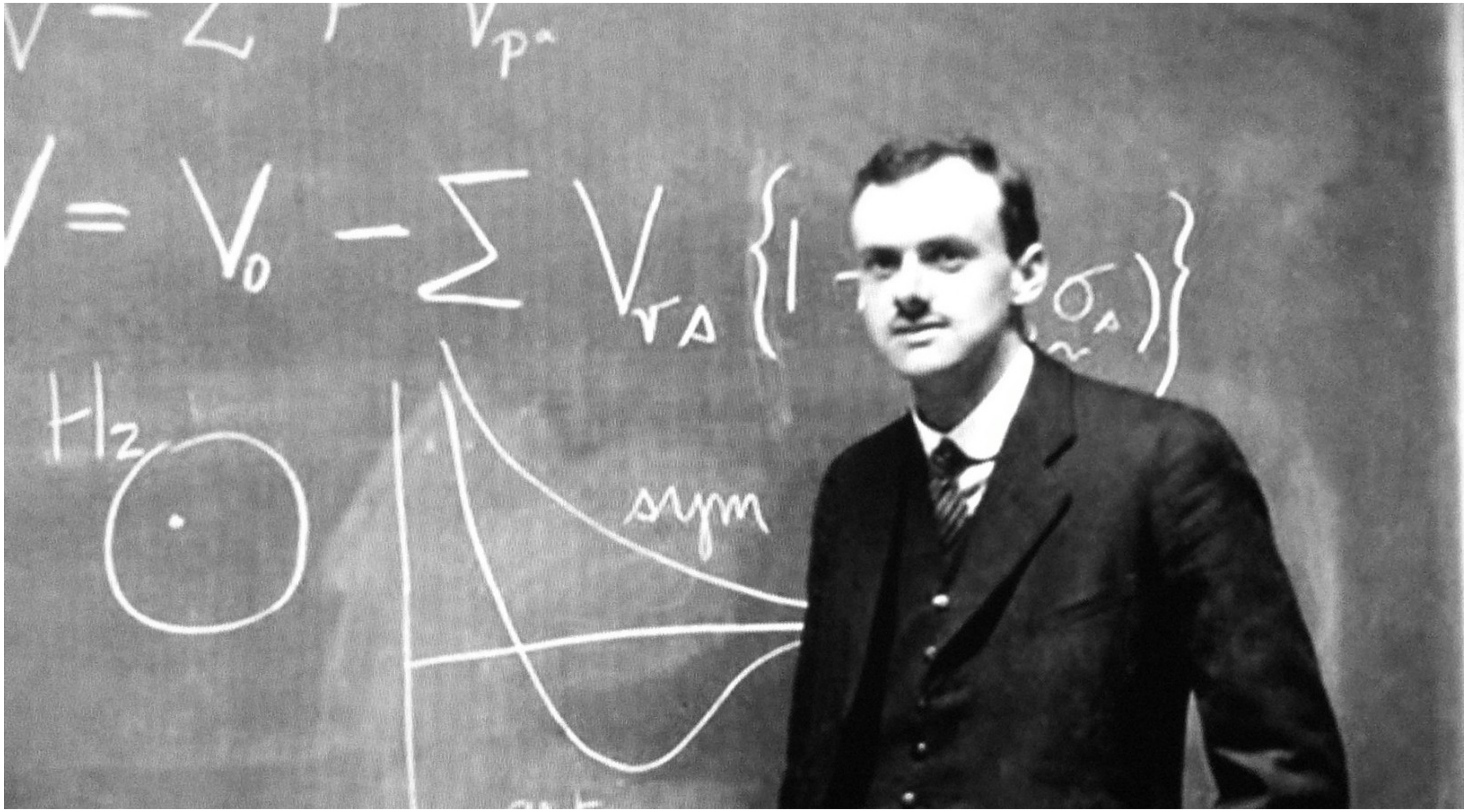
How can we couple

topological signal

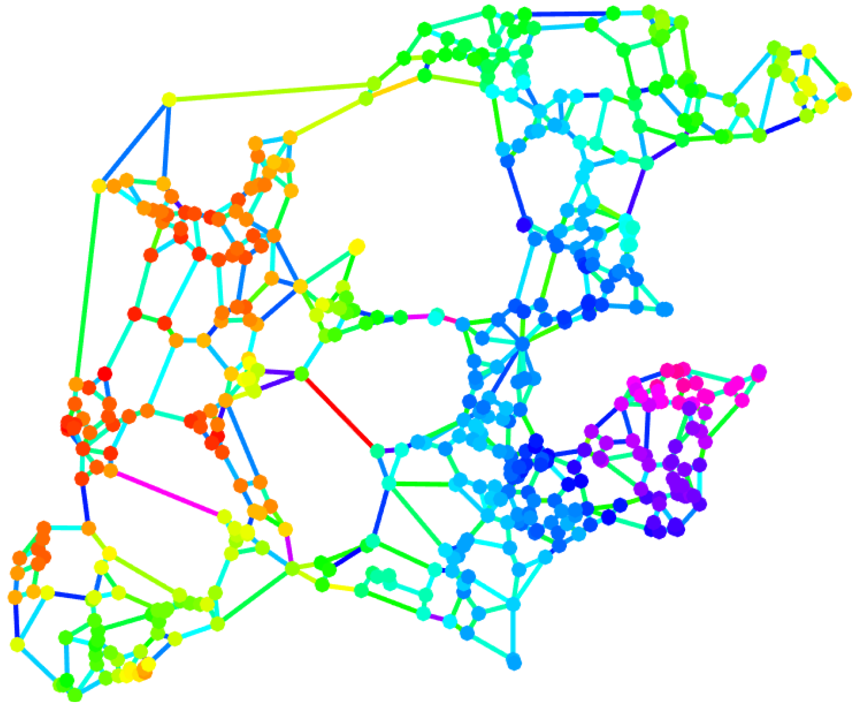
of different dimension

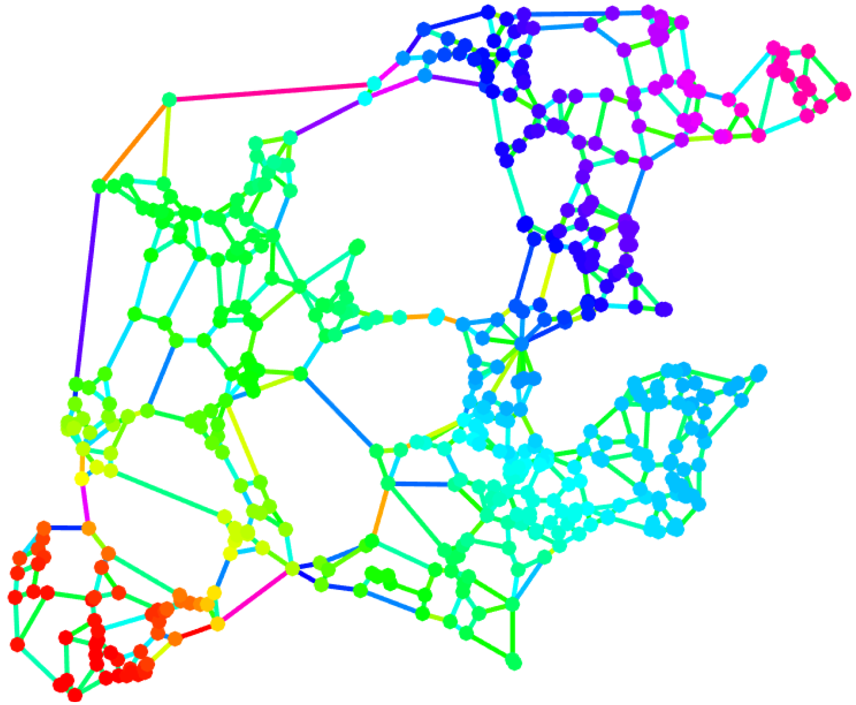
locally and topologically?

Dirac legacy



Dirac operator on graphs





Lesson III:
The Dirac operator
on networks and simplicial complexes

- Dirac operator on networks
  - Eigenvalues, Eigenvectors Chirality
  - Dirac equation version 1.1

- Dirac operator on simplicial complexes
- Dirac operator & Algebra
  - Topological Dirac equation in 3 dimensions
  - Topological Dirac equation in 3+1 dimensions

- Weighted and Normalised Dirac operator

The Dirac operator on graphs

Topological spinor

The topological spinor is defined on both nodes and edges of a graph $G = (V, E)$

as $\boldsymbol{\Psi} = \chi \oplus \psi \in C^0 \oplus C^1$ or equivalently

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\psi} \end{pmatrix}$$

with

- $\boldsymbol{\chi}$ defined on nodes, i.e. $\chi \in C^0$
- $\boldsymbol{\psi}$ defined on edges, i.e. $\psi \in C^1$

Exterior derivative and its dual

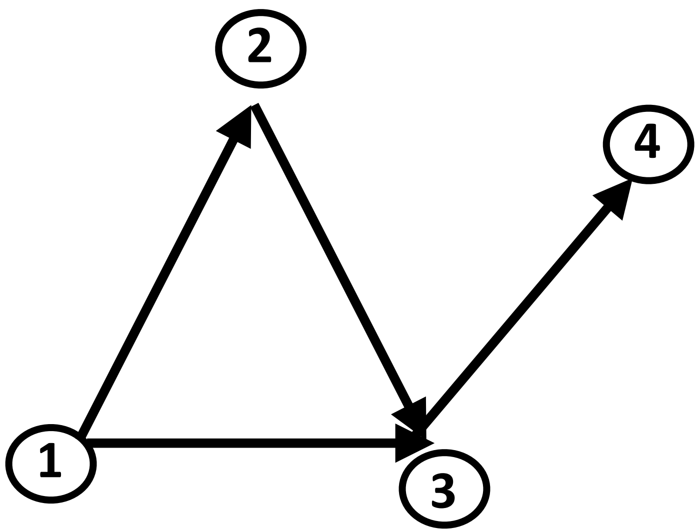
- The exterior derivative $d : C^0 \to C^1$ is defined as

$$(d\chi)_{e=[i,j]} = \chi_j - \chi_i \quad \textbf{gradient}$$

- It adjoint operator $d^* : C^1 \to C^0$ is defined as

$$(d^*\psi)_i = \sum_{e \in E_i^+} \psi_e - \sum_{e \in E_i^-} \psi_e \quad \textbf{divergence}$$

Boundary matrix



Boundary matrix

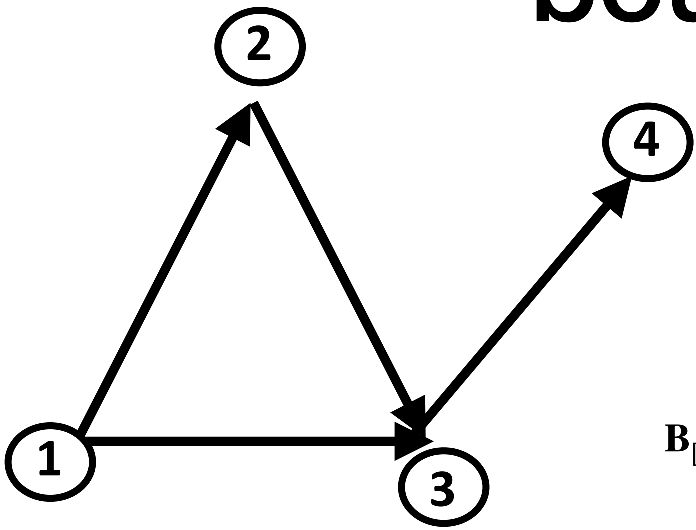
$\mathbf{B}_{[1]}$ is a $N \times L$ matrix of elements

$$\mathbf{B}_{[1]}(r,\ell) = \begin{cases} 1 \ \textbf{if} \ \ell = [s,r] \\ -1 \ \textbf{if} \ \ell = [r,s] \\ 0 \ \textbf{otherwise} \end{cases}$$

$\mathbf{B}_{[1]}^{\top}$	**Discrete gradient**
$\mathbf{B}_{[1]}$	**Discrete divergence**

The discrete gradient can be represented by the coboundary matrix $\bar{\mathbf{B}}_{[1]} = \mathbf{B}_{[1]}^{\top}$

