# Higher-order networks

*An introduction to simplicial complexes*
*Lesson II*

**Franqui Chair Lessons**

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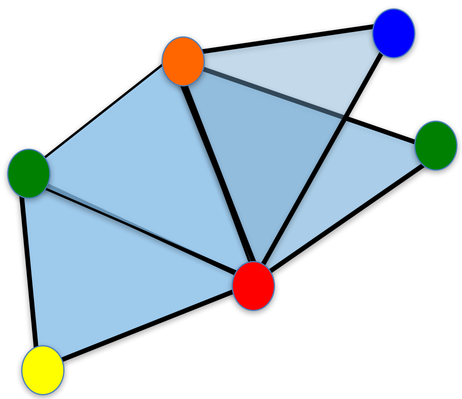
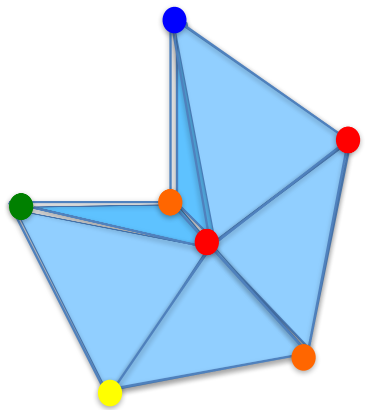
# Higher-order networks

**Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.**



**d=2 simplicial complex**

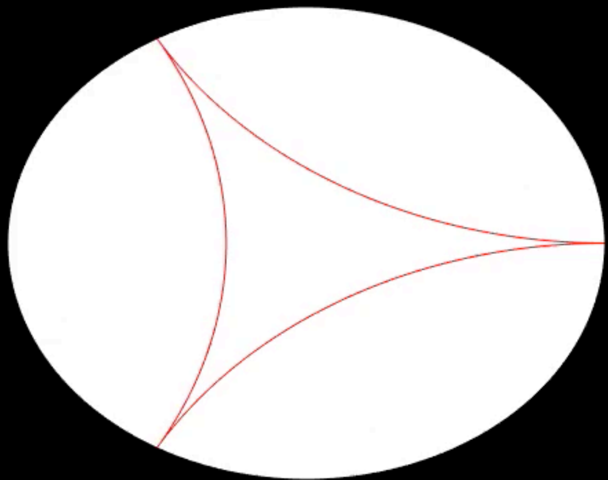


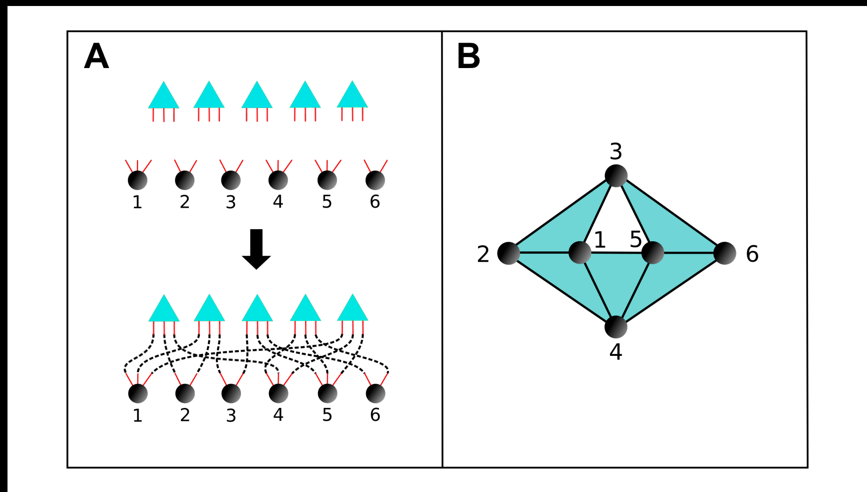
**d=3 simplicial complex**

# Simplicial complex models

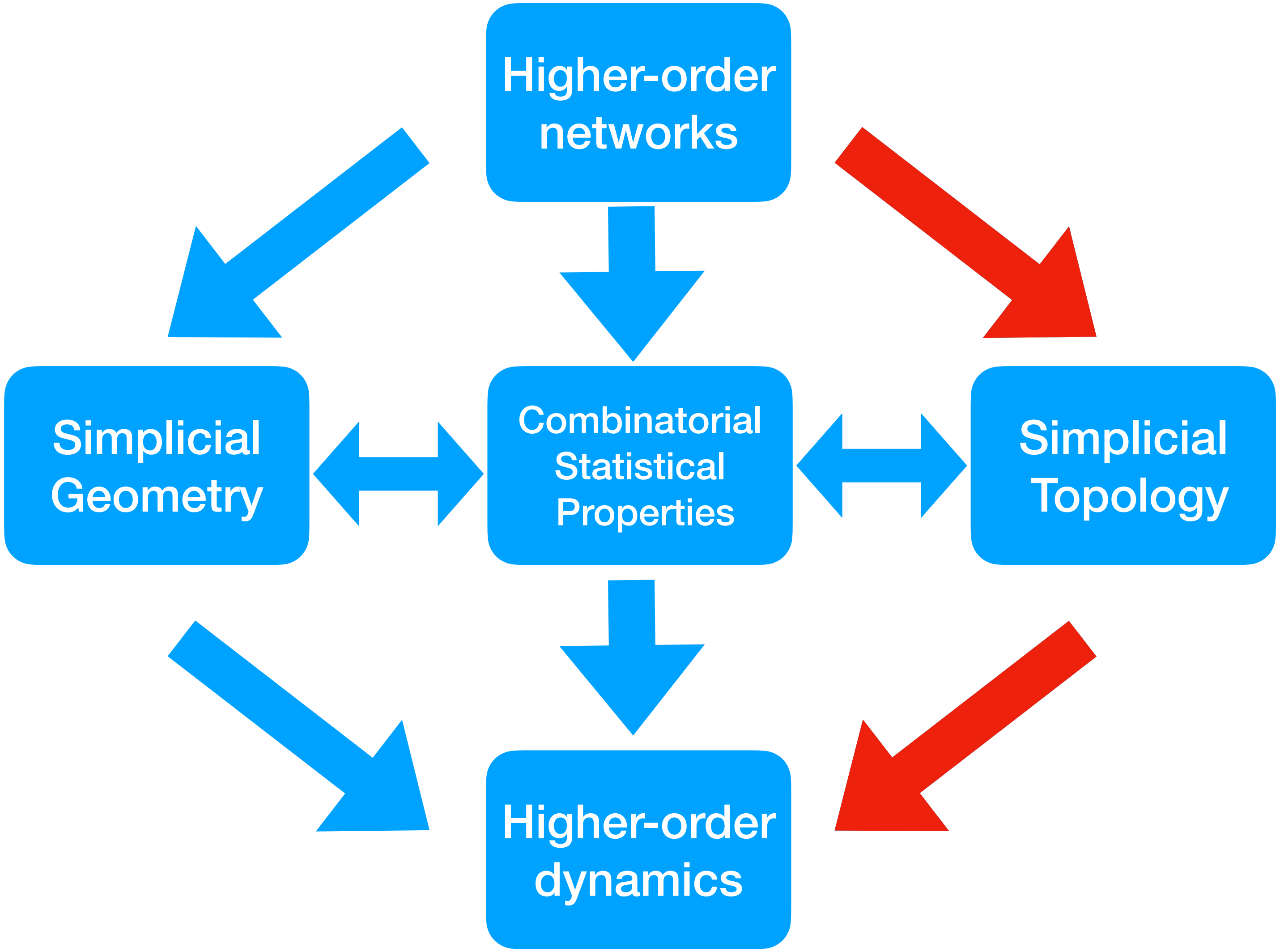
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





# Higher-order structure and dynamics

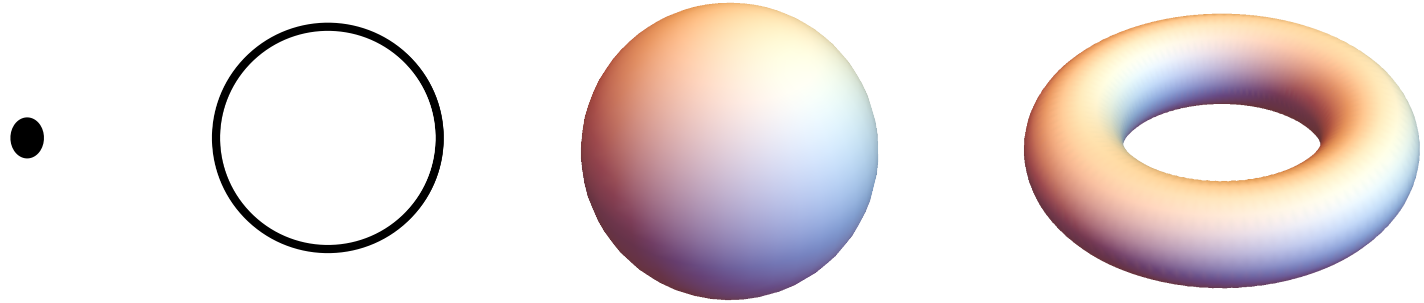


# Lesson II:
# Topological Kuramoto model

- **Spectral properties of the Laplacians**
- **Topological Kuramoto model**
  - **The Kuramoto model on graphs**
  - **The Topological Kuramoto model**

# Summary of
# Algebraic Topology

# Betti numbers

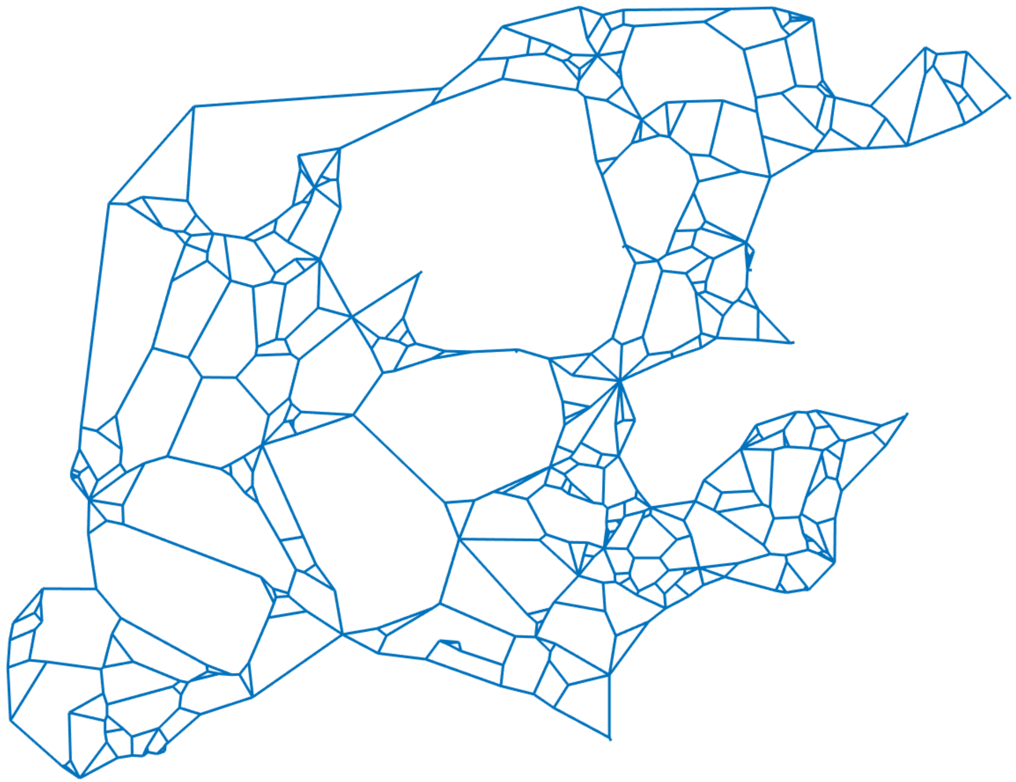


| Point | Circle | Sphere | Torus |
|---|---|---|---|
| $\beta_0 = 1$ | $\beta_0 = 1$ | $\beta_0 = 1$ | $\beta_0 = 1$ |
| $\beta_1 = 0$ | $\beta_1 = 1$ | $\beta_1 = 0$ | $\beta_1 = 2$ |
| $\beta_2 = 0$ | $\beta_2 = 0$ | $\beta_2 = 1$ | $\beta_2 = 1$ |

## Euler characteristic

$$\chi = \sum_n (-1)^n \beta_n$$

# Betti number 1

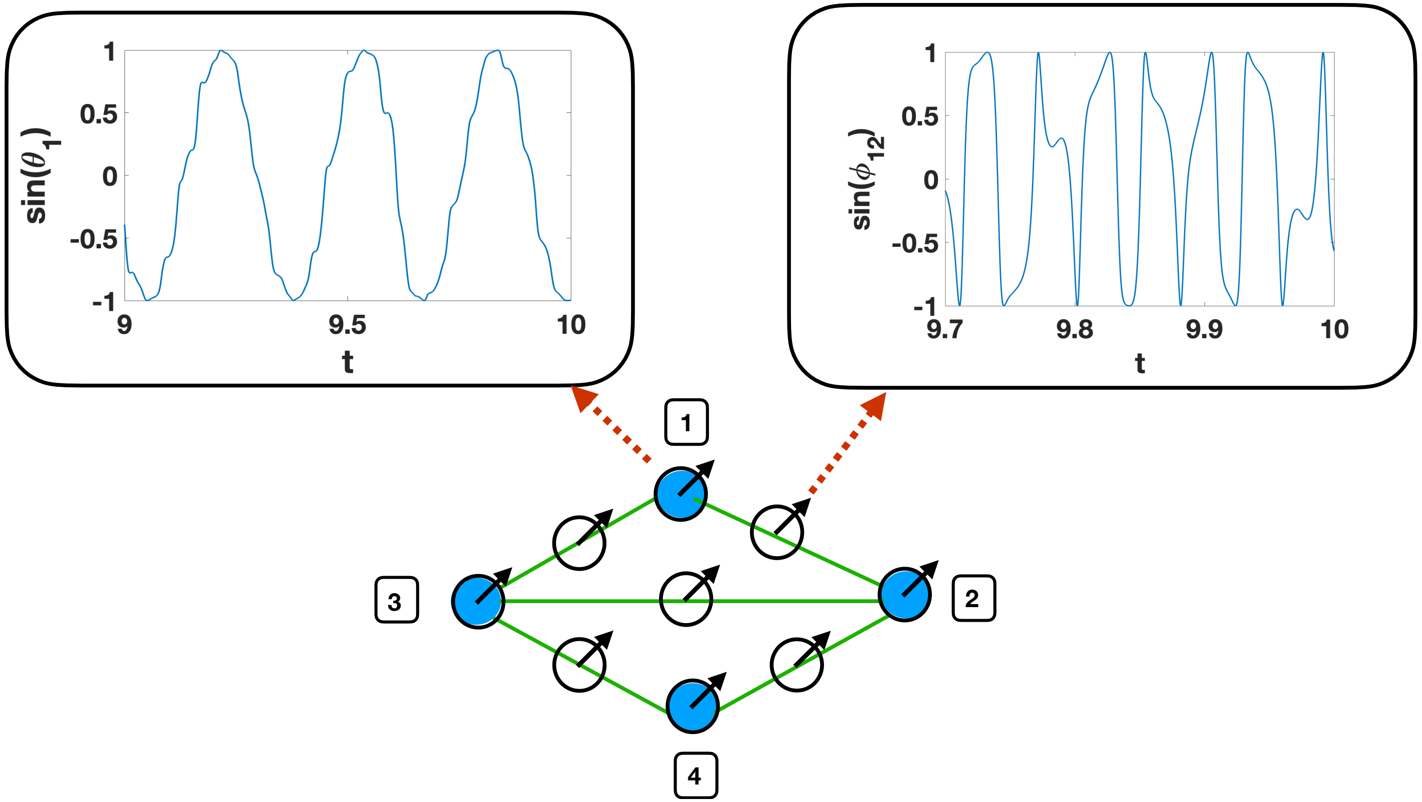


Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

# Topological signals, Hodge Laplacian

# Topological signals

**Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called *topological signals***



# Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

***Topological signals***
***are co-chains or vector fields***

# Comparison between spectral properties of graphs and simplicial complexes

# Graphs and networks

## Definition

A *graph* is an ordered pair $G = (V, E)$ comprising a set $V$ of *vertices* connected by the set $E$ of *edges*.

A graph is a 1-dimensional simplicial complex

## Definition

A *network* is the graph $G = (V, E)$ describing the set of interactions between the constituents of a complex system. The vertices of a network are called *nodes* and the edges *links*.

The *network size* $N$ is the total number of nodes in the network $N=|V|$.
The total number of edges $L$ is given by $L=|E|$.

# Simple networks

**Adjacency matrix**

A simple network is fully determined by its adjacency matrix.

The adjacency matrix $a$ of a simple network is a $N \times N$ matrix of elements given by

$$a_{rs} = \begin{cases} 1 & \textbf{if } r \textbf{ is linked to } s \\ 0 & \textbf{otherwise.} \end{cases}$$

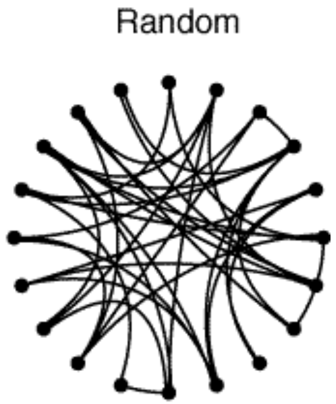
*The adjacency matrix of a simple network is symmetric.*

**Definition**

In a simple network the *degree* $k_i$ of node $i$ is given by the total number of links incident to node , i.e.

$$k_r = \sum_{s=1}^{N} a_{rs}$$

# Random graphs




Uncorrelated maximally random graphs with given degree sequence

Are generated by ensembles in which each edge $(r, s)$ is drawn independently with probability

$$\boxed{p_{rs} = \langle a_{rs} \rangle = \frac{k_r k_s}{\langle k \rangle N}}$$

# Graph Laplacian in terms of the boundary matrix

The graph Laplacian of elements

$$\left(L_{[0]}\right)_{rs} = \delta_{rs}k_r - a_{rs}$$

Can be expressed in terms of the 1-boundary matrix

as

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}$$

we have

dim ker$(\mathbf{L}_{[0]}) = \beta_0$

# Graph Laplacian

The graph Laplacian matrix is defined as

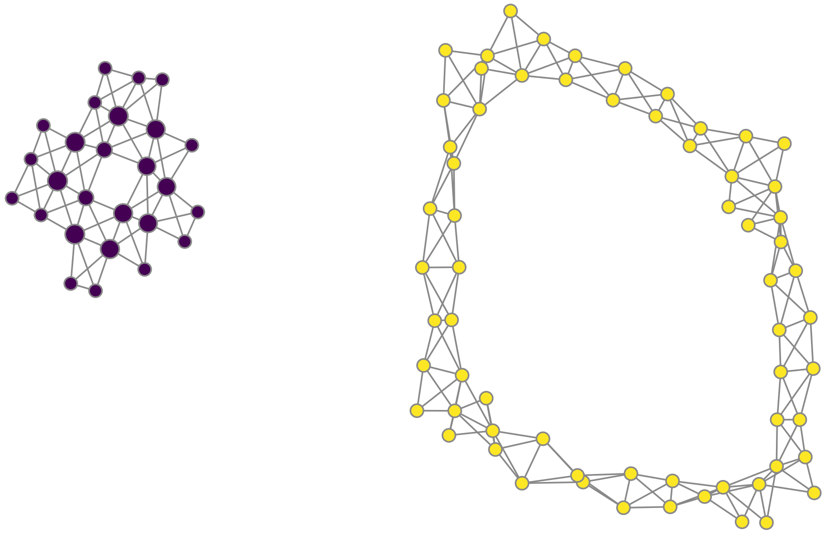
$$L_{ij} = \delta_{ij} k_i - a_{ij}$$

The graph Laplacian is a semi-definite positive matrix that in a connected network has eigenvalues

$$0 = \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_N$$

**The Laplacian is key for describing diffusion processes and the Kuramoto model on networks and constitutes a natural link between topology and dynamics**

# Harmonic eigenvectors of the graph Laplacian



The quadratic form of the graph Laplacian reads

$$\mathbf{X}^\top \mathbf{L}_{[0]} \mathbf{X} = \frac{1}{2} \sum_{r,s} a_{rs} (X_r - X_s)^2$$

Therefore the harmonic eigenvectors of the graph Laplacian are constant on each connected component of the graph and zero everywhere else.

# Connected network

**A connected network has a single eigenvector in the kernel of the graph Laplacian.**

**This eigenvector is constant on each node of the network, i.e.**

$$\mathbf{u} = \frac{1}{\sqrt{N}}\mathbf{1}$$

# Hodge Laplacians

The higher order Laplacians can be defined in terms of the incidence matrices as

$$\boxed{\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]} + \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.}$$

The higher order Laplacian can be decomposed as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up},$$

with

$$\mathbf{L}_{[n]}^{down} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]},$$
$$\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.$$

# Expression of the matrix elements of the Hodge Laplacians

$$\mathbf{L}_m^{\mathrm{up}}(r,s)=\begin{cases} k_{m+1,m}(\alpha_r^m), & r=s. \\ -1, & r\neq s, \alpha_r^m \frown \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ 1, & r\neq s, \alpha_r^m \frown \alpha_s^m, \alpha_r^m \nsim \alpha_s^m. \\ 0, & \text{otherwise.} \end{cases}$$

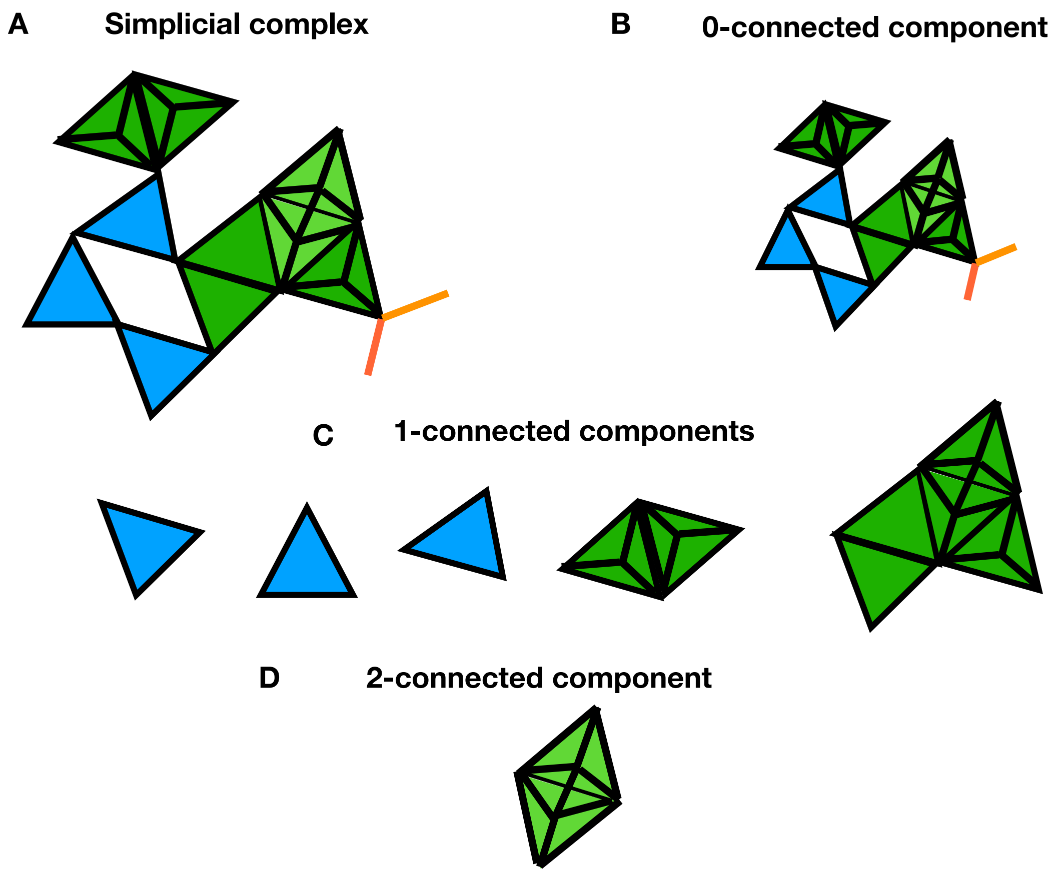
$$\mathbf{L}_m^{\mathrm{down}}(r,s)=\begin{cases} m+1, & r=s. \\ 1, & r\neq s, \alpha_r^p \smile \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ -1, & r\neq s, \alpha_r^p \smile \alpha_s^m, \alpha_r^p \nsim \alpha_s^m. \\ 0, & \text{otherwise.} \end{cases}$$