Boundary operator and co-boundary matrix



Boundary and co-boundary matrices

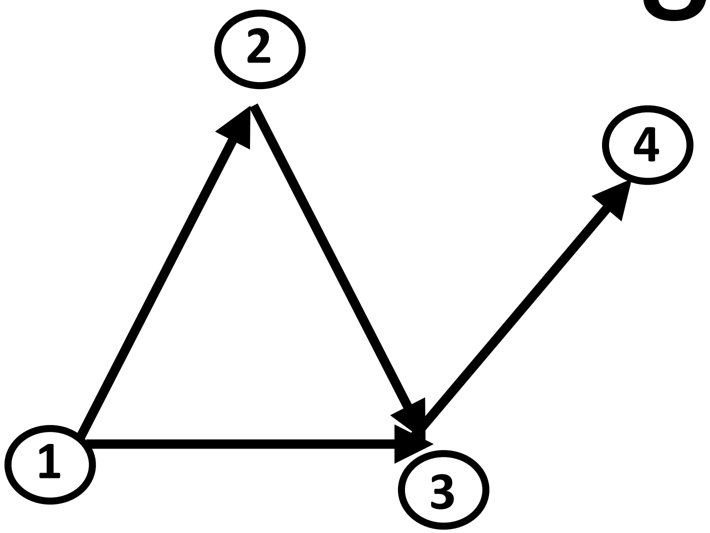
$$
\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \quad \mathbf{B}_{[1]}^{\top} = \begin{array}{c|cccc} & [1] & [2] & [3] & [4] \\ \hline [1,2] & -1 & 1 & 0 & 0 \\ [1,3] & -1 & 0 & 1 & 0 \\ [2,3] & 0 & -1 & 1 & 0 \\ [3,4] & 0 & 0 & -1 & 1 \end{array},
$$

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[1]}$ **Discrete divergence**

The discrete gradient can be represented by a coboundary matrix $\bar{\mathbf{B}}_{[1]} = \mathbf{B}_{[1]}^{\top}$

Hodge Laplacians



Betti numbers of a connected network

$\beta_0 = 1$ one connected component

$\beta_1 = L - (N-1)$ number of independent cycles

dim ker$(\mathbf{L}_{[m]}) = \beta_m$

Hodge Laplacians

The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices:

for a graph we have

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]}$$

Exterior derivation and its adjoint on a graph

The exterior derivative and its adjoint

$$d = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix} \qquad d^{*} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

act on a topological spinor

$$\mathbf{\Psi} = \begin{pmatrix} \chi \\ \psi \end{pmatrix}$$

Basic definition of the Dirac operator on graphs

The Dirac operator in its simplest form

is the self-adjoint operator $D : C^0 \bigoplus C^1 \to C^0 \bigoplus C^1$ defined as

$$D = d + d^*$$

satisfying

$$D(\chi \bigoplus \psi) = (d + d^*)(\chi \bigoplus \psi) = (d^*\psi) \bigoplus (d\chi)$$

Dirac operator on a network

Exterior divergence

$$d = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix} \qquad d^{*} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Dirac operator is a self-adjoint operator

$$D = d + d^{*}$$

Dirac operator on graph

Dirac operator on a graph

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

Action of the Dirac operator on the topological spinor

$$\mathbf{D}\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{[1]}\psi \\ \mathbf{B}_{[1]}^{\top}\chi \end{pmatrix}$$

The Dirac as the square-root of the Laplacian

The Dirac operator
can be interpreted as the
“square-root” of the Laplacian

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & 0 \end{pmatrix}, \qquad \mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}$$

The non-zero eigenvalues of the Dirac operator
are the square root of the non-zero eigenvalues of the graph Laplacian.

The spectrum of the Dirac operator

Since $\mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}$ and $\mathbf{L}_{[0]}, \mathbf{L}_{[1]}$ are isospectral, it follows that:

Spectrum: For every positive eigenvalue μ of $\mathbf{L}_{[0]}$ there is one positive and one negative eigenvalue λ of the Dirac operator $\mathbf{D}$ with

$$\lambda = \pm\sqrt{\mu}$$

Chirality

Let us define $\gamma_0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$

obeying the anti commutator relation $\{\mathbf{D}, \gamma_0\} = \mathbf{0}$

- **Chirality**:If $\boldsymbol{\Psi} = (\chi, \psi)^\top$ is an eigenvector of the Dirac operator with eigenvalue λ, i.e. if $\mathbf{D}\boldsymbol{\Psi} = \lambda\boldsymbol{\Psi}$ then $\gamma_0\boldsymbol{\Psi} = (\chi, -\psi)^\top$ is an eigenvector of $\mathbf{D}$ with eigenvalue $-\lambda$

- Indeed from the anti-commutator relation it follows that $\mathbf{D}\gamma_0\boldsymbol{\Psi} = -\gamma_0\mathbf{D}\boldsymbol{\Psi} = -\lambda\gamma_0\boldsymbol{\Psi}$

Eigenvectors of the Dirac operator

- It follows that the matrix of eigenvectors of the Dirac operator can be expressed as

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{U}^{[1]} & \mathbf{U}^{[1]} & \mathbf{U}_0^{harm} & \mathbf{0} \\ \mathbf{V}^{[1]} & -\mathbf{V}^{[1]} & \mathbf{0} & \mathbf{U}_1^{harm} \end{pmatrix}$$

- where $\mathbf{U}^{[1]}, \mathbf{V}^{[1]}$ Indicates the right and left singular vector of the coboundary operator and $\mathbf{U}_0^{harm}, \mathbf{U}_1^{harm}$ are the matrices of the harmonic eigenvectors of $\mathbf{L}_{[0]}, \mathbf{L}_{[1]}$ respectively.

Index of the Dirac operator

The index of the Dirac operator D is given

by the Euler number χ_E of the graph

$$\text{ind } D = \text{dim ker } d - \text{dim ker } d^* = \chi_E$$

Indeed

$$\text{ind } D = \chi_E = N - L$$

Introducing an algebra

Dirac operator on a network can be enriched by an algebra

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b\mathbf{B}_{[1]} \\ b^{\star}\mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

with $b \in \mathbb{C}, \quad |b| = 1$

Topological spinor

On a network we consider the topological spinor

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\psi} \end{pmatrix}$$

Characterising the dynamical state of the topological signals of the network, being a vector with a block structure formed by a 0-cochain and a 1-cochain

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_N \end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_{\ell_1} \\ \psi_{\ell_2} \\ \vdots \\ \psi_{\ell_L} \end{pmatrix}.$$

Topological Dirac equation

The topological Dirac equation is then given by

$$i\partial_t \boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$$

with Hamiltonian

$$\mathscr{H} = \mathbf{D} + m\boldsymbol{\beta}$$

Where $\boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$ leading to the anti-commutator $\{\mathbf{D}, \boldsymbol{\beta}\} = \mathbf{0}$

Sketch of the derivation

The eigenvalue problem $E\boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$ is equivalent to

$$E\boldsymbol{\chi} = b\mathbf{B}\boldsymbol{\psi} + m\boldsymbol{\chi},$$
$$E\boldsymbol{\psi} = b^{\star}\mathbf{B}^{\top}\boldsymbol{\chi} - m\boldsymbol{\psi}$$

Let us re-order obtaining

$$(E-m)\boldsymbol{\chi} = b\mathbf{B}\boldsymbol{\psi},$$
$$(E+m)\boldsymbol{\psi} = b^{\star}\mathbf{B}^{\top}\boldsymbol{\chi}$$

Therefore

$$(E-m)(E+m)\boldsymbol{\chi} = \mathbf{B}\mathbf{B}^{\top}\boldsymbol{\chi} = \mathbf{L}_{[0]}\boldsymbol{\chi},$$
$$(E+m)(E-m)\boldsymbol{\psi} = \mathbf{B}^{\top}\mathbf{B}\boldsymbol{\psi} = \mathbf{L}_{[1]}^{down}\boldsymbol{\psi}$$

→ This implies $E^2 = m^2 + |\lambda|^2$

Energy Eigenstates

The energy eigenstates satisfy $E\boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$ which leads to

$$E\boldsymbol{\chi} = b\mathbf{B}\boldsymbol{\psi} + m\boldsymbol{\chi},$$
$$E\boldsymbol{\psi} = b^\star\mathbf{B}^\top\boldsymbol{\chi} - m\boldsymbol{\psi}$$

It follows that $\boldsymbol{\chi}, \boldsymbol{\psi}$ are respectively the left and right singular vectors of $\mathbf{B}$ with singular value λ

and that the dispersion relation is relativistic $E^2 = |\lambda|^2 + m^2$,

i.e. the energy values are given by $E = \pm\sqrt{|\lambda|^2 + m^2}$

Eigenvectors of the Dirac equation

The eigenvectors of the Dirac operator are

$$\boldsymbol{\phi}_{\lambda}^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_{\lambda}\\ \mathbf{v}_{\lambda}\end{pmatrix} \qquad \boldsymbol{\phi}_{\lambda}^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_{\lambda}\\ -\mathbf{v}_{\lambda}\end{pmatrix}$$

where $\mathbf{u}_{\lambda}, \mathbf{v}_{\lambda}$ are the right and left singular vector of $\mathbf{B}_{[1]}$ corresponding to singular value λ and $\mathscr{C}$ indicates the normalisation constants.

The eigenvectors of the topological Dirac equation are instead

$$\boldsymbol{\phi}_{\lambda}^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_{\lambda}\\ \frac{b^{*}\lambda^{*}}{|E|+m}\mathbf{v}_{\lambda}\end{pmatrix} \qquad \boldsymbol{\phi}_{\lambda}^{[+]} = \mathscr{C}\begin{pmatrix}\frac{b\lambda}{|E|+m}\mathbf{u}_{\lambda}\\ -\mathbf{v}_{\lambda}\end{pmatrix}$$

Therefore the overall normalisation of the nodes signal changes with respect

to the normalisation of the edge signal.

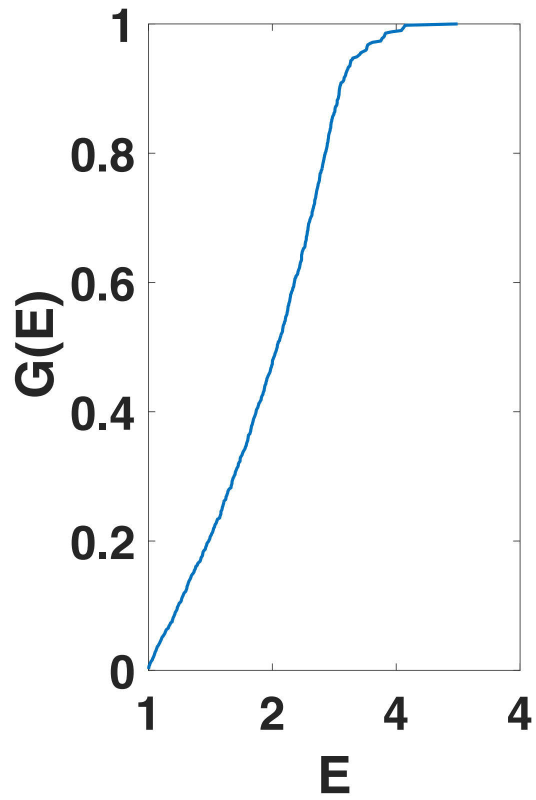
Matter-Antimatter asymmetry and homology

For $E^2 > m^2$ there is symmetry between positive energy eigenstates and negative energy eigenstates.

However the symmetry between positive energy states and negative energy states breaks down for $|E| = m$

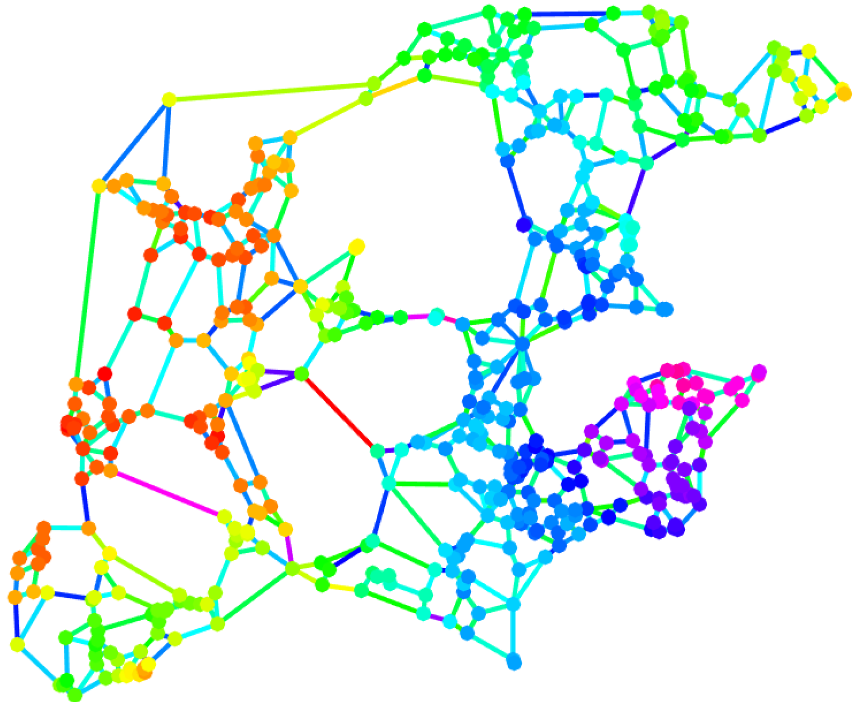
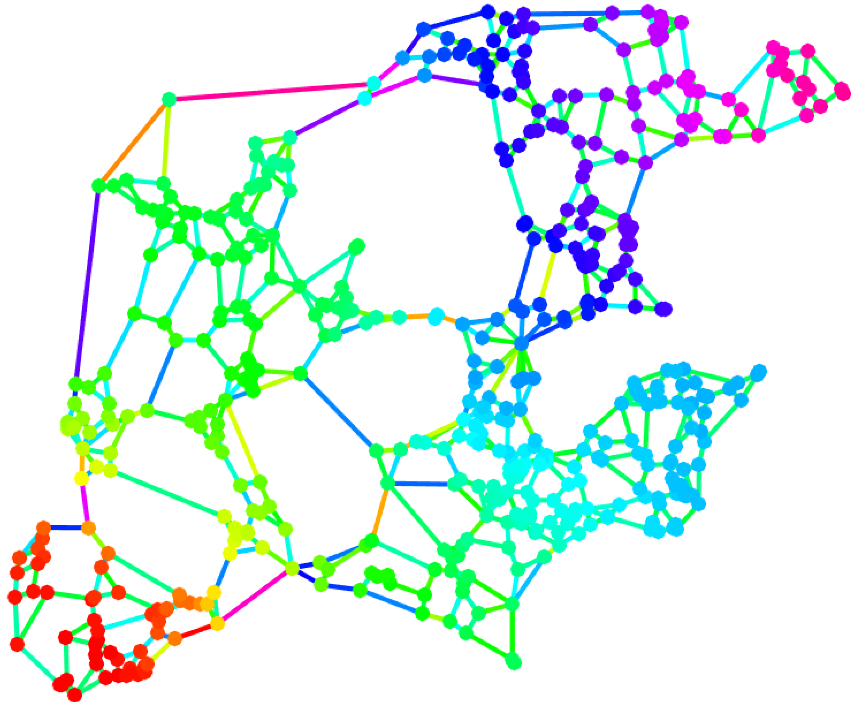
The states at energy states at $E = m$ are localised on nodes and they have a degeneracy given by the Betti number β_0

The energy states $E = -m$ are localised on links and they have a degeneracy given by the Betti number β_1

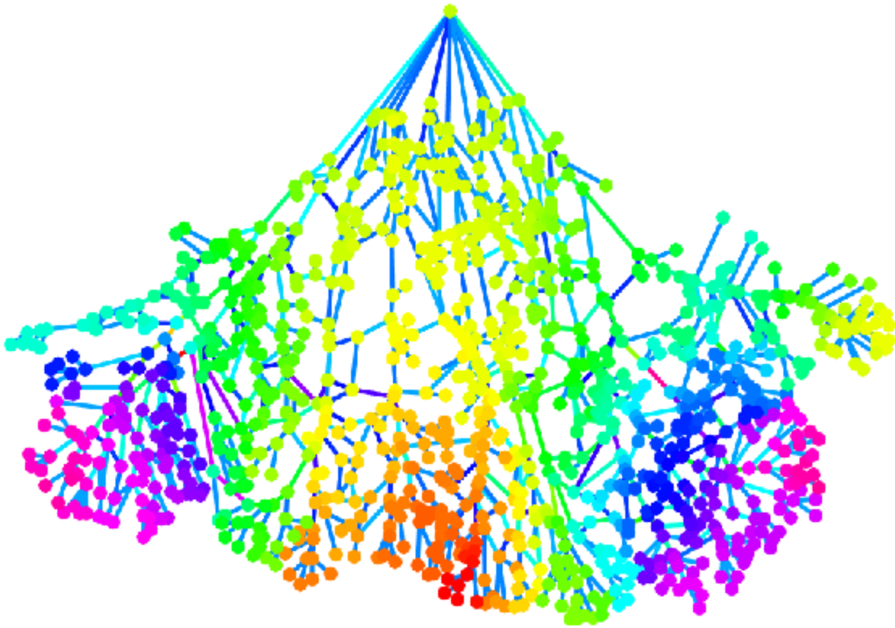


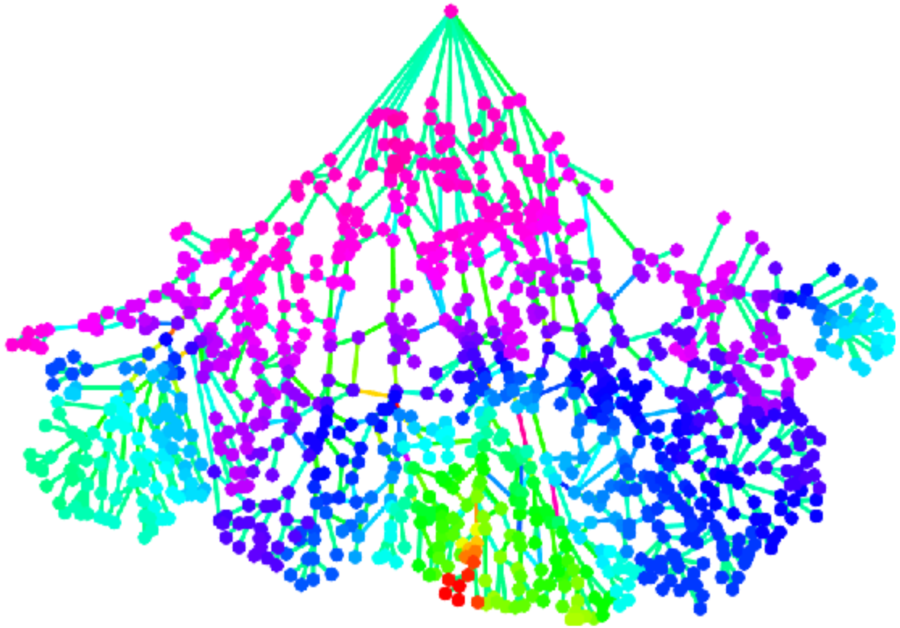
Density of states

Eigenvectors of the Dirac operator on real networks

Eigenvectors of the Dirac Operator
on real networks





The Dirac operator on simplicial complexes

The Dirac operator on simplicial complexes

The Dirac operator allows to study interacting topological signals of different dimensions coexisting in the same network topology

Dirac operator

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix},$$

Topological signal "spinor" $\mathbf{s} = \bigoplus_{m=0}^{d} C^d$

$$\mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

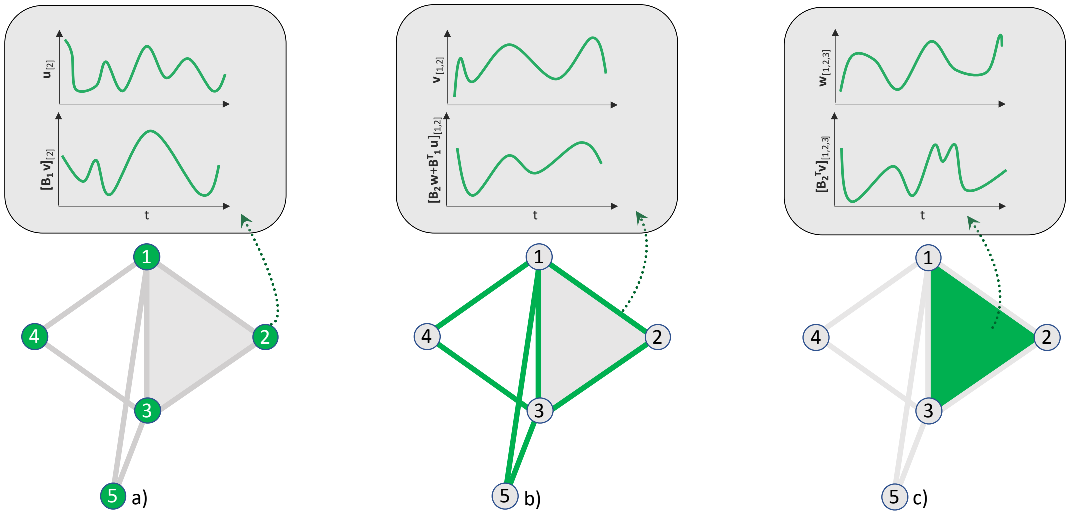
$\mathbf{s}_0$ **Node signal**

$\mathbf{s}_1$ **Link signal**

$\mathbf{s}_2$ **Triangle signal**

The action of the Dirac operator

The Dirac operator allows cross-talking between signals of different dimension



$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]} & 0 \end{pmatrix}, \text{ acts on } \mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}\mathbf{s} = \begin{pmatrix} \mathbf{B}_{[1]}\mathbf{s}_1 \\ \mathbf{B}_{[1]}^\top \mathbf{s}_0 + \mathbf{B}_{[2]}\mathbf{s}_2 \\ \mathbf{B}_{[2]}^\top \mathbf{s}_1 \end{pmatrix}$$

Dirac decomposition

$$\mathbf{D} = \mathbf{D}_{[1]} + \mathbf{D}_{[2]}$$

Here

$\mathbf{D}_{[1]}$ only couples node and link signals and

$\mathbf{D}_{[2]}$only couples link and triangle signals

$$\mathbf{D}_{[1]} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^{\top} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{D}_{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^{\top} & 0 \end{pmatrix}$$

$$\mathbf{D}_{[1]}^2 = \mathscr{L}_{[1]} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]}^{down} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{D}_{[2]}^2 = \mathscr{L}_{[2]} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]}^{up} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{[2]}^{down} \end{pmatrix}$$

Dirac decomposition

Since the boundary of the boundary is null we obtain

$$\mathbf{D}_{[1]}\mathbf{D}_{[2]} = \mathbf{D}_{[2]}\mathbf{D}_{[1]} = \mathbf{0}$$

which implies

$$\text{ker}(\mathbf{D}_{[1]}) \supseteq \text{im}(\mathbf{D}_{[2]})$$
$$\text{ker}(\mathbf{D}_{[2]}) \supseteq \text{im}(\mathbf{D}_{[1]})$$

Dirac decomposition

Every topological signal can be decomposed in a unique way thanks to the Dirac decomposition

$$\mathbb{R}^{D_S} = \text{im}(\mathbf{D}_{[1]}) \oplus \text{ker}(\mathbf{D}) \oplus \text{im}(\mathbf{D}_{[2]})$$

therefore every signals defined on nodes, links and triangles can be decomposed in a unique way as

$$\mathbf{s} = \mathbf{s}^{[1]} + \mathbf{s}^{[2]} + \mathbf{s}^{harm}$$

With

$$\mathbf{s}^{[1]} = \mathbf{D}_{[1]}\mathbf{D}_{[1]}^{+}\mathbf{s}$$

$$\mathbf{s}^{[2]} = \mathbf{D}_{[2]}\mathbf{D}_{[2]}^{+}\mathbf{s}$$

Eigenvalues of the Dirac operator

Due to the Dirac decomposition

the eigenvalues of the Dirac operator $\mathbf{D}$

are the direct sum

of the non-zero eigenvalues

of $\mathbf{D}_{[1]}$ and of $\mathbf{D}_{[2]}$

plus the zero eigenvalue

with degeneracy $\beta_0 + \beta_1 + \beta_2$

Eigenvectors of the Dirac operator

Due to the Dirac decomposition

the eigenvectors of the Dirac operator $\mathbf{D}$

are the eigenvectors

corresponding to non-zero eigenvalues

of $\mathbf{D}_{[1]}$ or of $\mathbf{D}_{[2]}$

r the harmonic eigenvectors of $\mathbf{D}$

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}^{[1]} & \mathbf{\Phi}^{[2]} & \mathbf{\Phi}^{harm} \end{pmatrix}$$

With $\mathbf{\Phi}^{[1]}$ localised on nodes and links and

$\mathbf{\Phi}^{[2]}$ localised on links and triangles

Chirality

Let us define $\gamma_0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$

obeying the anti commutator relation $\{\mathbf{D}, \gamma_0\} = \mathbf{0}$, $\{\mathbf{D}_{[n]}, \gamma_0\} = \mathbf{0}$,