**The m-dimensional up- Hodge Laplacian has nonzero elements only among upper incident m-simplices (simplices which are faces of a common m+1 simplex) The eigenvectors have support on the m-connected components**

**The m-dimensional down-Hodge Laplacian has nonzero elements only among lower incident m-simplices (simplifies sharing a m-1 face) The eigenvectors have support on the (m-1)—connected components**

**Here ~ indicates similar orientation with respect to the lower-simplices**

# m-connected components



# Expression of the matrix elements of the Hodge Laplacians

$$\mathbf{L}_m(r,s) = \begin{cases} k_{m+1,m}(\alpha_r^m) + m + 1, & r = s. \\ 1, & r \neq s, \alpha_r^m \not\frown \alpha_s^m, \alpha_r^m \smile \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ -1, & r \neq s, \alpha_r^m \not\frown \alpha_s^m, \alpha_r^m \smile \alpha_s^m, \alpha_r^m \nsim \alpha_s^m. \\ 0 & \text{otherwise.} \end{cases}$$

**for** $0 < m < d$

**The matrix elements of the Hodge Laplacian is only non zero among lower adjacent simplices that are not upper-adjacent**

# Harmonic eigenvectors of the Hodge Laplacian

**The dimension of the kernel of the Hodge Laplacian**

**is given by the corresponding Betti number**

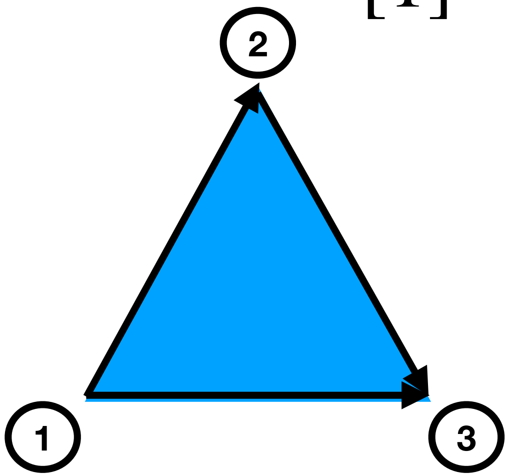
$$\textbf{dim ker}(\mathbf{L}_{[m]}) = \beta_m$$

**The harmonic eigenvectors**

**are associated to the generators of the homology**

**They are in general non-uniform over the $m$-simplices of the simplicial complex**

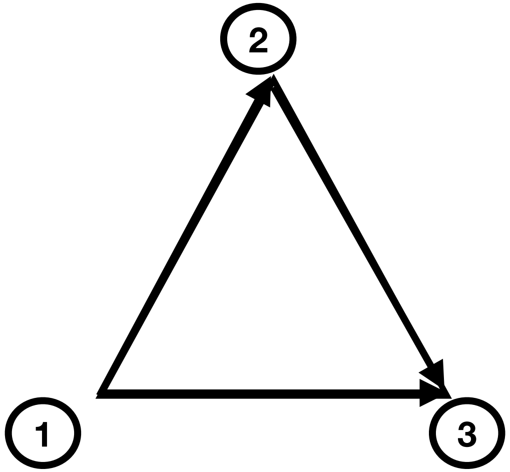
# Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian




$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \end{array} \begin{array}{ccc} [1,2] & [2,3] & [1,3] \\ -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}, \quad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [2,3] \\ [1,3] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ 1 \\ -1 \end{array}. \quad \mathbf{L}_{[1]} = \begin{array}{c} \\ [1,2] \\ [2,3] \\ [1,3] \end{array} \begin{array}{ccc} [1,2] & [1,3] & [1,3] \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{c} \\ [1,2] \\ [2,3] \\ [1,3] \end{array} \begin{array}{ccc} \mu = 3 & \mu = 3 & \mu = 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}, \quad \boldsymbol{\Sigma}_{[1]} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

# Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian

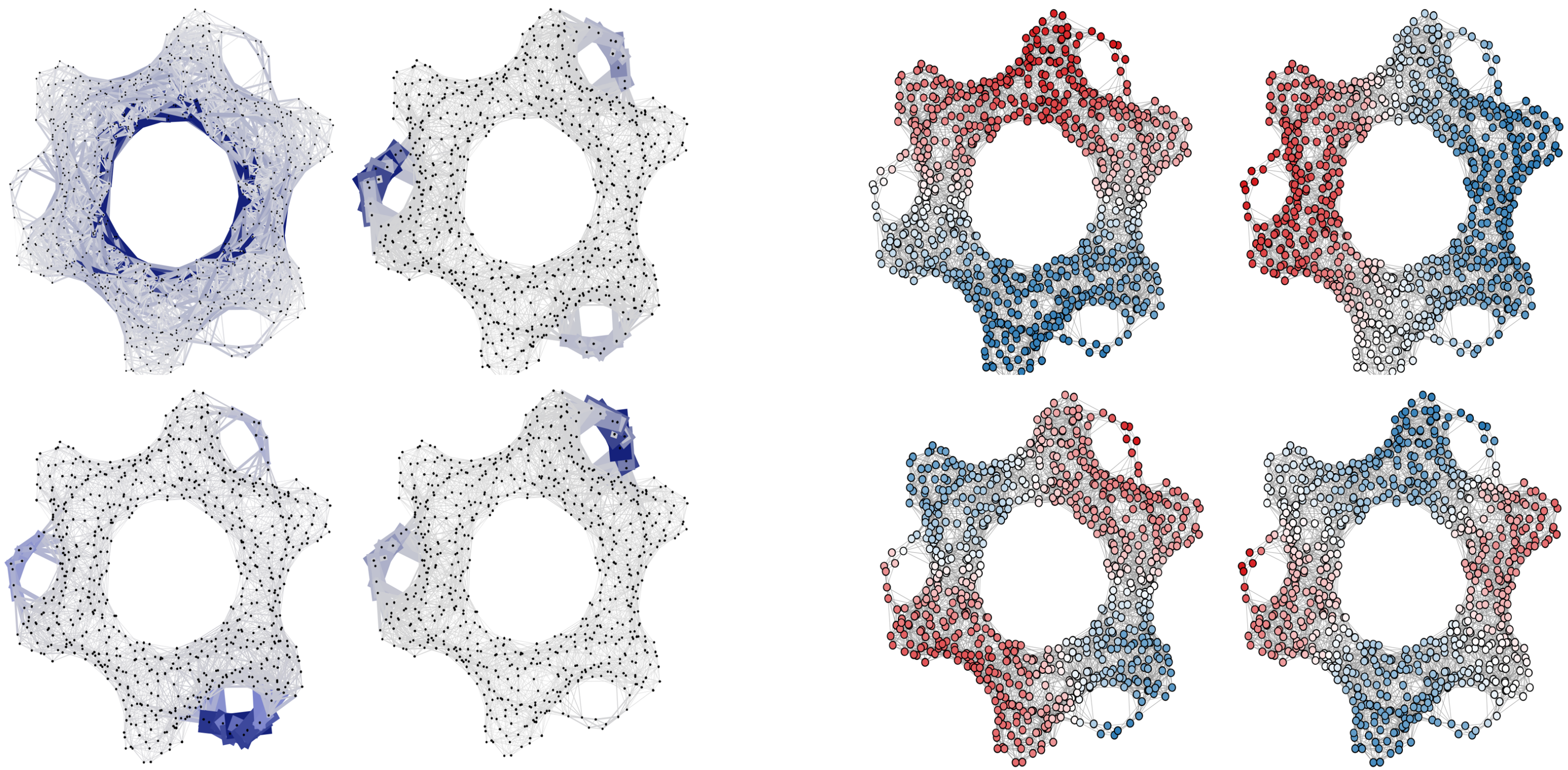



$$\mathbf{B}_{[1]} = \begin{array}{c|ccc} & [1,2] & [2,3] & [1,3] \\ \hline [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \mathbf{0} \qquad \mathbf{L}_{[1]} = \begin{array}{c|ccc} & [1,2] & [1,3] & [1,3] \\ \hline [1,2] & 2 & -1 & 1 \\ [2,3] & -1 & 2 & 1 \\ [1,3] & 1 & 1 & 2 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{c|ccc} & \mu = 0 & \mu = 3 & \mu = 3 \\ \hline [1,2] & 1/\sqrt{3} & 1/\sqrt{2} & 0 \\ [2,3] & 1/\sqrt{3} & 0 & 1/\sqrt{2} \\ [1,3] & -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{2} \end{array}, \qquad \mathbf{\Sigma}_{[1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

# Homology of molecular molecules

**Harmonic eigenvectors** **Non-harmonic eigenvectors**



Wee et al. (2023)

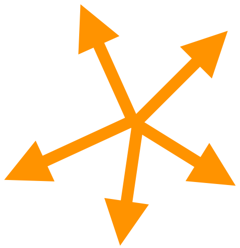
# Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

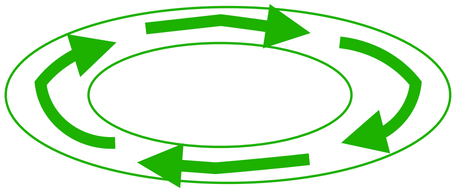
in a irrotational, harmonic and solenoidal components

$$\mathbb{R}^{D_m} = \textbf{im}(\mathbf{B}_{[m]}^{\top}) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})$$

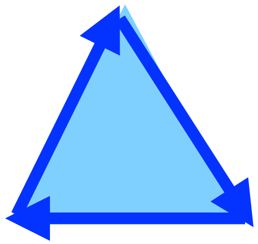
which in the case of topological signals of the links can be sketched as



**Irrotational component**
**Gradient Flow**



**Harmonic component**



**Solenoidal component**
**Curl Flow**

# Hodge-decomposition

**Given that** $\mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0} \quad \mathbf{B}_{[m-1]}^{\top}\mathbf{B}_{[m]}^{\top} = \mathbf{0}$

**and that** $\mathbf{L}_{[m]}^{up} = \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}, \quad \mathbf{L}_{[m]}^{down} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]}$

We have:

$$\mathbf{L}_{[m]}^{down}\mathbf{L}_{[m]}^{up} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}_{[m]}^{up} \subseteq \mathrm{ker}\mathbf{L}_{[m]}^{down}$$

$$\mathbf{L}_{[m]}^{up}\mathbf{L}_{[m]}^{down} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}_{[m]}^{down} \subseteq \mathrm{ker}\mathbf{L}_{[m]}^{up}$$

# Hodge decomposition

The Hodge decomposition can be summarised as

$$C^m = \textbf{im}(\mathbf{B}_{[m]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})$$

This means that $\mathbf{L}_{[m]}, \mathbf{L}_{[m]}^{up}, \mathbf{L}_{[m]}^{down}$ are commuting and can be diagonalised simultaneously. In this basis these matrices have the block structure

$$\mathbf{U}^{-1}\mathbf{L}_{[m]}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[m]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[m]}^{up} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[m]}^{down}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[m]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[m]}^{up}\mathbf{U} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[m]}^{up} \end{pmatrix}$$

- **Therefore an eigenvector in the ker of $\mathbf{L}_{[m]}$ is also in the ker of both $\mathbf{L}_{[m]}^{up}, \mathbf{L}_{[m]}^{down}$**
- **An eigenvector corresponding to an non-zero eigenvalue of $\mathbf{L}_{[m]}$ is either a non-zero eigenvector of $\mathbf{L}_{[m]}^{up}$ or a non-zero eigenvector of $\mathbf{L}_{[m]}^{down}$**