- **Chirality**:If $\boldsymbol{\Psi} = (\boldsymbol{\chi}, \boldsymbol{\psi}, \mathbf{0})^\top$ is an eigenvector of the Dirac operator with eigenvalue λ, i.e. if $\mathbf{D}\boldsymbol{\Psi} = \lambda\boldsymbol{\Psi}$ then $\gamma_0\boldsymbol{\Psi} = (\boldsymbol{\chi}, -\boldsymbol{\psi}, \mathbf{0})^\top$ is an eigenvector of $\mathbf{D}$ with eigenvalue $-\lambda$. Likewise if $\boldsymbol{\Psi} = (\mathbf{0}, \boldsymbol{\chi}, \boldsymbol{\psi})^\top$ is an eigenvector of the Dirac operator with eigenvalue λ, i.e. if $\mathbf{D}\boldsymbol{\Psi} = \lambda\boldsymbol{\Psi}$ then $\gamma_0\boldsymbol{\Psi} = (\mathbf{0}, -\boldsymbol{\chi}, \boldsymbol{\psi})^\top$ is an eigenvector of $\mathbf{D}$ with eigenvalue $-\lambda$

- Indeed from the anti-commutator relation it follows that $\mathbf{D}\gamma_0\boldsymbol{\Psi} = -\gamma_0\mathbf{D}\boldsymbol{\Psi} = -\lambda\gamma_0\boldsymbol{\Psi}$

Eigenvalues of $\mathbf{D}_{[n]}$

The eigenstates of $\mathbf{D}_{[n]}$ satisfy

$$\mu \mathbf{s} = \mathbf{D}_{[n]} \mathbf{s}$$

with $\mathbf{s} = (\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2)^\top$ which leads to

$$\mu \mathbf{s}_{n-1} = \mathbf{B}_{[n]} \mathbf{s}_n$$

$$\mu \mathbf{s}_n = \mathbf{B}_{[n]}^\top \mathbf{s}_{n-1}$$

It follows that $\mathbf{s}_{n-1}, \mathbf{s}_n$ are respectively the left and right singular vectors of $\mathbf{B}_n$ with eigenvalue λ and $\mu = \pm |\lambda|$

Matter-antimatter symmetry…

For every singular value $\lambda \neq 0$ of $\mathbf{B}_{[n]}$

corresponding to the singular vectors $\mathbf{u}_\lambda, \mathbf{v}_\lambda$

the Dirac operator admits

a positive eigenvalue $\mu = |\lambda|$ with eigenvector $\boldsymbol{\phi}_\lambda^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_\lambda \\ \mathbf{v}_\lambda\end{pmatrix}$

and

a negative eigenvalue $\mu = -|\lambda|$ with eigenvector $\boldsymbol{\phi}_\lambda^{[-]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_\lambda \\ -\mathbf{v}_\lambda\end{pmatrix}$

…and its violation

The zero eigenvectors of $\mathbf{D}_{[n]}$

are linear combinations of the zero eigenvectors of $\mathbf{B}_{[n]}$

they can be only localised on n-dimensional

or on (n-1)-dimensional simplices

The degeneracy the zero eigenvalue is given by

the sum of the Betti numbers $\beta_{n-1} + \beta_n$

Eigenvectors or the Dirac operator

In summary the eigenvectors of the Dirac operator

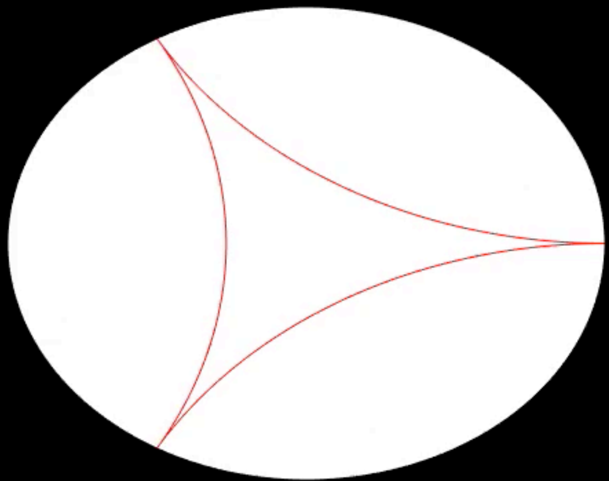
defined on a simplicial complex of dimension 2 have the structure

$$\mathbf{\Phi}=\begin{pmatrix}\mathbf{U}^{[1]} & \mathbf{U}^{[1]} & \mathbf{0} & \mathbf{0} & \mathbf{U}_0^{harm} & \mathbf{0} & \mathbf{0}\\ \mathbf{V}^{[1]} & -\mathbf{V}^{[1]} & \mathbf{U}^{[2]} & \mathbf{U}^{[2]} & \mathbf{0} & \mathbf{U}_1^{harm} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}^{[2]} & -\mathbf{V}^{[2]} & \mathbf{0} & \mathbf{0} & \mathbf{U}_2^{harm}\end{pmatrix}$$

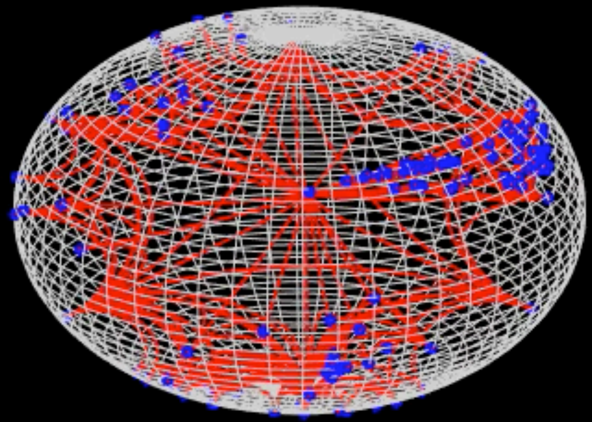
Simplicial complex models of arbitrary dimension

Emergent Hyperbolic Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

d=2



d=3



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Topological Dirac operator on a simplicial complex

The Topological Dirac operator can be extended to higher-dimensional simplices. For instance on a 3-dimensional simplex it is given by

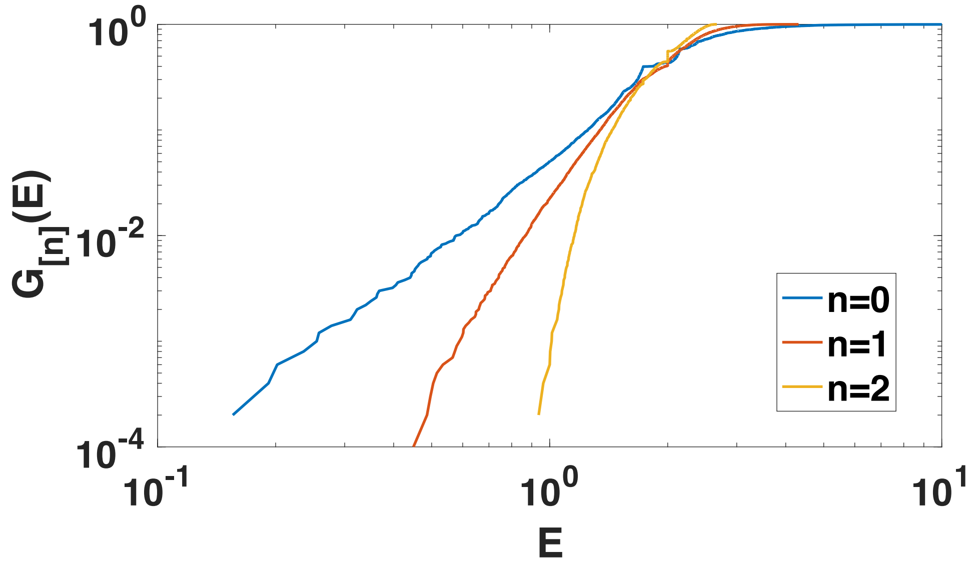
$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b_{[1]}\mathbf{B}_{[1]} & \mathbf{0} & \mathbf{0} \\ b^{\star}_{[1]}\mathbf{B}^{\top}_{[1]} & \mathbf{0} & b_{[2]}\mathbf{B}_{[2]} & \mathbf{0} \\ \mathbf{0} & b^{\star}_{[2]}\mathbf{B}^{\top}_{[2]} & \mathbf{0} & b_{[3]}\mathbf{B}_{[3]} \\ \mathbf{0} & \mathbf{0} & b^{\star}_{[3]}\mathbf{B}^{\top}_{[3]} & \mathbf{0} \end{pmatrix}$$

Topological Dirac equation on simplicial complexes

- The topological Dirac equation can be extended to simplicial complexes, in the case of zero mass it is given by

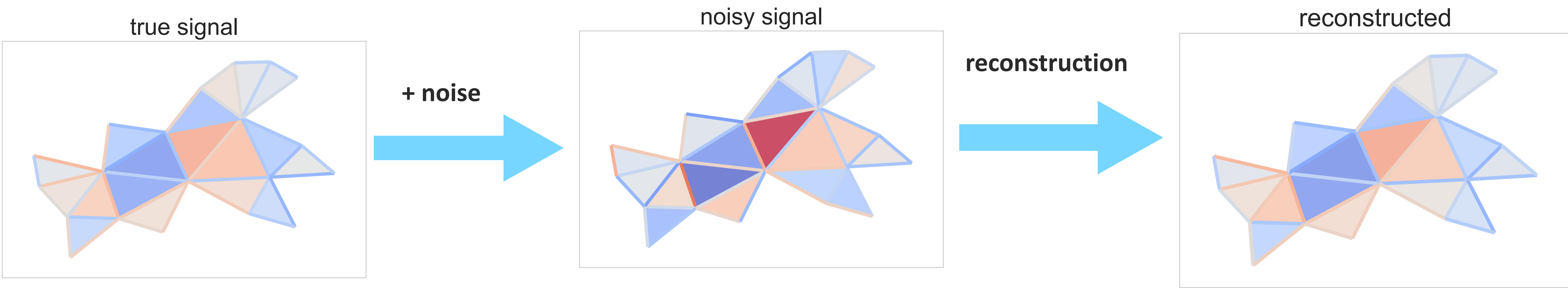
$$i\partial_t \psi = \mathbf{D}\psi$$

- It can be shown that thanks to the Hodge decomposition this equation leads to a multi-band spectrum of the energy states.



Multi-band eigenspectrum of the Topological Dirac equation on a 3-dimensional NGF

Dirac Signal Processing



The Dirac operator allows us to filter out nodes and links signals **jointly**

L. Calmon, M. Schaub and G. Bianconi
Dirac signal processing of topological signals
(2023)

Processing with the Dirac operator

Given a **noisy topological signal defined on simplices of different dimension**

$\tilde{\mathbf{s}} = \mathbf{s} + \boldsymbol{\epsilon}$ with $\boldsymbol{\epsilon}$ noise

Joint-filtering with the Dirac:

$$\hat{\mathbf{s}} = \text{argmin} \left\{ ||\tilde{\mathbf{s}} - \hat{\mathbf{s}}||_2^2 + \gamma \hat{\mathbf{s}}^T (\mathbf{D} - m\mathbf{I})^2 \hat{\mathbf{s}} \right\}$$

$m > 0 \rightarrow$ higher cost negative components

$m < 0 \rightarrow$ higher cost to positive components

Interpretation of the parameter m

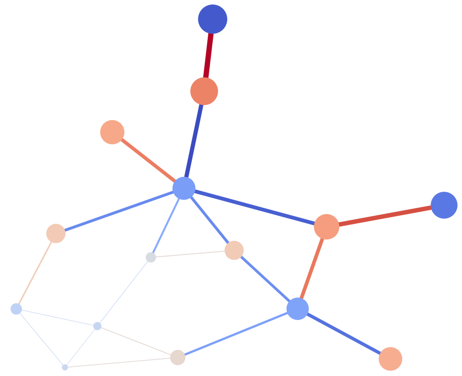
The parameter m can be interpreted as

$$m = \frac{\mathbf{s}^\top \mathbf{D} \mathbf{s}}{\mathbf{s}^\top \mathbf{s}}$$

Which allow us to interpret the regularisation as a minimization of the mean square error of the signal around m

The parameter m can be learned from data

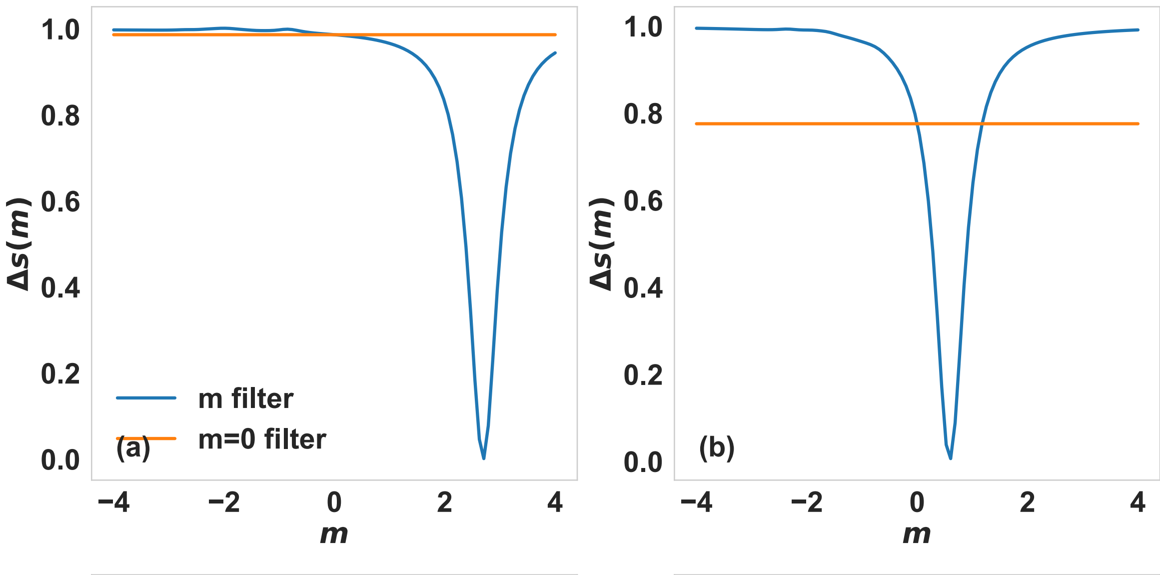
The Florentine Families network:



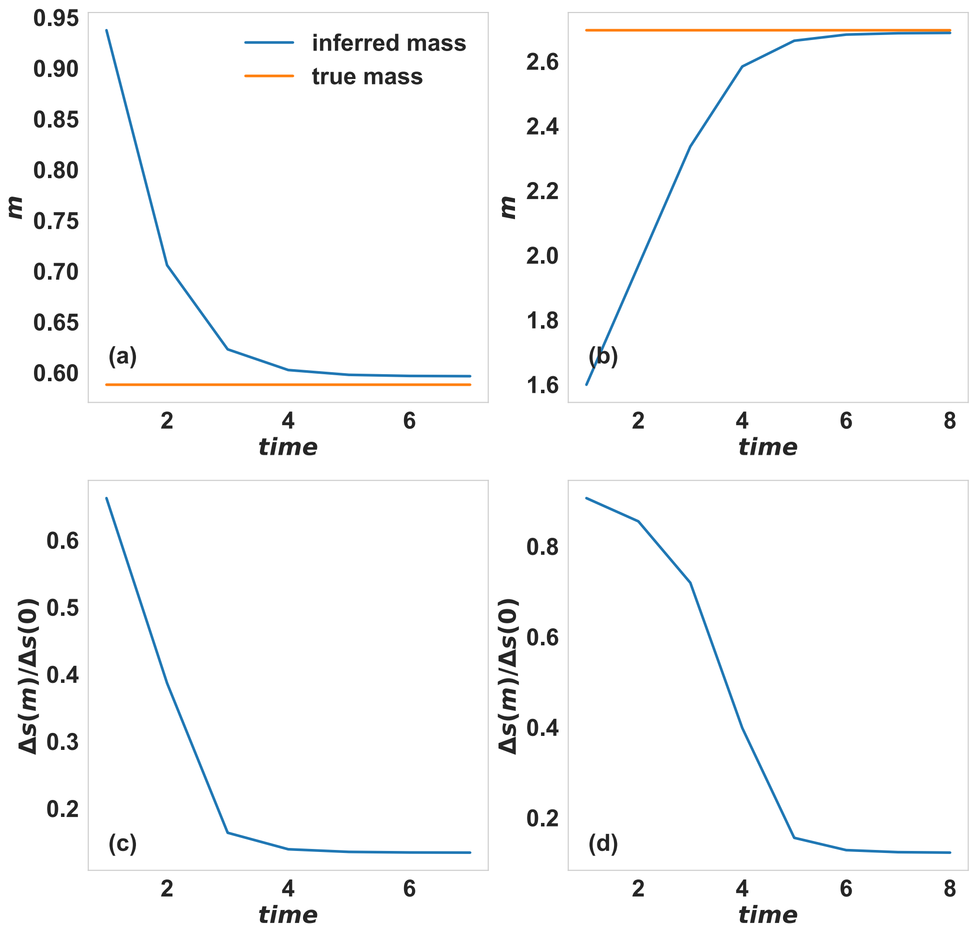
- Simple **network structure**, true signal aligned with an eigenvector of $\mathbf{D}$

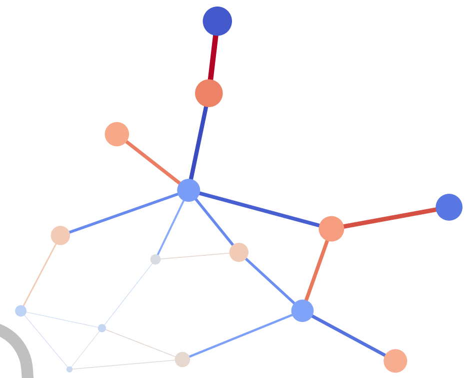
Dirac signal processing

$$\hat{\mathbf{s}} = \text{argmin}\left\{ ||\tilde{\mathbf{s}} - \hat{\mathbf{s}}||_2^2 + \gamma \hat{\mathbf{s}}^T (\mathbf{D} - m\mathbf{I})^2 \hat{\mathbf{s}} \right\}$$



Learning m





$$m = \frac{\mathbf{s}^\top \mathbf{D} \mathbf{s}}{\mathbf{s}^\top \mathbf{s}}$$

Learning m

Require: m_n, T, η, $\tilde{\boldsymbol{s}}_n$

$t \leftarrow 0$

while $t < T$ **do**

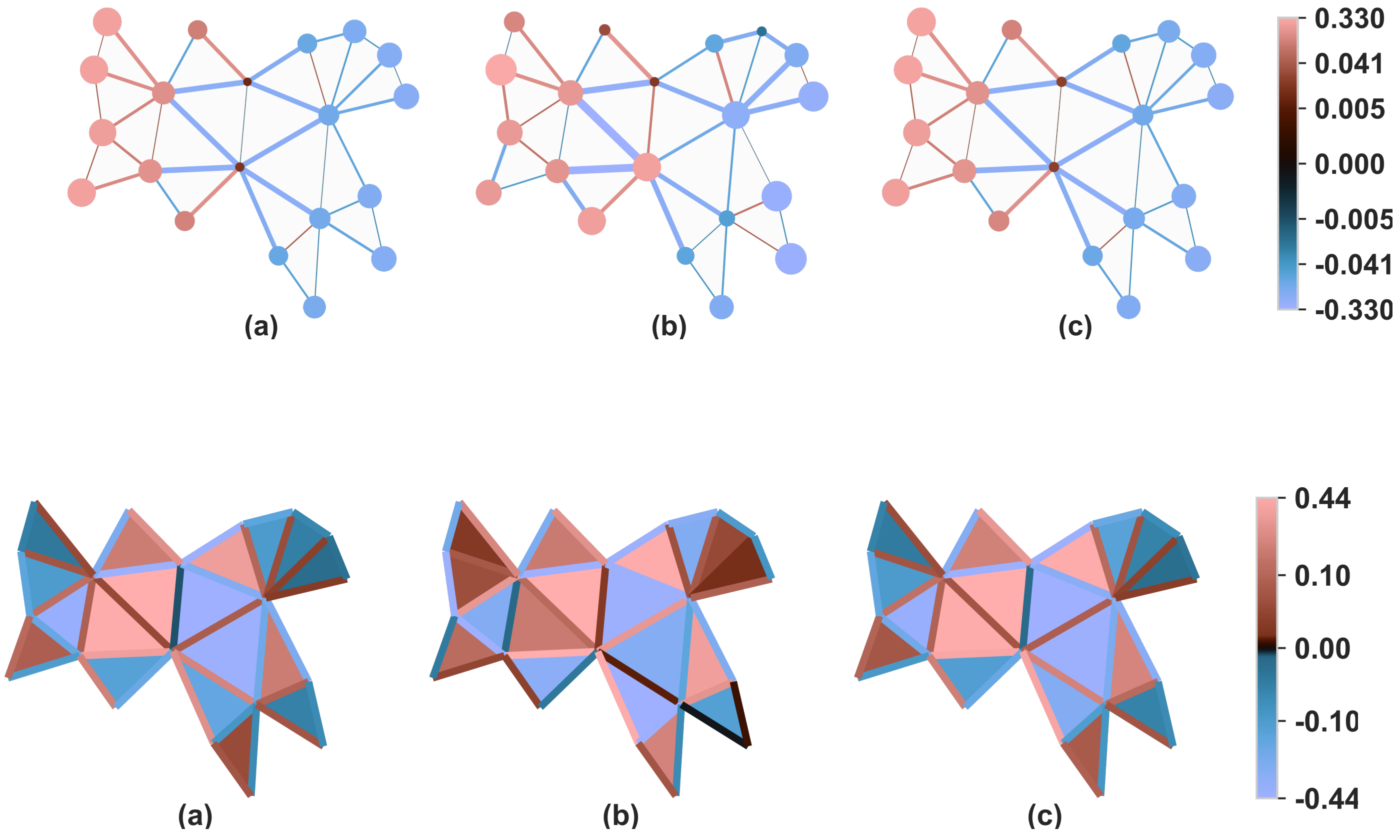
$t \leftarrow t + 1$

$\hat{\boldsymbol{s}}_n \leftarrow [\mathbf{I} - \gamma(\mathbf{D}_n - M_n \mathbf{I})^2]^{-1} \tilde{\boldsymbol{s}}_n$

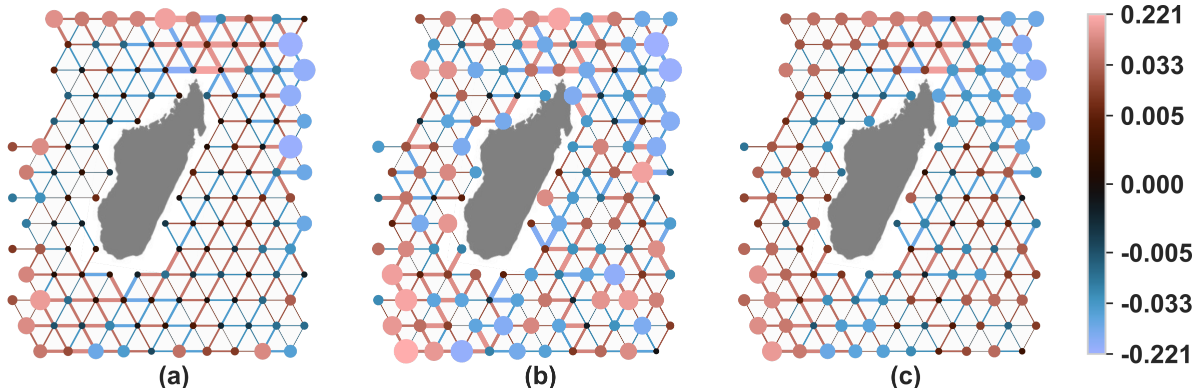
$M_n \leftarrow (1-\eta) M_n + \eta \frac{\hat{\boldsymbol{s}}_n^\top \mathbf{D}_n \hat{\boldsymbol{s}}_n}{\hat{\boldsymbol{s}}_n^\top \hat{\boldsymbol{s}}_n}$