# Hodge decomposition

Every $m$-cochain (topological signal) can be decomposed in a unique way thanks to the Hodge decomposition as

$$\mathbb{R}^{D_m} = \text{im}(\mathbf{B}_{[m]}^{\top}) \oplus \text{ker}(\mathbf{L}_{[m]}) \oplus \text{im}(\mathbf{B}_{[m+1]})$$

therefore every $m$-cochain can be decomposed in a unique way as

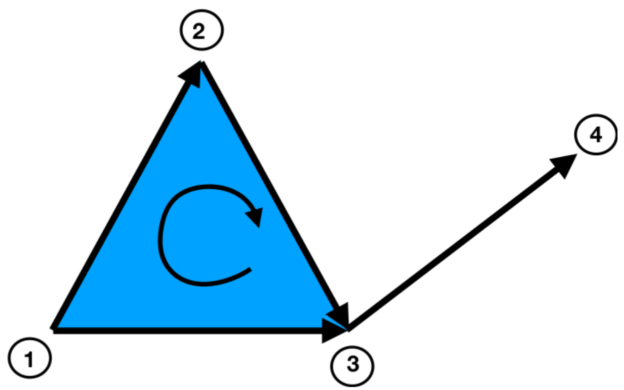
$$\mathbf{x} = \mathbf{x}^{[1]} + \mathbf{x}^{[2]} + \mathbf{x}^{harm}$$

With

$$\mathbf{x}^{[1]} = \mathbf{L}_{[m]}^{up} \mathbf{L}_{[m]}^{up,+} \mathbf{x}$$

$$\mathbf{x}^{[2]} = \mathbf{L}_{[m]}^{down} \mathbf{D}_{[m]}^{down,+} \mathbf{x}$$

# Boundary Operators




**Boundary operators**

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

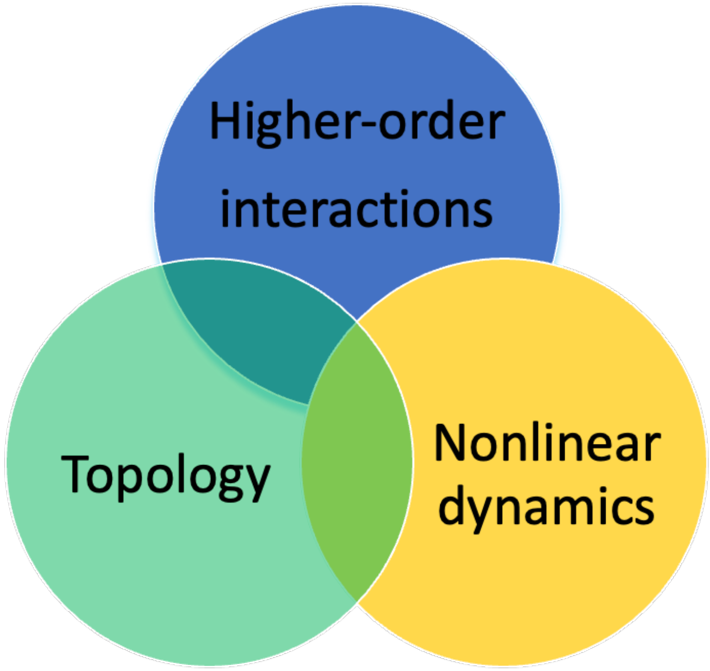
$\mathbf{B}_{[1]}^\top$ **Discrete gradient**

$\mathbf{B}_{[2]}^\top$ **Discrete Curl**

**The boundary of the boundary is null**

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^\top \mathbf{B}_{[m-1]}^\top = \mathbf{0}$$

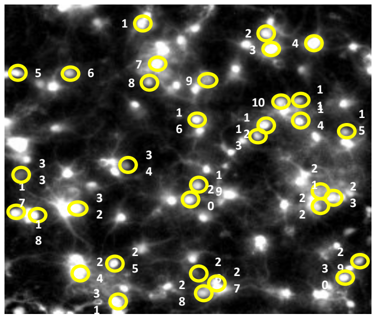
# Complexity challenge

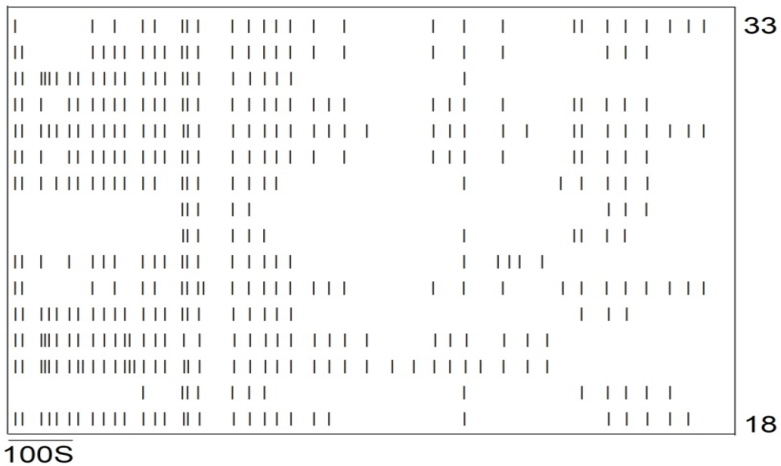


# Kuramoto model on a graph

# Synchronization is a fundamental dynamical process

**NEURONS**

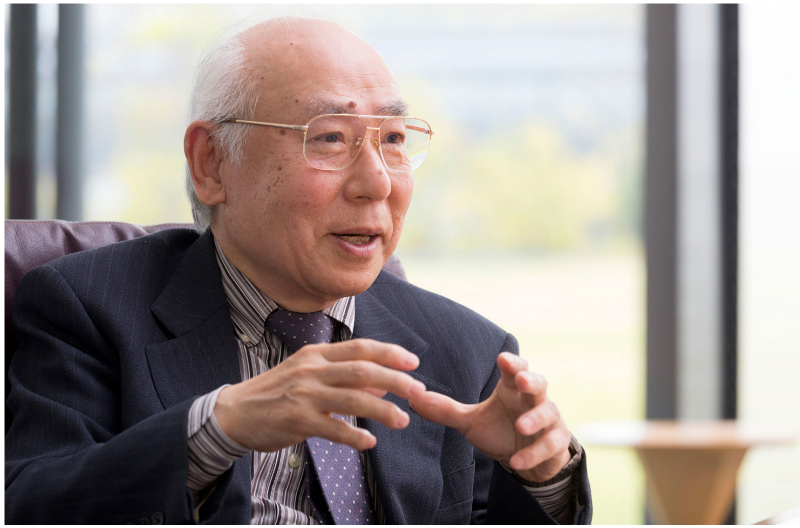





**FIREFLIES**

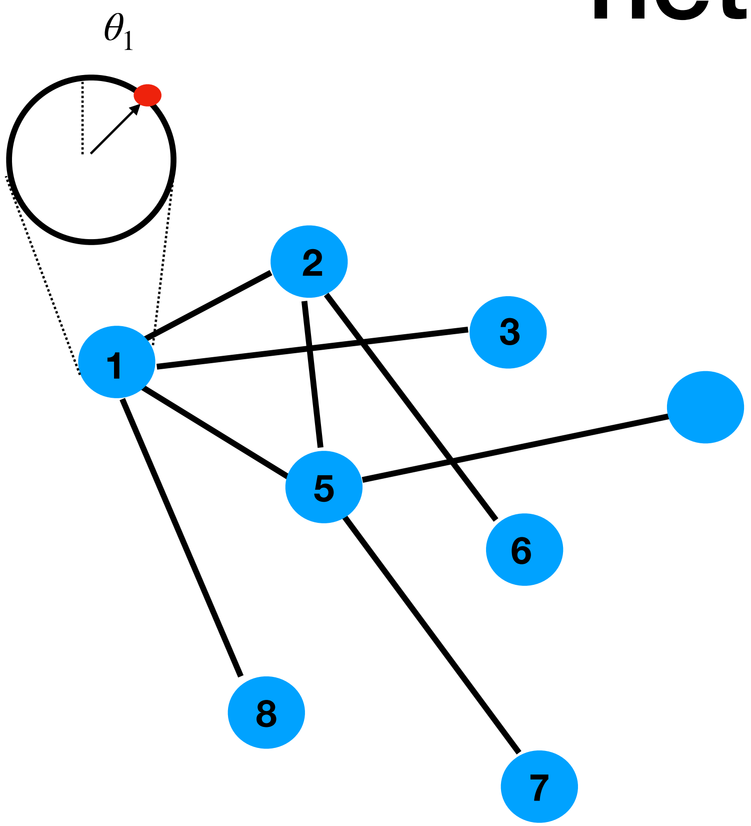
# Founding fathers of synchronisation

Christiaan Huygens



Yoshiki Kuramoto

# Kuramoto model on a network




**Given a network of N nodes**
**defined by an adjacency matrix a**
**we assign to each node a phase obeying**

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

**where the internal frequencies of the nodes**
**are drawn randomly from**

$$\omega \sim \mathcal{N}(\Omega, 1)$$

**and the coupling constant is $\sigma$**

**The oscillators are non-identical**

# Order parameter for synchronization

We consider the global order parameter $R$

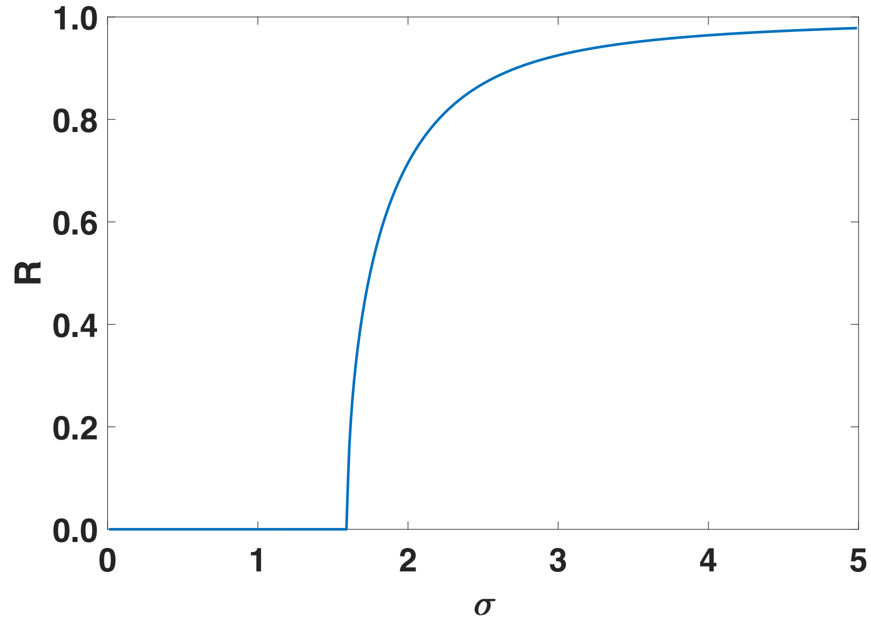
$$X = Re^{\mathrm{i}\hat{\Psi}} = \frac{1}{N}\sum_{r=1}^{N} e^{\mathrm{i}\theta_{\mathrm{r}}}$$

which indicates the

synchronisation transition such that for

$$|\sigma - \sigma_c| \ll 1$$

$$R = \begin{cases} 0 & \textbf{for } \sigma < \sigma_c \\ c(\sigma - \sigma_c)^{1/2} & \textbf{for } \sigma \geq \sigma_c \end{cases}$$




**Kuramoto (1975)**

# Gauge invariance of the Kuramoto equation

Given the Kuramoto dynamics

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

If we perform the transformation

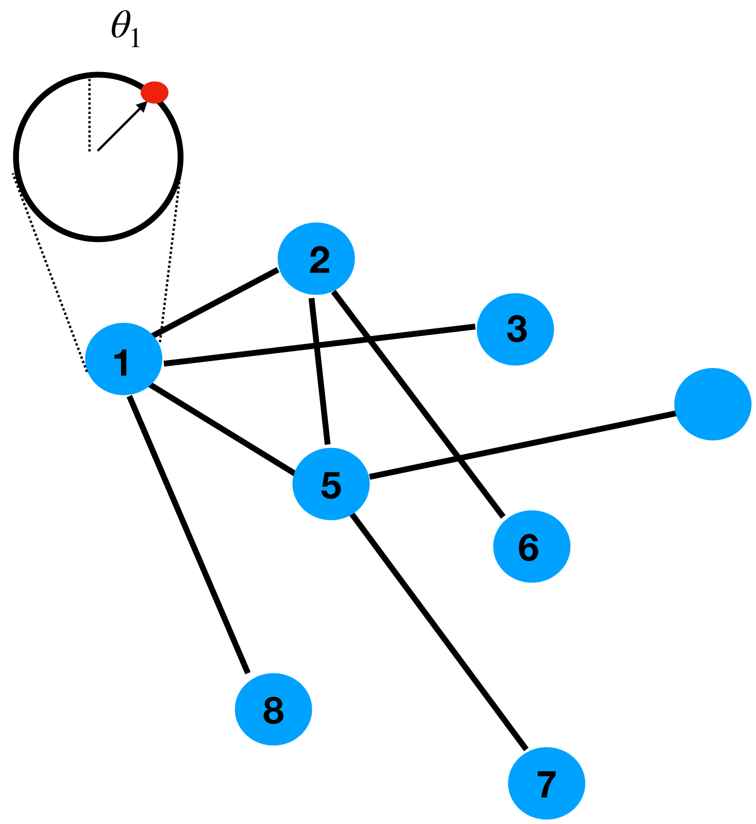
$$\theta_r \rightarrow \theta_r - \hat{\Omega} t - c$$