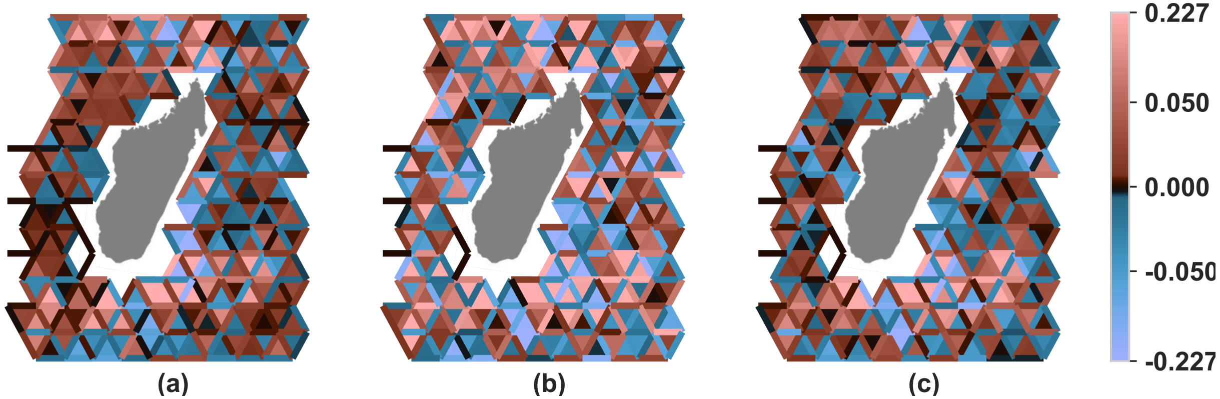
end while

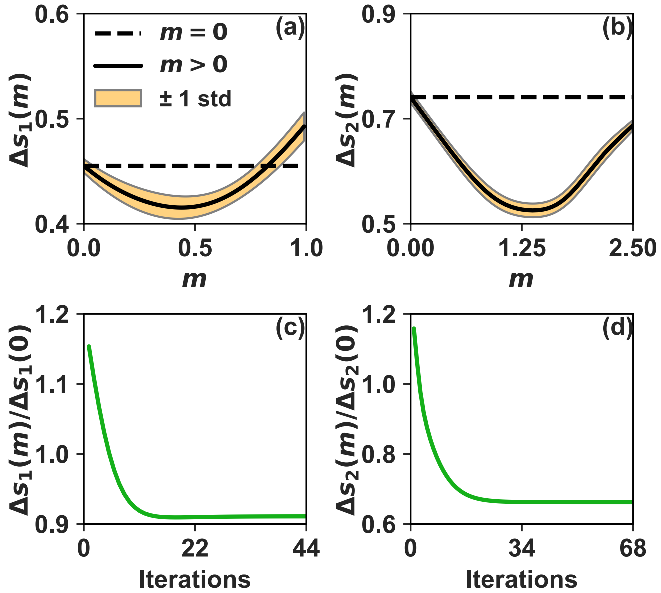
Dirac signal processing on the Network Geometry with Flavor



Dirac signal processing on buoys data







Combining the Dirac operator with algebra
Topological Dirac equation on 3 dimensional lattice

G. Bianconi,
Topological Dirac equation on networks and simplicial complexes
JPhys Complexity (2021)

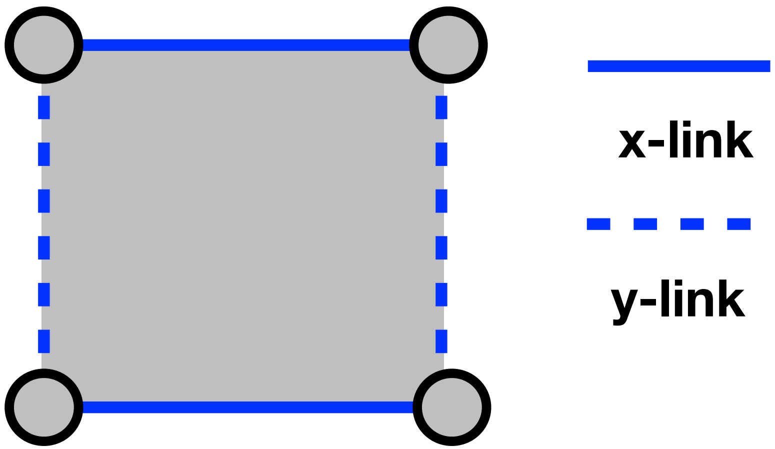
G.Bianconi,
Dirac gauge theory for topological spinors in 3+1 dimensional networks.
***arXiv preprint arXiv:2212.05621 (2022)*.**

Directional Dirac operator on lattices

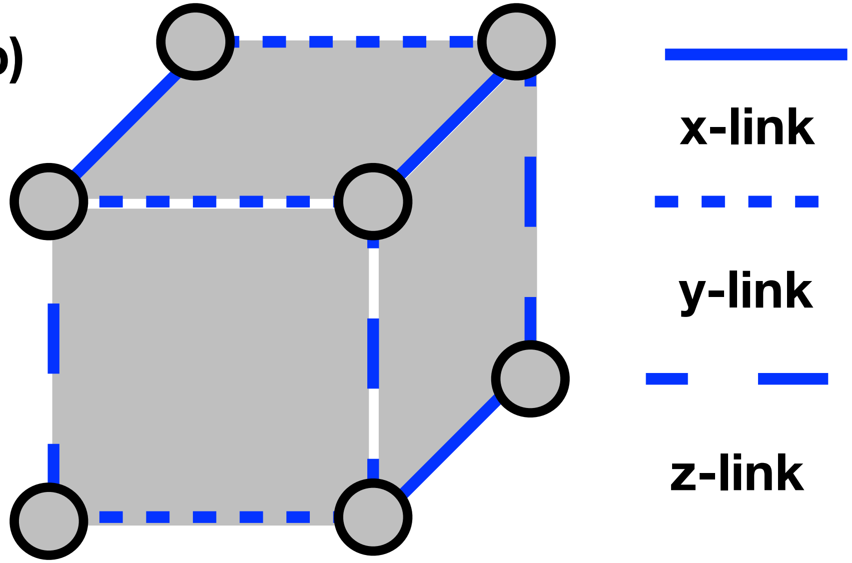
On a lattice links have different directions

The Directional Dirac operator induces a
phase rotation of the topological signal depending on the direction of the links

(a)



(b)



Introducing an algebra

Dirac operator on a network can be enriched by an algebra

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b\mathbf{B}_{[1]} \\ b^{\star}\mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

with $b \in \mathbb{C}, \quad |b| = 1$

Topological spinor for 3-dimensional lattice

In order to treat every type of link differently

by inducing different rotations of the topological spinor,

in 3-d we need to consider the spinor Ψ formed by two 0-cochains and two 1-cochains, i.e.

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Xi} \\ \hat{\boldsymbol{\Psi}} \end{pmatrix},$$

with

$$\boldsymbol{\Xi} = \begin{pmatrix} \chi^{(1)} \\ \chi^{(2)} \end{pmatrix}, \hat{\boldsymbol{\Psi}} = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}$$

Directional Boundary operators and graph Laplacians on 3-dimensional lattice

We consider directional boundary operators only acting between nodes and w-type links

$$[\mathbf{B}_{(w)}]_{r\ell} = \begin{cases} 1 \text{ if } \ell = [s,r] \text{ and } \ell \text{ is a type } w-\text{link} \\ -1 \text{ if } \ell = [r,s] \text{ and } \ell \text{ is a type } w-\text{link} \\ 0 \text{ otherwise} \end{cases}$$

This allows to define the directional graph Laplacians

$$\mathbf{L}_{(w)} = \mathbf{B}_{(w)}\mathbf{B}_{(w)}^{\top}$$

whose sum gives the graph Laplacian of the network

$$\mathbf{L} = \mathbf{L}_{(x)} + \mathbf{L}_{(y)} + \mathbf{L}_{(z)}$$

Note that on square lattices we have that the directional Laplacian commute

$$[\mathbf{L}_{(w)}, \mathbf{L}_{(w')}] = \mathbf{0}$$

Directional Dirac operators on 3-dimensional lattice

In 3d the Directional Dirac operators are defined as

$$\mathbf{D}_{(w)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{w})} \\ \mathscr{B}^{\dagger}_{(\mathbf{w})} & \mathbf{0} \end{pmatrix}$$

with

$$\mathscr{B}_{(x)} = \boldsymbol{\sigma}_1(\mathbf{B}_{(x)}), \qquad \mathscr{B}_{(y)} = \boldsymbol{\sigma}_2(\mathbf{B}_{(y)}), \qquad \mathscr{B}_{(z)} = \boldsymbol{\sigma}_3(\mathbf{B}_{(z)}),$$

where we make use of the Pauli matrices

$$\boldsymbol{\sigma}_1(\mathbf{F}) = \begin{pmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{F} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\sigma}_2(\mathbf{F}) = \begin{pmatrix} \mathbf{0} & -\mathrm{i}\mathbf{F} \\ \mathrm{i}\mathbf{F} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\sigma}_3(\mathbf{F}) = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\mathbf{F} \end{pmatrix}.$$

Spatial directional Dirac operators

The spatial directional Dirac operators

$$\mathbf{D}_{(w)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{w})} \\ \mathscr{B}^{\dagger}_{(\mathbf{w})} & \mathbf{0} \end{pmatrix}$$

are Hermitian

and their square is given by the directional Laplacians

$$[\mathbf{D}_{(w)}]^2 = \mathscr{L}_{(w)} = \begin{pmatrix} \boldsymbol{\sigma}_0(\mathbf{B}_{(w)}\mathbf{B}^{\top}_{(w)}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_0(\mathbf{B}^{\top}_{(w)}\mathbf{B}_{(w)}) \end{pmatrix}$$

Topological Dirac equation on 3-dimensional lattice

The Topological Dirac equation in 3d lattice is given by

$$i\partial_t \boldsymbol{\Psi} = (\not{\mathbf{D}} + m\boldsymbol{\beta})\boldsymbol{\Psi}$$

where

$$\not{\mathbf{D}} = \sum_{w\in\{x,y,z\}} \mathbf{D}_{(w)}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

Dispersion relations and anti-commutation relations

The dispersion relation remain relativistic

$$E^2 = m^2 + |\lambda_x|^2 + |\lambda_y|^2 + |\lambda_z|^2$$

with $\lambda_{(w)}$ indicating the eigenvalue of the directional boundary operator $\mathbf{B}_{(w)}$

despite the directional Dirac operators do not anti-commute (or commute)

$$\{\mathbf{D}_{(x)}, \mathbf{D}_{(y)}\} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & i\epsilon_{1,2,3}\boldsymbol{\sigma}_3(\mathbf{B}^\dagger_{(x)}\mathbf{B}_{(y)} - \mathbf{B}^\dagger_{(y)}\mathbf{B}_{(x)}) \end{pmatrix}$$

Sketch of the derivation

The eigenvalue problem $E\boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$ is equivalent to

$$E\boldsymbol{\Xi} = \sum_{w\in(x,y,z)} \mathscr{B}_{(w)}\hat{\boldsymbol{\Psi}}_{(w)} + m\boldsymbol{\Xi},$$

$$E\hat{\boldsymbol{\Psi}}_{(w)} = \mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi} - m\hat{\boldsymbol{\Psi}}_{(w)}$$

Let us re-order obtaining

$$(E-m)\boldsymbol{\Xi} = \sum_{w\in(x,y,z)} \mathscr{B}_{(w)}\boldsymbol{\hat{P}}si_{(w)},$$

$$(E+m)\hat{\boldsymbol{\Psi}}_{(w)} = \mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi}$$

Therefore

$$(E-m)(E+m)\boldsymbol{\Xi} = \sum_{(w)} \mathscr{B}_{(w)}\mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi} = \boldsymbol{\sigma}_0(\mathbf{L}_{(x)} + \mathbf{L}_{(y)} + \mathbf{L}_{(z)})\boldsymbol{\Xi} = \boldsymbol{\sigma}_0(\mathbf{L}_{[0]})\boldsymbol{\Xi}$$

→ This implies

$$E^2 = |\lambda|^2 + m^2$$

Eigenvalues $\lambda_{(w)}$

Let us consider a regular square lattice of dimension d.

For any direction w the network formed by the nodes and the w-links is a set of disconnected chains. Therefore the eigenvectors are the Fourier modes.

Therefore the eigenvalues of the directional Laplacian $\mathbf{L}_{(w)}$ satisfy

$$|\lambda_{(w)}| = 2|\sin q_{(w)}/2|$$

Where $q_{(w)}$ is the w-component of the wave-number.