We obtain

$$\dot{\theta}_r = \omega_r - \hat{\Omega} + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right),$$

i.e. the dynamics is invariant under rescaling

of the average of the intrinsic frequencies , i.e. $\hat{\Omega} = \Omega$



# Solution of the Kuramoto model on a fully connected network

On a fully connected network the coupling constant  is rescaled as

$$\sigma \to \frac{\sigma}{N}$$

The Kuramoto equation

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

can be written in terms of the complex order parameter $X$ as

$$\dot{\theta}_r = \omega_r - \Omega - \sigma \mathrm{Im}(X e^{-\mathrm{i}\theta_\mathrm{r}})$$

Thanks to the gauge invariance we can study the dynamics in the rotating frame which reads

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

# Solution of the Kuramoto model on a fully connected network

Looking for the stationary states $\dot{\theta}_r = 0$ of

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

We obtain $\sin(\theta_r) = \dfrac{\omega_r - \Omega}{\sigma R}$ only valid for nodes such that

$$\left| \frac{\omega_r - \Omega}{\sigma R} \right| \leq 1$$

(frozen nodes)

# Solution of the Kuramoto model on a fully connected network

Assuming that only the frozen nodes contribute to the order parameter, since $X = R$ in the rotating frame, we obtain the self-consistent equation for the order parameter

$$R = \frac{1}{N} \sum_{r|r\ are\ frozen} \cos\theta_r = \frac{1}{N} \sum_{r|r\ are\ frozen} \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R}\right)^2}$$

Or, equivalently considering the probability density distribution $g(\omega)$ for the intrinsic frequencies,

$$R = \int_{\left|\frac{\omega-\Omega}{\sigma R}\right| \leq 1} g(\omega) \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R}\right)^2} d\omega$$

# Solution of the Kuramoto model in the annealed approximation

The e Kuramoto model on a random graph with given degree distribution can be studied within the annealed approximation obtained by making the substitution

$$a_{rs} \to p_{rs} = \frac{k_r k_s}{\langle k \rangle N}$$

Therefore the Kuramoto model becomes

$$\dot{\theta}_r = \omega_r - \sigma \sum_s \frac{k_r k_s}{\langle k \rangle N} \sin(\theta_s - \theta_r)$$

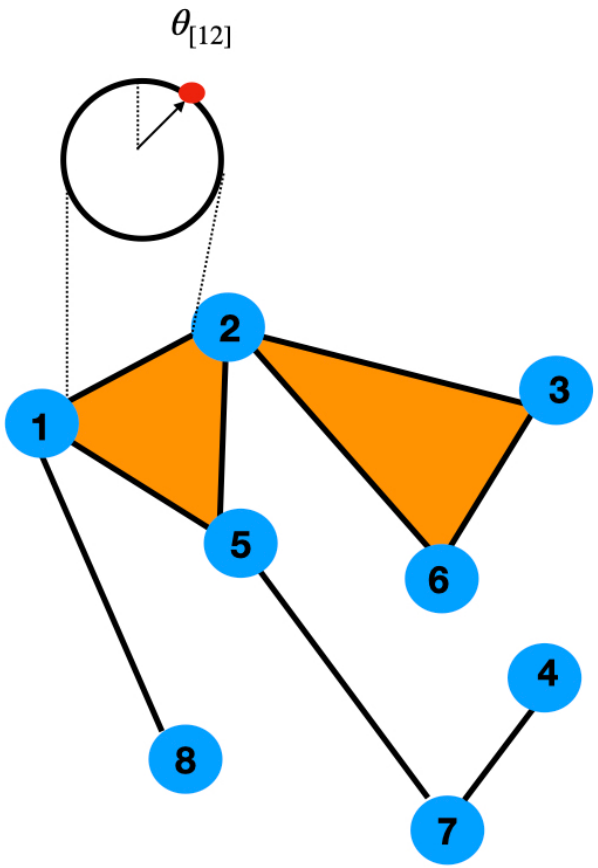
Which can be written as

$$\dot{\theta}_r = \omega_r - \sigma k_r \mathrm{Im} \hat{X} e^{-\mathrm{i}\theta_\mathrm{r}} \text{ with } \hat{X} = \frac{1}{\langle k \rangle N} \sum_s k_s e^{-\mathrm{i}\theta_\mathrm{s}}$$

which can be studied following similar steps detailed for the fully connected case.

# Topological Kuramoto model on simplicial complexes

# The higher-order simplicial Kuramoto model

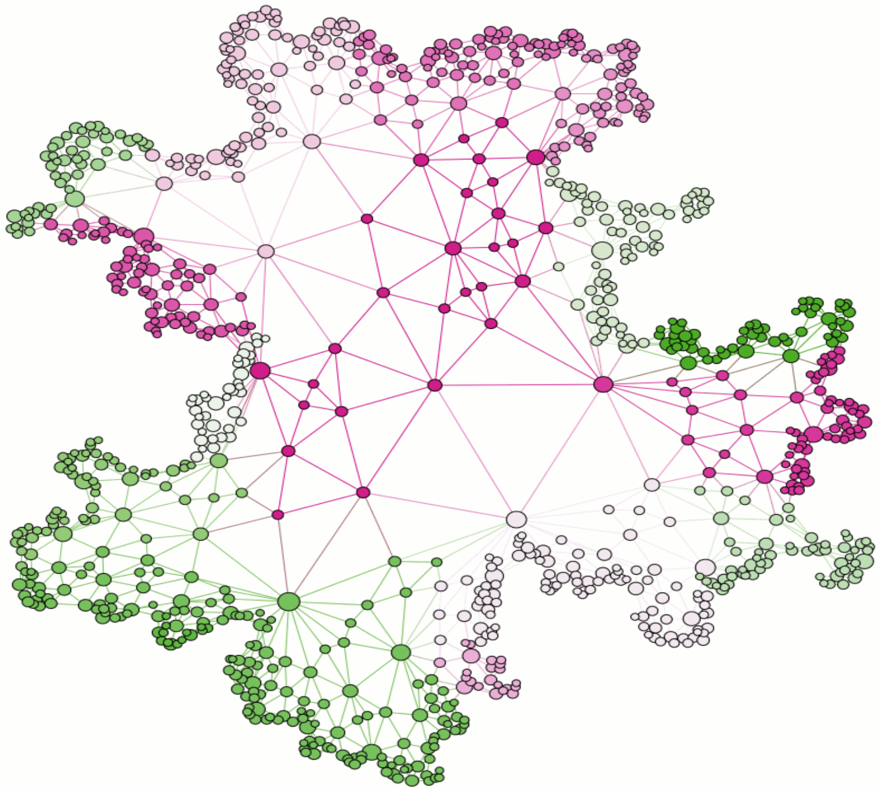



**How to define the higher-order Kuramoto model coupling higher dimensional topological signals?**

**A. P. Millán, J. J. Torres, and G.Bianconi,**
***Physical Review Letters*, *124*, 218301 (2020)**

# Topological signals

**Simplicial complexes can sustain dynamical variables (signals)**
**not only defined on nodes but also defined on higher order simplices**
**these signals are called *topological signals***



# Standard Kuramoto model in terms of boundary matrices

The standard Kuramoto model, can be expressed in terms

of the boundary matrix $\mathbf{B}_{[1]}$ as

$$\boxed{\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

# The standard Kuramoto model in terms of boundary matrices

Let us show that the Kuramoto equations

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

can be also written in matrix form as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using the explicit expression of the elements of the boundary matrix $\mathbf{B}_{[1]}$

$$[B_{[1]}]_{r\ell} = \begin{cases} -1 & \text{if } \ell = [r, s] \\ 1 & \text{if } \ell = [s, r] \\ 0 & \text{otherwise} \end{cases}$$

# Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_r = \omega_r - \sigma \sum_{\ell} [B_{[1]}]_{r\ell} \sin\left( \sum_{s} [B_{[1]}]_{\ell s} \theta_s \right)$$

For the link $\ell = [r, s]$ we obtain

$$[B_{[1]}]_{r\ell} \sin\left( \sum_{s} [B_{[1]}]_{\ell s} \theta_s \right) = -\, a_{rs} \sin(\theta_s - \theta_r)$$

# Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_r = \omega_r - \sigma \sum_{\ell} [B_{[1]}]_{r\ell} \sin\left( \sum_{s} [B_{[1]}]_{\ell s} \theta_s \right)$$

For the link $\ell = [s, r]$ we obtain

$$[B_{[1]}]_{r\ell} \sin\left( \sum_{s} [B_{[1]}]_{\ell s} \theta_s \right) = a_{rs} \sin(\theta_r - \theta_s) = - a_{rs} \sin(\theta_s - \theta_r)$$

# The harmonic mode of the Kuramoto model

Let us now study the full nonlinear Kuramoto equation

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta} \qquad (1)$$

Let us consider the harmonic eigenvector $\mathbf{u}_{harm}^\top \propto \mathbf{1}^T$ of the graph Laplacian

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]} \mathbf{B}_{[1]}^\top.$$

Since the graph Laplacian is symmetric we have $\mathbf{u}_{harm}^\top \mathbf{B}_{[1]} = \mathbf{0}$

By multiplying (1) by $\mathbf{u}_{harm}^\top$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\theta} \rangle}{dt} = \langle \mathbf{u}_{harm}, \boldsymbol{\omega} \rangle$

Therefore the harmonic mode oscillates at constant frequency also in the nonlinear Kuramoto model.

# Linearised dynamics

Let us study the linearisation of the Kuramoto dynamics.

Let us start from the nonlinear system

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using $\sin \mathbf{x} \simeq \mathbf{x}$ we get the linearised dynamics

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{L}_{[0]} \boldsymbol{\theta}$$

# Linearised Dynamics

The linearised dynamics is dictated by the graph

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{L}_{[0]} \boldsymbol{\theta} \, .$$

The phases and the intrinsic frequencies can be decomposed in the basis of the eigenvectors of the graph Laplacian

$$\boldsymbol{\theta}(t) = \sum_{\mu} c_{\mu}(t) \mathbf{u}_{\mu}$$

$$\boldsymbol{\omega} = \sum_{\mu} \omega_{\mu} \mathbf{u}_{\mu}$$

The dynamical equation in this basis reduce to

$$\dot{c}_{\mu} = \omega_{\mu} - \sigma \mu c_{\mu}$$

# Linearised Dynamics (continuation)

The dynamical equations

$$\dot{c}_\mu = \omega_\mu - \sigma\mu c_\mu$$

have solution

$$c_{harm}(t) = c_{harm}(0) + \omega_{harm} t$$
$$c_\mu(t) = \frac{\omega_\mu}{\sigma\mu}\left(1 - e^{-\sigma\mu t}\right) + c_\mu(0) e^{-\sigma\mu t}$$

Therefore the harmonic mode undergoes  an unperturbed motion,

while the non-harmonic modes are freezing with time.

# The harmonic mode of the Kuramoto model

Let us now study the full nonlinear Kuramoto equation

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Since the graph Laplacian is symmetric we have $\mathbf{u}_{harm}^{\top} \mathbf{B}_{[1]} = \mathbf{0}$

By multiplying (1) by $\mathbf{u}_{harm}^{\top}$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\theta} \rangle}{dt} = \langle \mathbf{u}_{harm}, \boldsymbol{\omega} \rangle$

Therefore the harmonic mode oscillates at constant frequency also in the nonlinear Kuramoto model.