Directional Dirac operators on 3+1-dimensional lattice

In 3+1-dimensions the Directional Dirac operators are defined as

$$\mathbf{D}_{(w)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{w})} \\ \mathscr{B}^{\dagger}_{(\mathbf{w})} & \mathbf{0} \end{pmatrix} \textbf{ for } w \in \{x, y, z\} \quad \mathbf{D}_{(t)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{t})} \\ -\mathscr{B}^{\dagger}_{(\mathbf{t})} & \mathbf{0} \end{pmatrix}$$

with

$$\mathscr{B}_{(t)} = i\boldsymbol{\sigma}_0(\mathbf{B}_{(x)}), \qquad \mathscr{B}_{(x)} = \boldsymbol{\sigma}_1(\mathbf{B}_{(x)}), \qquad \mathscr{B}_{(y)} = \boldsymbol{\sigma}_2(\mathbf{B}_{(y)}), \qquad \mathscr{B}_{(z)} = \boldsymbol{\sigma}_3(\mathbf{B}_{(z)}),$$

where we make use of the Pauli matrices

$$\boldsymbol{\sigma}_0(\mathbf{F}) = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{pmatrix}, \qquad \boldsymbol{\sigma}_1(\mathbf{F}) = \begin{pmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{F} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\sigma}_2(\mathbf{F}) = \begin{pmatrix} \mathbf{0} & -\mathrm{i}\mathbf{F} \\ \mathrm{i}\mathbf{F} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\sigma}_3(\mathbf{F}) = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\mathbf{F} \end{pmatrix}.$$

Temporal directional Dirac operator

The temporal directional Dirac operators

$$\mathbf{D}_{(t)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{t})} \\ -\mathscr{B}^{\dagger}_{(\mathbf{t})} & \mathbf{0} \end{pmatrix}$$

Is anti-Hermitian

and its square is given by the directional temporal Laplacian with negative sign

$$[\mathbf{D}_{(t)}]^2 = -\mathscr{L}_{(t)} = -\begin{pmatrix} \boldsymbol{\sigma}_0(\mathbf{B}_{(t)}\mathbf{B}^{\top}_{(t)}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_0(\mathbf{B}^{\top}_{(t)}\mathbf{B}_{(t)}) \end{pmatrix}$$

Topological Dirac equation on 3-dimensional lattice

The Topological Dirac equation in 3d lattice is given by

$$(\not{\mathbf{D}} + m\boldsymbol{\beta})\boldsymbol{\Psi} = 0$$

where

$$\not{\mathbf{D}} = \sum_{w \in \{t,x,y,z\}} \mathbf{D}_{(w)}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

Dispersion relations and anti-commutation relations

The dispersion relation remain relativistic

$$|\lambda_t|^2 = m^2 + |\lambda_x|^2 + |\lambda_y|^2 + |\lambda_z|^2$$

with $\lambda_{(w)}$ indicating the eigenvalue of the directional boundary operator $\mathbf{B}_{(w)}$

despite the directional Dirac operators do not anti-commute (or commute)

$$\{\mathbf{D}_{(x)}, \mathbf{D}_{(y)}\} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & i\epsilon_{1,2,3}\boldsymbol{\sigma}_3(\mathbf{B}^\dagger_{(x)}\mathbf{B}_{(y)} - \mathbf{B}^\dagger_{(y)}\mathbf{B}_{(x)}) \end{pmatrix} \qquad \{\mathbf{D}_{(t)}, \mathbf{D}_{(x)}\} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -i\boldsymbol{\sigma}_1(\mathbf{B}^\dagger_{(t)}\mathbf{B}_{(x)} + \mathbf{B}^\dagger_{(x)}\mathbf{B}_{(t)}) \end{pmatrix}$$

Sketch of the derivation

The eigenvalue problem $(\not{D} + m\boldsymbol{\beta})\boldsymbol{\Psi} = 0$ is equivalent to

$$\mathscr{B}_{(t)}\hat{\boldsymbol{\Psi}}_{(t)} + \sum_{w\in(x,y,z)} \mathscr{B}_{(w)}\hat{\boldsymbol{\Psi}}_{(w)} + m\boldsymbol{\Xi} = 0$$

$$-\mathscr{B}^{\dagger}_{(t)}\boldsymbol{\Xi} - m\hat{\boldsymbol{\Psi}}_{(t)} = 0$$

$$\mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi} - m\hat{\boldsymbol{\Psi}}_{(w)} = 0$$

Let us re-order obtaining

$$m\boldsymbol{\Xi} = -\mathscr{B}_{(t)}\hat{\boldsymbol{\Psi}}_{(t)} - \sum_{w\in(x,y,z)} \mathscr{B}_{(w)}\hat{\boldsymbol{\Psi}}_{(w)}$$

$$m\hat{\boldsymbol{\Psi}}_{(t)} = -\mathscr{B}^{\dagger}_{(t)}\boldsymbol{\Xi}$$

$$m\hat{\boldsymbol{\Psi}}_{(w)} = \mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi}$$

Therefore

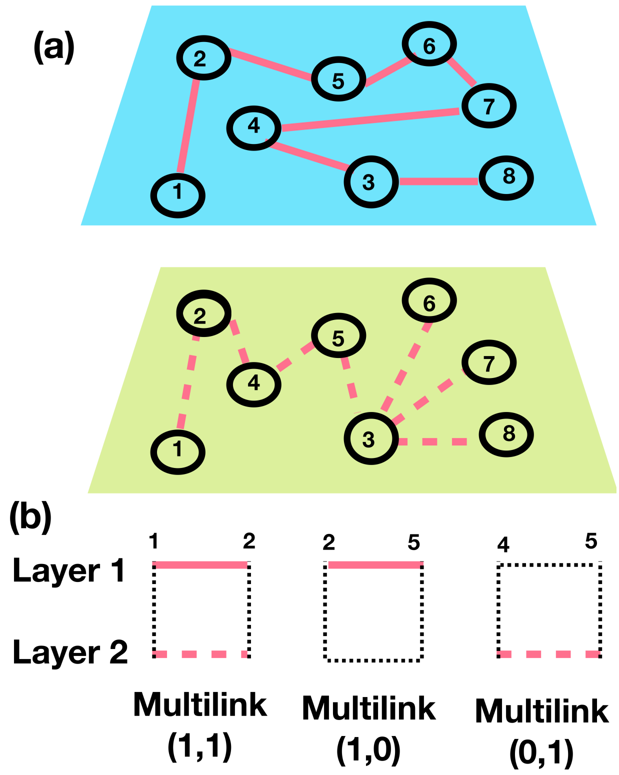
$$m^2\boldsymbol{\Xi} = \mathscr{B}_{(t)}\mathscr{B}^{\dagger}_{(t)}\boldsymbol{\Xi} - \sum_{(w)} \mathscr{B}_{(w)}\mathscr{B}^{\dagger}_{(w)}\boldsymbol{\Xi} = \boldsymbol{\sigma}_0(\mathbf{L}_{(t)} - \mathbf{L}_{(x)} - \mathbf{L}_{(y)} - \mathbf{L}_{(z)})\boldsymbol{\Xi} = \Box\,\boldsymbol{\Xi}$$

→ This implies

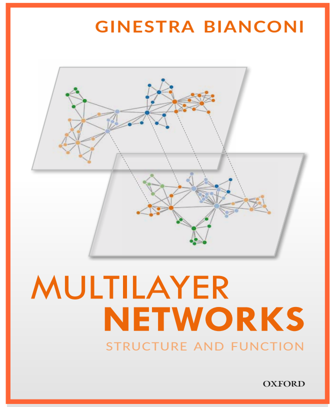
$$|\lambda_t|^2 = m^2 + |\lambda_x|^2 + |\lambda_y|^2 + |\lambda_z|^2$$

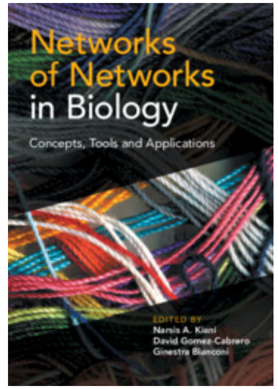
Application to Multiplex Networks

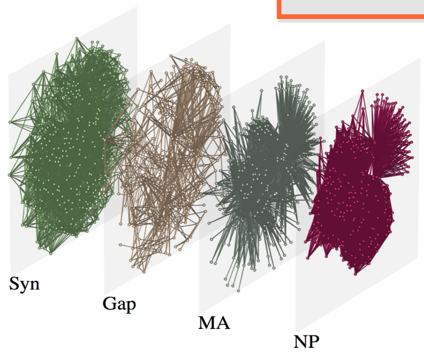
Multilayer Networks

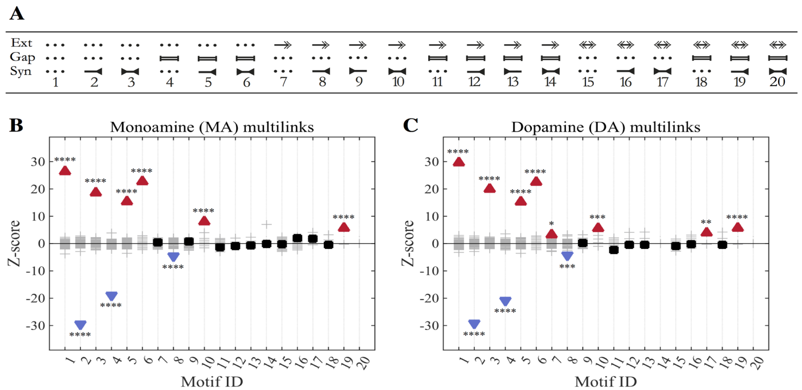


G. Bianconi PRE (2013)









Multilayer connectome of c.elegans, Bentley et al (2016)

Application to multiplex networks

We can "blindly" use the directional Dirac operators of 3d lattices for multiplex networks where one distinguish between different types of multilinks

The dispersion relation is relativistic

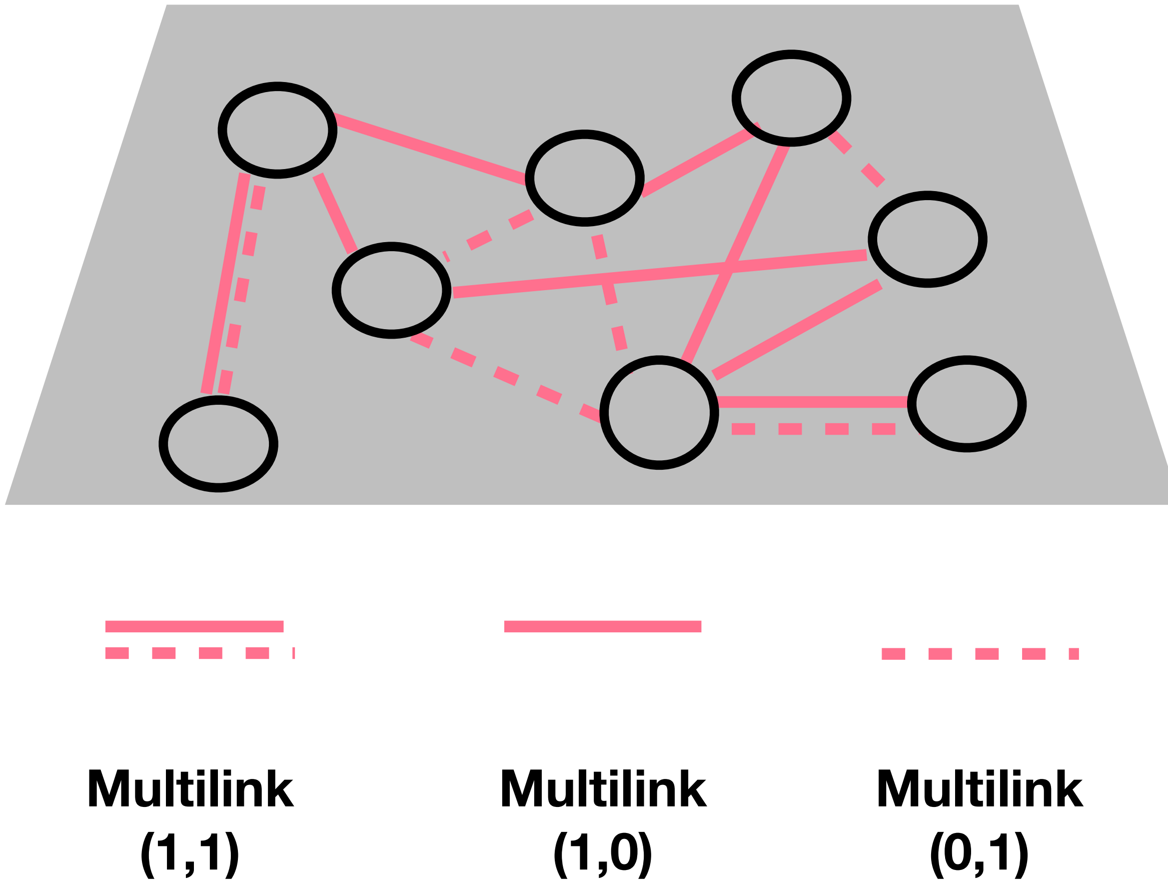
$$E^2 = m^2 + \mu$$

With μ indicating the eigenvalue of

$$\mathbf{L} = \mathbf{L}_{(1,0)} + \mathbf{L}_{(0,1)} + \mathbf{L}_{(1,1)}$$

Note however that in practically all multiplex networks the graphical Laplacians do not commute

$$[\mathbf{L}_{\vec{m}}, \mathbf{L}_{\vec{m}'}] \neq \mathbf{0}$$



Weighted and Normalised Dirac operator

Recall from lecture 1 the definition of coboundary operator and its dual

Cochains

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \rightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

L^2 norm between cochains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathcal{K})} f_r^2$$

This scalar product can be generalised by introducing metric matrices (see next)

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{[m+1]}^{\top} \mathbf{f} \equiv \bar{\mathbf{B}}_{[m+1]} \mathbf{f}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \rightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

If follows that if $f' \in C^m$ is given by $f' = \delta_m^* g$.

Then $\mathbf{f}' = \bar{\mathbf{B}}_{[m+1]}^\top \mathbf{g} = \mathbf{B}_{[m+1]} \mathbf{g}$

How this definition change is we introduce a non-trivial metric?

Metric matrices

We introduce the $N_m \times N_m$ metric matrices $\mathbf{G}^{-1}_{[m]}$ typically taken to be diagonal with elements

$$\mathbf{G}^{-1}_{[m]}(\alpha_r, \alpha_r) = w(\alpha_r)$$

where $w(\alpha_r)$ indicates the affinity weight (inverse of a "distance") associated to the simplex α_r

For a graph, typical choices of these matrices are

$$\mathbf{G}^{-1}_{[1]}([r,s],[r,s]) = w([r,s])$$ weight of the link

$$\mathbf{G}^{-1}_{[0]}([r],[r]) = \sum_{s \in Q_0(\mathscr{K})} w([r,s])$$ strength (weighted degree) of the node

Scalar product between co-chains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{G}_{[m]}^{-1} \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathcal{K})} f_r [G_{[m]}^{-1}]_{rs} f_s$$

For $\mathbf{G}_{[m]} = \mathbf{I}$ we recover the standard L^2 norm.

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{[m+1]}^\top \mathbf{f} \equiv \bar{\mathbf{B}}_{[m+1]} \mathbf{f}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \longrightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

Ajoint operator δ_m^*

We define the matrix $\bar{\mathbf{B}}^*_{[m+1]}$ as the matrix representing δ_m^*,

i.e. if $f' = \delta_m^* g,$ then $\mathbf{f}' = \bar{\mathbf{B}}^*_{[m+1]}\mathbf{g}$

From the definition it follows that

$$\bar{\mathbf{B}}^*_{[m+1]} = \mathbf{G}_{[m]}\bar{\mathbf{B}}^\top_{[m+1]}\mathbf{G}^{-1}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}$$

Hence if $\mathbf{G}_{[m]} = \mathbf{I}, \mathbf{G}_{[m+1]} = \mathbf{I}$ then $\mathbf{B}^*_{[m+1]} = \mathbf{B}_{[m+1]}$

Proof

We define the matrix $\mathbf{B}^*_{[m+1]}$ as the matrix representing δ^*_m,

i.e. if $f' = \delta^*_m g$, then $\mathbf{f}' = \mathbf{B}^*_{[m+1]}\mathbf{g}$

We have the scalar product

$$\langle g, \delta_m f \rangle = \mathbf{g}\mathbf{G}^{-1}_{[m+1]}\bar{\mathbf{B}}_{[m+1]}\mathbf{f}$$

$$\langle \delta^*_m g, f \rangle = \mathbf{g}(\mathbf{B}^*)^\top_{[m+1]}\mathbf{G}^{-1}_{[m]}\mathbf{f}$$

If follows that for any **f** and **g**

$$\mathbf{g}\mathbf{G}^{-1}_{[m+1]}\bar{\mathbf{B}}_{[m+1]}\mathbf{f} = \mathbf{g}\left(\bar{\mathbf{B}}^*_{[m+1]}\right)^\top \mathbf{G}^{-1}_{[m]}\mathbf{f}$$

Hence

$$\mathbf{B}^*_{[m+1]} = \mathbf{G}_{[m]}\bar{\mathbf{B}}^\top_{[m+1]}\mathbf{G}^{-1}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}$$

Weighed Hodge Laplacian

The Hodge-Laplacians

The m-dimensional Hodge-Laplacian L_m is defined as

$$L_m = L_m^{up} + L_m^{down}$$

where up and down m-dimensional Hodge Laplacians are given by

$$\begin{aligned} L_m^{up} &= \delta_m^* \delta_m, \\ L_m^{down} &= \delta_{m-1} \delta_{m-1}^*. \end{aligned}$$

The weighted Hodge Laplacian obeys Hodge decomposition

Hodge decomposition for weighted Hodge Laplacians

The weighted Hodge Laplacians

$$\mathbf{L}^{up}_{[m]} = \bar{\mathbf{B}}^{*}_{[m+1]}\bar{\mathbf{B}}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}$$

$$\mathbf{L}^{down}_{[m]} = \bar{\mathbf{B}}_{[m]}\bar{\mathbf{B}}^{*}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]}$$

obey Hodge decomposition, i.e. $\mathbf{L}^{up}_{[m]}\mathbf{L}^{down}_{[m]} = \mathbf{0}, \quad \mathbf{L}^{down}_{[m]}\mathbf{L}^{up}_{[m]} = \mathbf{0}$

Proof:

$$\mathbf{L}^{up}_{[m]}\mathbf{L}^{down}_{[m]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]} = \mathbf{0}$$

$$\mathbf{L}^{down}_{[m]}\mathbf{L}^{up}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]}\mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]} = \mathbf{0}$$

Weighted Dirac operator

Weighted Dirac operator on a network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & b^* \bar{\mathbf{B}}^*_{[1]} \\ b\bar{\mathbf{B}}_{[1]} & \mathbf{0} \end{pmatrix}$$

with $b \in \mathbb{C}, \ |b| = 1$ **and** $\bar{\mathbf{B}}^*_{[1]} = \mathbf{G}_{[0]} \bar{\mathbf{B}}^\top_{[1]} \mathbf{G}^{-1}_{[1]}$

$$\hat{\mathbf{D}}^2 = \mathscr{L} = \begin{pmatrix} \hat{\mathbf{L}}_{[0]} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}_{[1]} \end{pmatrix}$$

with

$$\hat{\mathbf{L}}_{[0]} = \bar{\mathbf{B}}^*_{[1]} \bar{\mathbf{B}}_{[1]}, \hat{\mathbf{L}}_{[1]} = \bar{\mathbf{B}}_{[1]} \bar{\mathbf{B}}^*_{[1]}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Normalised Dirac operator

If the matrix $\mathbf{G}_{[1]}^{-1}$,$\mathbf{G}_{[0]}^{-1}$ are the diagonal matrices with elements

$$\mathbf{G}_{[1]}^{-1}(\ell,\ell) = w_\ell/2$$
$$\mathbf{G}_{[0]}^{-1}(r,r) = \sum_{\ell \in E_r} w_\ell$$

The weighted Dirac operator is also called normalised Dirac operator and has eigenvalues bounded in absolute value by one $|\lambda| \leq 1$

F. Baccini, F. Geraci and G. Bianconi (2022)

Normalised Dirac operator of unweighted networks

If the weights of all the links are one, i.e. $w_\ell = 1$ we have

That the matrices $\mathbf{G}_{[1]}^{-1}$,$\mathbf{G}_{[0]}^{-1}$ are the diagonal matrices with elements

$$\mathbf{G}_{[1]}^{-1}(\ell,\ell) = 1/2 \quad \mathbf{G}_{[0]}^{-1}(r,r) = k_r$$

The normalised Dirac operator is given by

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \mathbf{K}^{-1}\mathbf{B}_{[1]}/2 \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Symmetric Normalised Dirac operator

The normalised Dirac operator of an unweighted network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \mathbf{K}^{-1}\mathbf{B}_{[1]}/2 \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

Can be symmetrized obtaining the Dirac operator with the same spectrum given by

$$\tilde{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \mathbf{K}^{-1/2}\mathbf{B}_{[1]}/\sqrt{2} \\ \mathbf{B}_{[1]}^{\top}\mathbf{K}^{-1/2}/\sqrt{2} & \mathbf{0} \end{pmatrix}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Normalised Dirac operator on a network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \bar{\mathbf{B}}^*_{[1]} \\ \bar{\mathbf{B}}_{[1]}/2 & \mathbf{0} \end{pmatrix}$$

with $$\bar{\mathbf{B}}^*_{[1]} = \mathbf{G}_{[0]}\bar{\mathbf{B}}^\top_{[1]}\mathbf{G}^{-1}_{[1]} = \mathbf{K}_0^{-1}\mathbf{B}_{[1]}$$

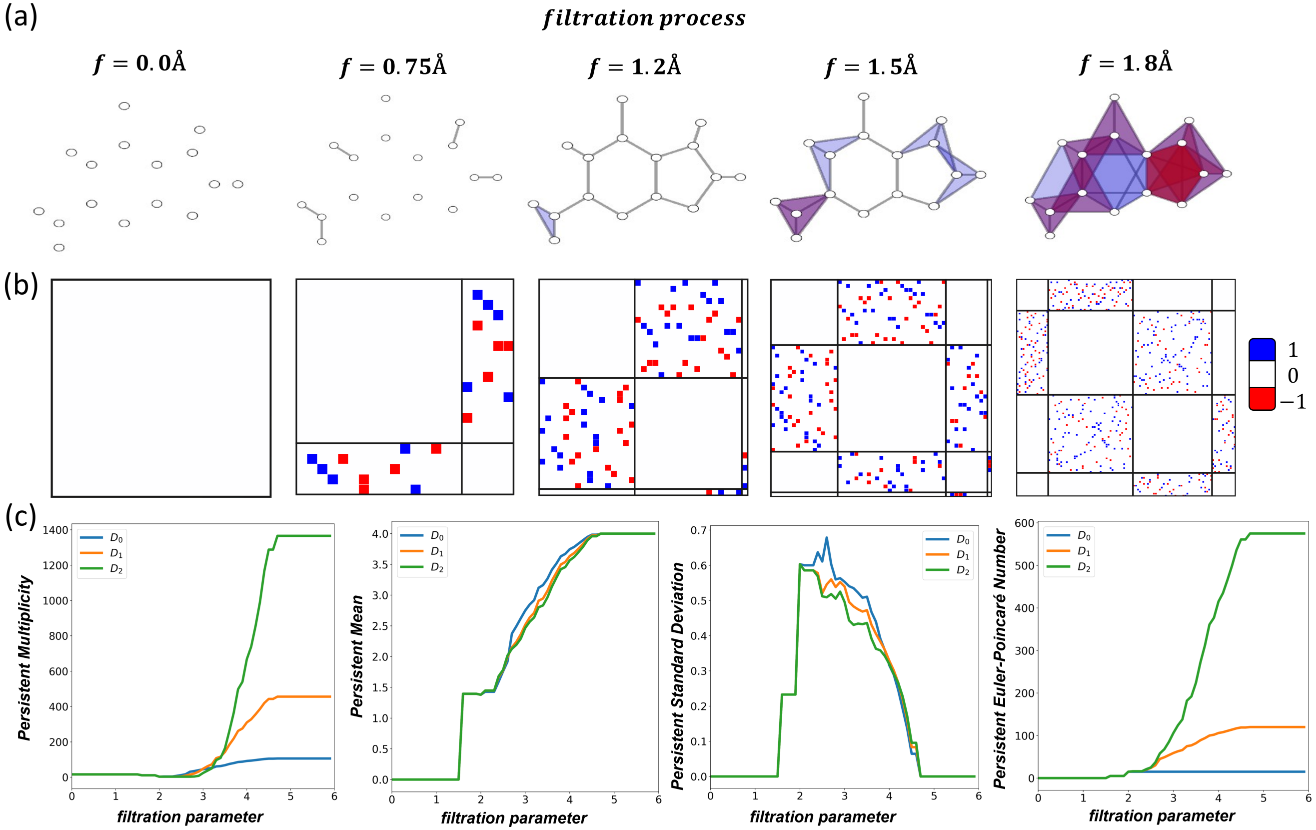
$$\hat{\mathbf{D}}^2 = \mathscr{L} = \begin{pmatrix} \hat{\mathbf{L}}_{[0]} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}_{[1]} \end{pmatrix}$$

with

$$\hat{\mathbf{L}}_{[0]} = \bar{\mathbf{B}}^*_{[1]}\bar{\mathbf{B}}_{[1]}, \hat{\mathbf{L}}_{[1]} = \bar{\mathbf{B}}_{[1]}\bar{\mathbf{B}}^*_{[1]}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Topological Data analysis
Persistent Dirac for molecular representations



JJ Wee, G. Bianconi, K. Xia (2023)

Lesson III:
The Dirac operator
on networks and simplicial complexes

- Dirac operator on networks
  - Eigenvalues, Eigenvectors Chirality
  - Dirac equation (basic version with no associated algebra)

- Dirac operator on simplicial complexes
- Dirac operator & Algebra
  - Topological Dirac equation in 3 dimension
  - Topological Dirac equation in 3+1 dimensions

- Weighted and Normalised Dirac operator

References: Applications

The Dirac operator

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Online workshop on the Dirac operator

Dirac equation between discrete and continuous: new trends and applications

Online workshop
May 3-4
co-organized by Delio Mugnolo (FernUniversität in Hagen) and myself

we expect an exciting interplay of operator theory, noncommutative geometry, and network theory and applied topology.

Website
https://mat-dyn-net.eu/en/news/dirac-discrete-continuous-23