# Topological Kuramoto model

# Topological signals

**We associate to each**

**m-dimensional simplex $\alpha$ a phase $\phi_\alpha$**

For instance for m=1 we might associate to each link a oscillating flux

The vector of phases is indicated by

$$\boxed{\boldsymbol{\phi} = (\ldots, \phi_\alpha \ldots)^\top}$$

# Standard Kuramoto model in terms of boundary matrices

The standard Kuramoto model, can be expressed in terms

of the boundary matrix $\mathbf{B}_{[1]}$ as

$$\boxed{\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

# Topological synchronisation

We propose to study the higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},}$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

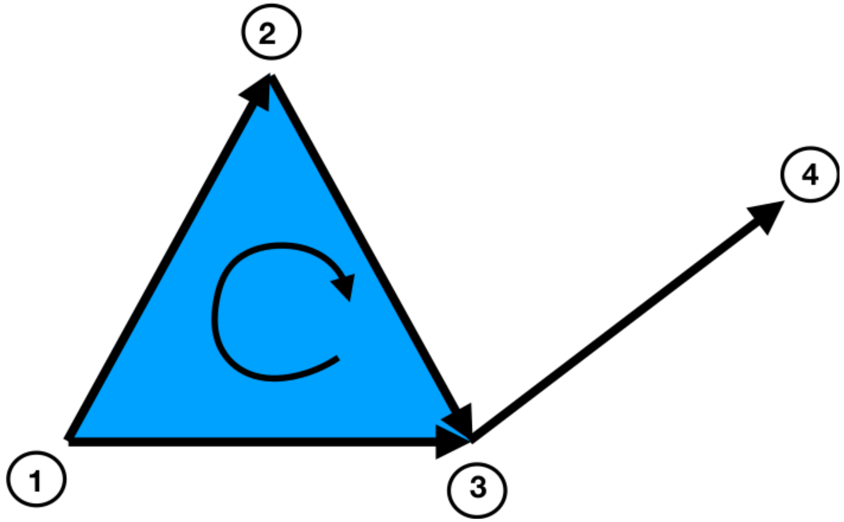
$$\boldsymbol{\phi} = (\ldots, \theta_\alpha \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_\alpha \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_\alpha \sim \mathcal{N}(\Omega, 1)$$

# Topologically induced many-body interactions




$$\dot{\phi}_{[12]} = \hat{\omega}_{[12]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[12]} - \phi_{[23]}) + \sin(\phi_{[13]} + \phi_{[12]})\right],$$
$$\dot{\phi}_{[13]} = \hat{\omega}_{[13]} + \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[13]} + \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$
$$\dot{\phi}_{[23]} = \hat{\omega}_{[23]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[23]} - \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$
$$\dot{\phi}_{[34]} = \hat{\omega}_{[34]} - \sigma \left[\sin(\phi_{[34]}) - \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

In the Topological Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

the dynamics of the synchronised state
is localised on the
$n$-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

# The harmonic mode of the non-linear Kuramoto model

Let us now study the full nonlinear Topological Kuramoto equation

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi}, \quad (2)$$

Let us consider any harmonic eigenvector $\mathbf{u}_{harm}^{\top}$ of the Hodge Laplacian $\mathbf{L}_{[n]} = \mathbf{B}_{[n+1]} \mathbf{B}_{[n+1]}^{\top} + \mathbf{B}_{[n]}^{\top} \mathbf{B}_{[n]}$.

Since Hodge decomposition applies $\mathbf{u}_{harm}^{\top} \mathbf{B}_{[n+1]} = \mathbf{u}_{harm}^{\top} \mathbf{B}_{[n]}^{\top} = \mathbf{0}$

By multiplying (2) by $\mathbf{u}_{harm}^{\top}$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$

Therefore the harmonic modes oscillate at constant frequency also in the nonlinear Topological Kuramoto model.

# Linearised Dynamics

The linearised dynamics is dictated by the Hodge-Laplacian

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m]} \boldsymbol{\phi},$$

The harmonic component of the signal oscillates freely

The other modes freeze asymptotically in time

# Linearised Dynamics

The linearised dynamics is dictated by the graph

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m]} \boldsymbol{\phi} \, .$$

The phases and the intrinsic frequencies can be decomposed in the basis of the eigenvectors of the $m$-Hodge Laplacian

$$\boldsymbol{\phi}(t) = \sum_{\mu} c_{\mu}(t) \mathbf{u}_{\mu}$$

$$\hat{\boldsymbol{\omega}} = \sum_{\mu} \omega_{\mu} \mathbf{u}_{\mu}$$

The dynamical equation in this basis reduce to

$$\dot{c}_{\mu} = \omega_{\mu} - \sigma \mu c_{\mu}$$

# Linearised Dynamics (continuation)

The dynamical equations

$$\dot{c}_\mu = \omega_\mu - \sigma\mu c_\mu$$

have solution

$$c_{harm}(t) = c_{harm}(0) + \omega_{harm} t$$
$$c_\mu(t) = \frac{\omega_\mu}{\sigma\mu}\left(1 - e^{-\sigma\mu t}\right) + c_\mu(0) e^{-\sigma\mu t}$$

Therefore the harmonic mode undergoes an unperturbed motion,

while the non-harmonic modes are freezing with time.

# Properties of linearized dynamics

**The linearised dynamics stabilises on the homological eigenvectors**

- The homological eigenvectors are localised on holes
- The Betti number can be zero or greater than one.

*Therefore a non-trivial steady state is reached only if the Betti number is positive.*

*In presence of more than one hole the stabilisation of the flow on one more more holes will depend on the initial condition*

**If we define a higher-order Kuramoto model on**

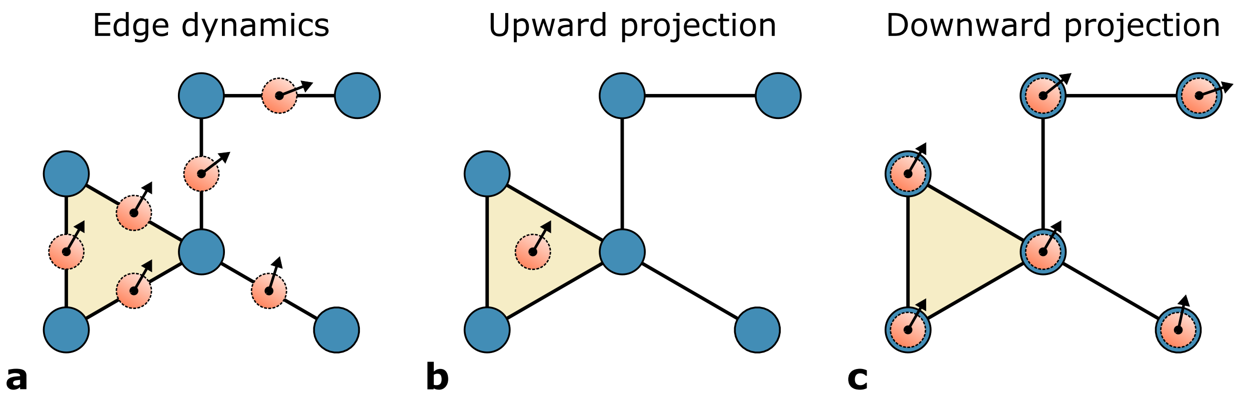
**m-simplices,**

*(let us say links, m=1) a key question is:*

**What is the dynamics induced**

**on (m-1) faces and (m+1) faces?**

*i.e. what is the dynamics induced on nodes and triangles?*



# Projected dynamics on m-1 and m+1 faces

A natural way to project the dynamics is to use the incidence matrices obtaining

$$\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[m+1]}^{\top}\boldsymbol{\phi}$$ **Discrete curl**

$$\boldsymbol{\phi}^{[-]} = \mathbf{B}_{[m]}\boldsymbol{\phi}$$ **Discrete divergence**

# Projected dynamics on m-1 and m+1 faces

Thanks to Hodge decomposition,

the projected dynamics

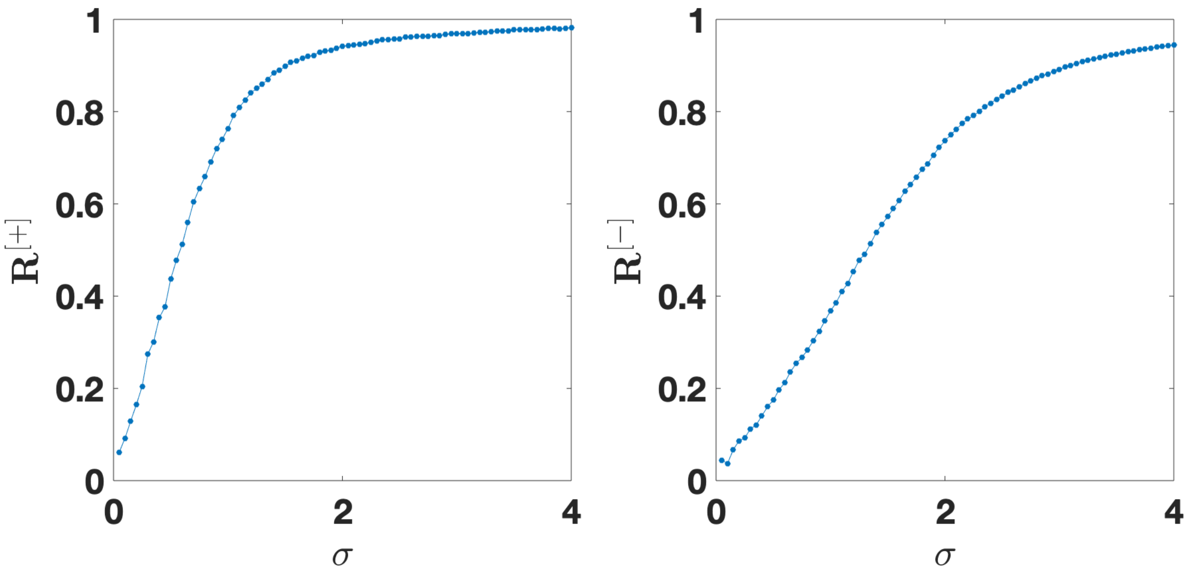
on the (m-1) and (m+1) faces

decouple

$$\boxed{\begin{aligned} \dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[m+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m+1]}^{[down]} \sin(\boldsymbol{\phi}^{[+]}) \\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[m]}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m-1]}^{[up]} \sin(\boldsymbol{\phi}^{[-]}) \end{aligned}}$$

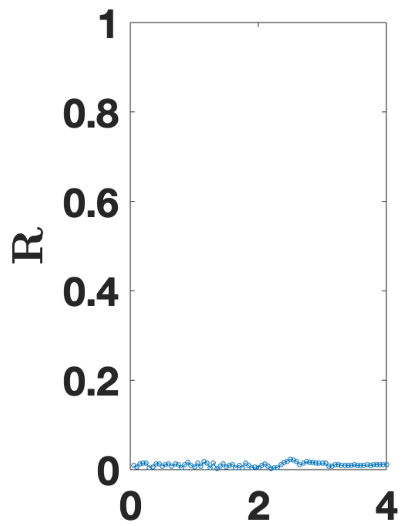
# Simplicial Synchronization transition

$$R^{[+]} = \frac{1}{N_{m+1}} \left| \sum_{\alpha=1}^{N_{m+1}} e^{\mathrm{i}\phi_\alpha^{[+]}} \right| \qquad R^{[-]} = \frac{1}{N_{m-1}} \left| \sum_{\alpha=1}^{N_{m-1}} e^{\mathrm{i}\phi_\alpha^{[-]}} \right|$$



# Order parameters using the n-dimensional phases

$$\boxed{R = \frac{1}{N_m} \left| \sum_{\alpha=1}^{N_m} e^{\mathrm{i}\phi_\alpha} \right|}$$



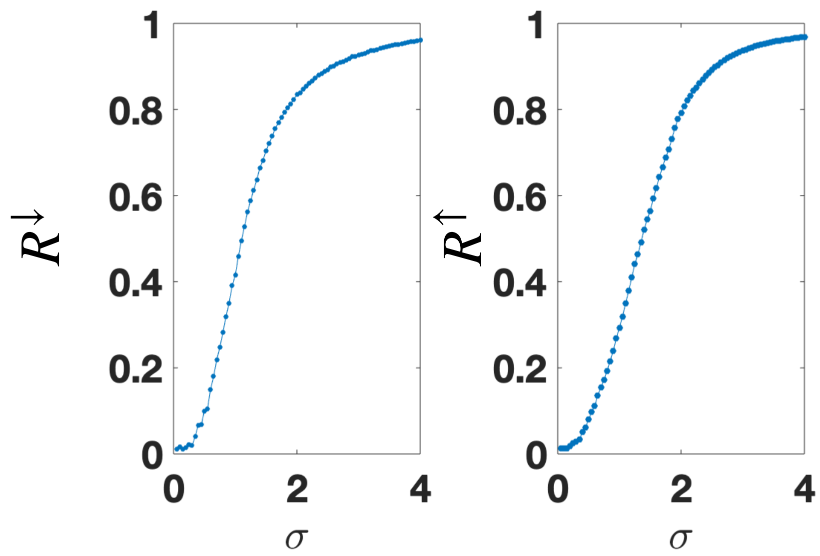
# Order parameters using the n-dimensional phases

$$\boldsymbol{\phi}^{\downarrow} = \mathbf{L}_{[n]}^{down}\boldsymbol{\phi}$$

$$\boxed{R^{\downarrow} = \frac{1}{N_m}\left|\sum_{\alpha=1}^{N_m} e^{\mathrm{i}\phi_\alpha^{\downarrow}}\right|}$$

$$\boldsymbol{\phi}^{\uparrow} = \mathbf{L}_{[n]}^{up}\boldsymbol{\phi}$$

$$\boxed{R^{\uparrow} = \frac{1}{N_m}\left|\sum_{\alpha=1}^{N_m} e^{\mathrm{i}\phi_\alpha^{\uparrow}}\right|}$$



**Only if we perform**

**the correct topological filtering**

**of the topological signal**

**we can reveal higher-order topological synchronisation**

# Explosive topological synchronisation

We propose the Explosive Topological  Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top}\boldsymbol{\phi} - \sigma R^{[+]}\mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]}\boldsymbol{\phi}}$$

# Projected dynamics

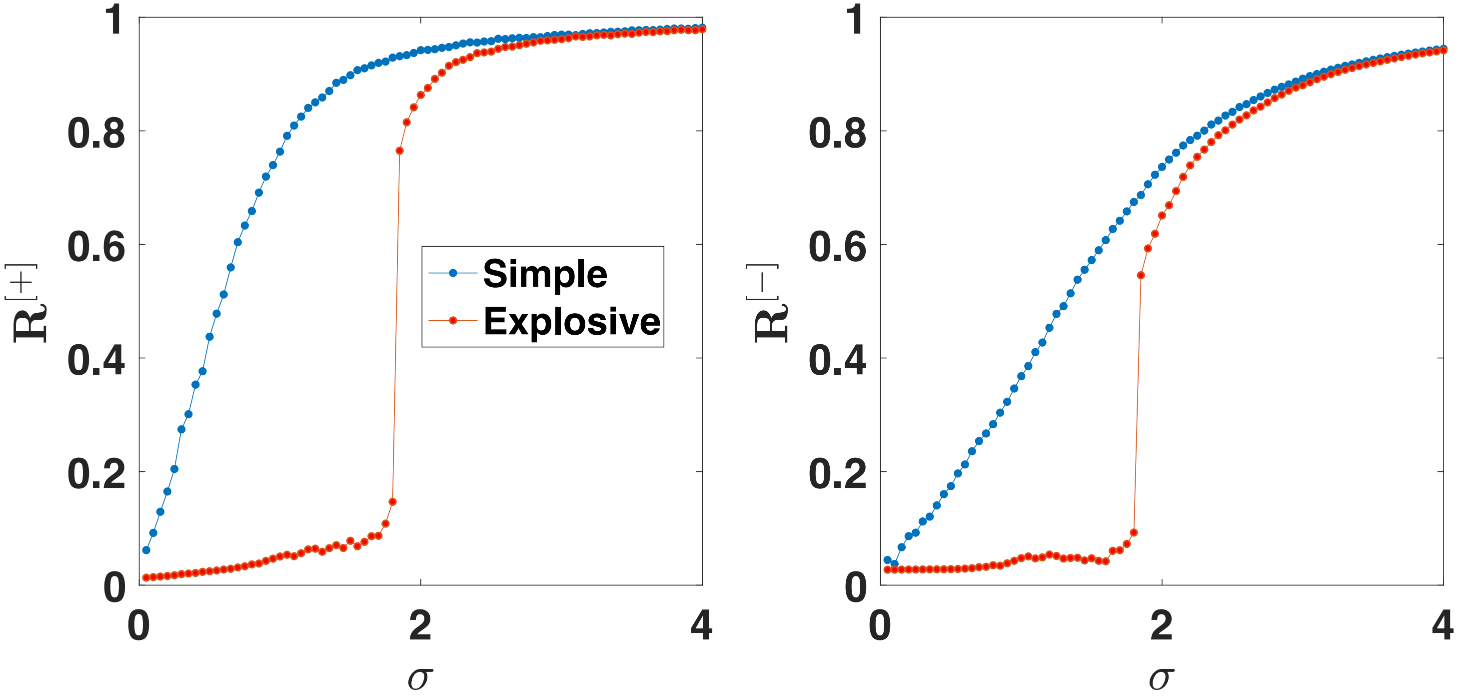
The projected dynamics on

(m+1) and (m-1) are now coupled

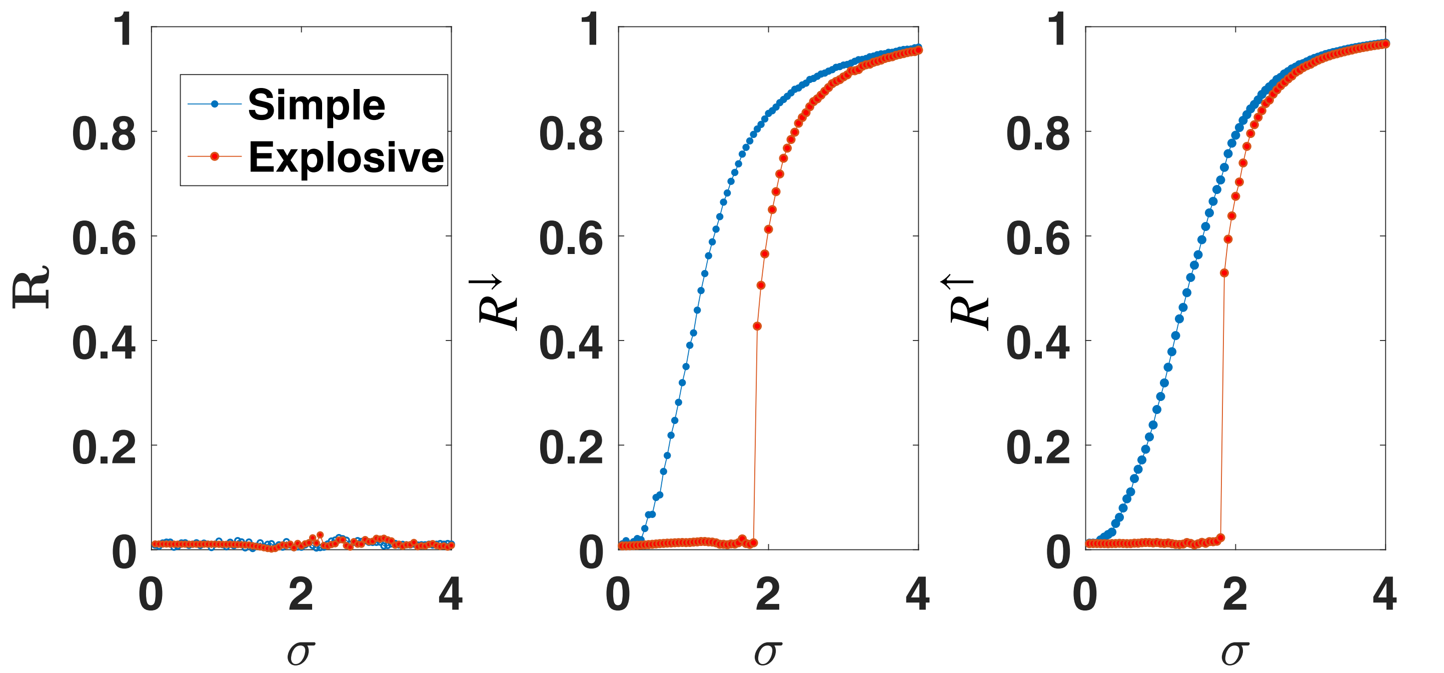
by their order parameters

$$
\begin{aligned}
\dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[m+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{L}_{[m+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]}) \\
\dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[m]}\hat{\boldsymbol{\omega}} - \sigma R^{[+]}\mathbf{L}_{[m-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]})
\end{aligned}
$$

# The explosive simplicial synchronisation transition



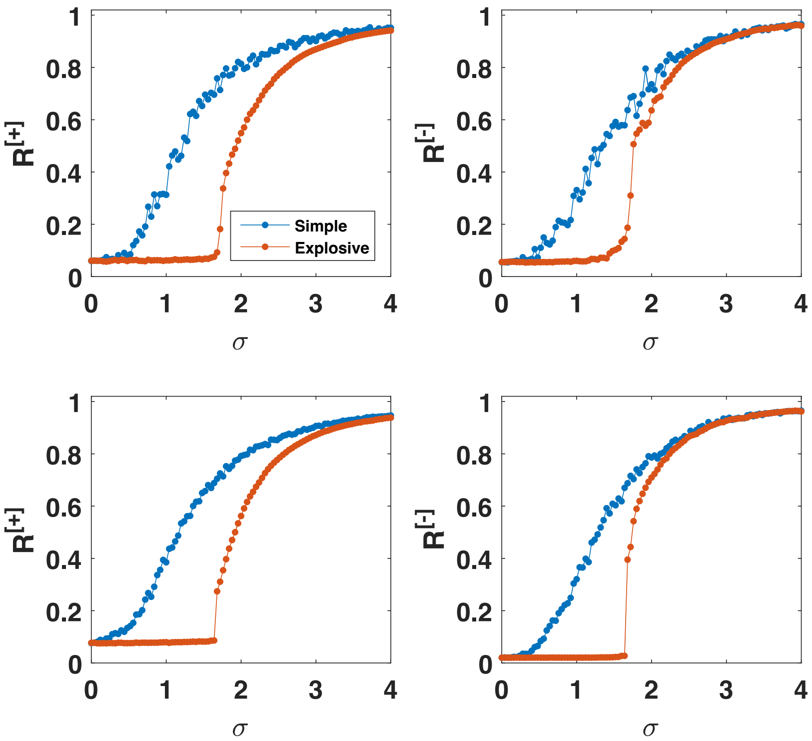
# Order parameters associated to n-faces



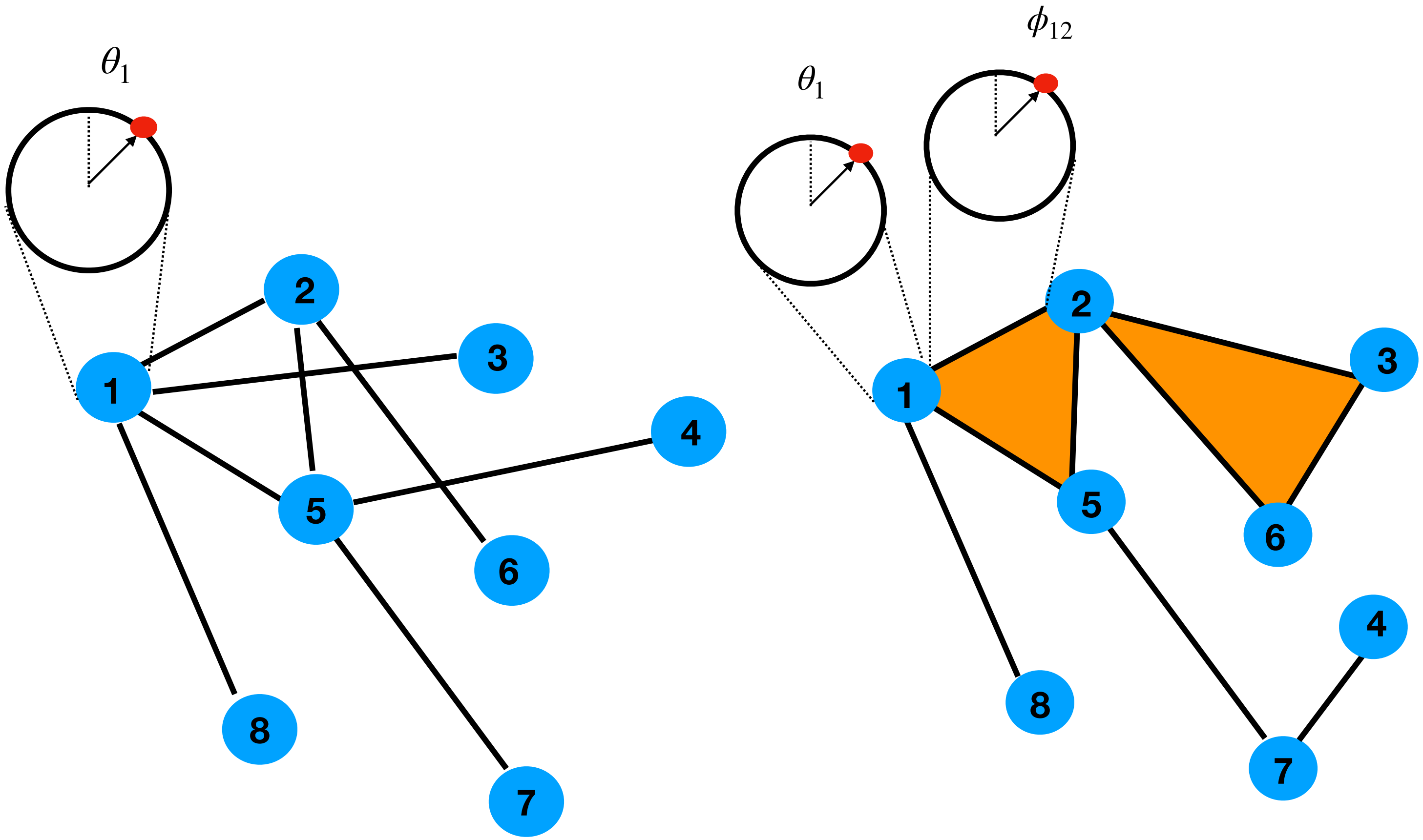
# Higher-order synchronisation on real Connectomes

**Homo sapiens Connectome**

**C.elegans Connectome**



# Coupling topological signals of different dimension

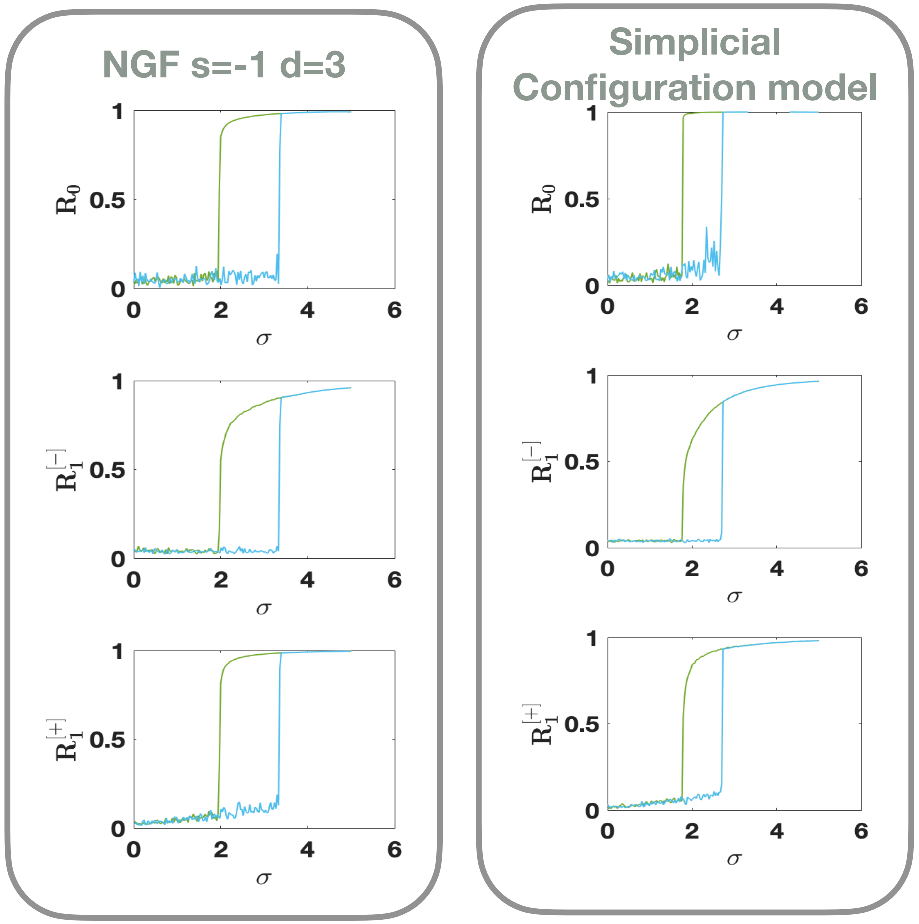



R. Ghorbanchian, J. Restrepo, J.J. Torres and G. Bianconi (2020)

# Explosive synchronisation of globally coupled topological signals

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]} \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R_1^{[+]} R_0 \mathbf{B}_{[1]}^{\top} \sin \mathbf{B}_{[1]} \boldsymbol{\phi} - \sigma R_1^{[-]} \mathbf{B}_{[2]} \sin \mathbf{B}_{[2]}^{\top} \boldsymbol{\phi}$$



# Coupled node and link topological signals on networks

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]} \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta}$$
$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R_0 \mathbf{B}_{[1]}^\top \sin \mathbf{B}_{[1]} \boldsymbol{\phi}$$

$$\omega_i \sim \mathcal{N}(\Omega_0, 1/\tau_0)$$
$$\hat{\omega}_i \sim \mathcal{N}(\Omega_1, 1/\tau_1)$$

**Dynamics projected on the nodes**

$$\boldsymbol{\psi} = \mathbf{B}_{[n]} \boldsymbol{\phi}$$
$$\tilde{\boldsymbol{\omega}} = \mathbf{B}_{[1]} \hat{\boldsymbol{\omega}}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]} \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta}$$
$$\dot{\boldsymbol{\psi}} = \tilde{\boldsymbol{\omega}} - \sigma R_0 \mathbf{L}_{[0]} \sin \boldsymbol{\psi}$$

**Correlation of projected frequencies**

$$\langle \tilde{\omega}_r \rangle = \left[ \sum_{s<r} a_{rs} - \sum_{s>r} a_{rs} \right] \Omega_1$$

$$\langle \tilde{\omega}_r \tilde{\omega}_s \rangle - \langle \tilde{\omega}_r \rangle \langle \tilde{\omega}_s \rangle = [\mathbf{L}_{[0]}]_{rs} \frac{1}{\tau_1^2}$$

# The node order parameter on a fully connected network

The node order parameter can be obtained similarly as for the standard Kuramoto model

$$R_0 = \frac{1}{N} \sum_{r|r \text{ are frozen}} \cos\theta_r = \frac{1}{N} \sum_{r|r \text{ are frozen}} \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R_0 R_1^{[-]}}\right)^2}$$

Or, equivalently considering the probability density distribution $g(\omega)$ for the intrinsic frequencies,

$$R_0 = \int_{\left|\frac{\omega-\Omega}{\sigma R_0 R_1^{[-]}}\right| \leq 1} g(\omega) \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R_0 R_1^{[-]}}\right)^2} d\omega$$

# The link order parameter on a fully connected network

The link order parameter can be obtained following similar steps obtaining

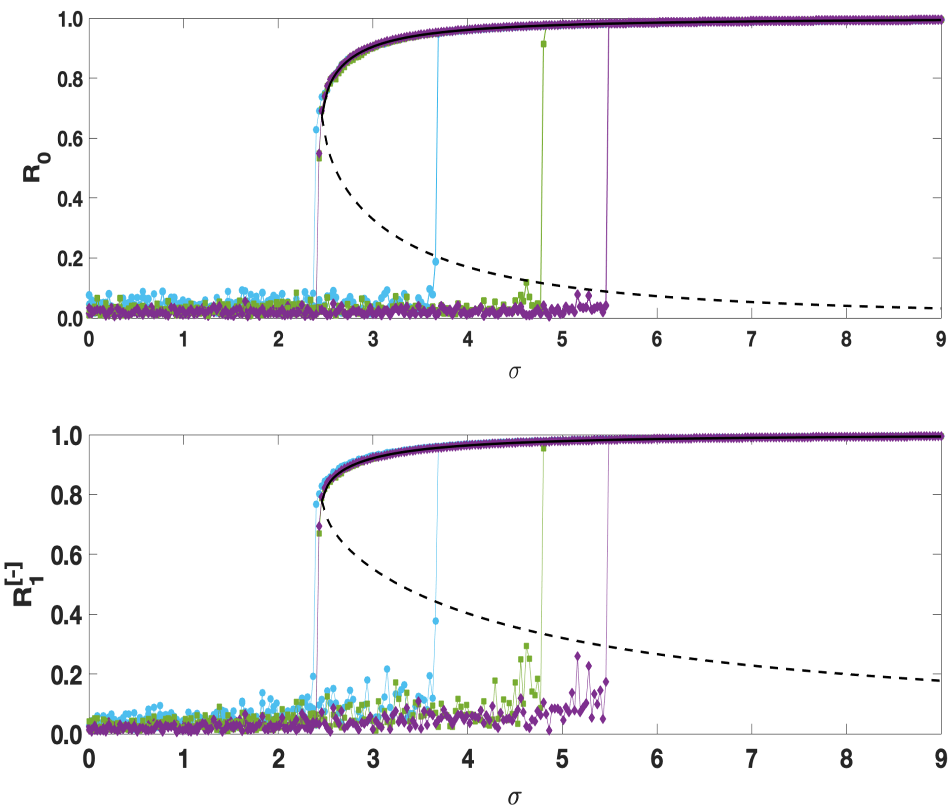
$$R_1^{[-]} = \frac{1}{N} \sum_{r|r\ are\ frozen} \cos\psi_r = \frac{1}{N} \sum_{r|r\ are\ frozen} \sqrt{1 - \left(\frac{\tilde{\omega}}{\sigma R_0}\right)^2}$$

Or, equivalently considering the probability density distribution $\tilde{g}(\tilde{\omega})$ for the intrinsic frequencies,

$$R_1^{[-]} = \int_{\left|\frac{\tilde{\omega}}{\sigma R_0}\right| \leq 1} \tilde{g}(\tilde{\omega}) \sqrt{1 - \left(\frac{\tilde{\omega}}{\sigma R_0}\right)^2} d\tilde{\omega}$$

# Solution on a fully connected network

Fully connected networks undergo a discontinuous synchronisation transition of topological signals defined on nodes and links



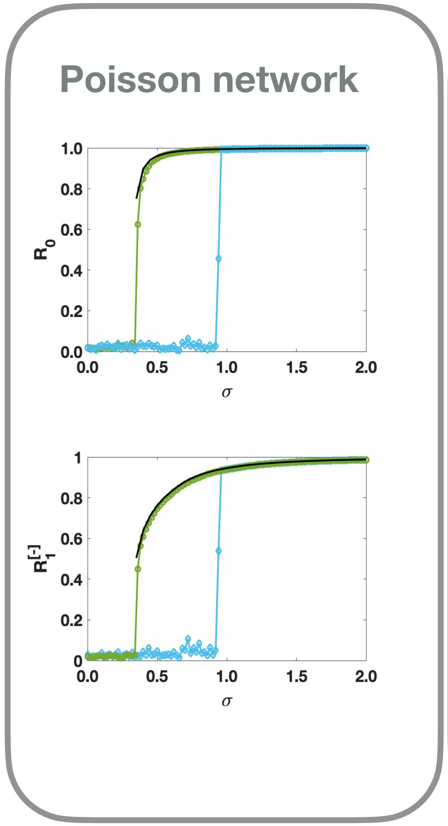

The hysteresis loop is not closed in the infinite network limit and on finite size networks
is driven by finite size effects

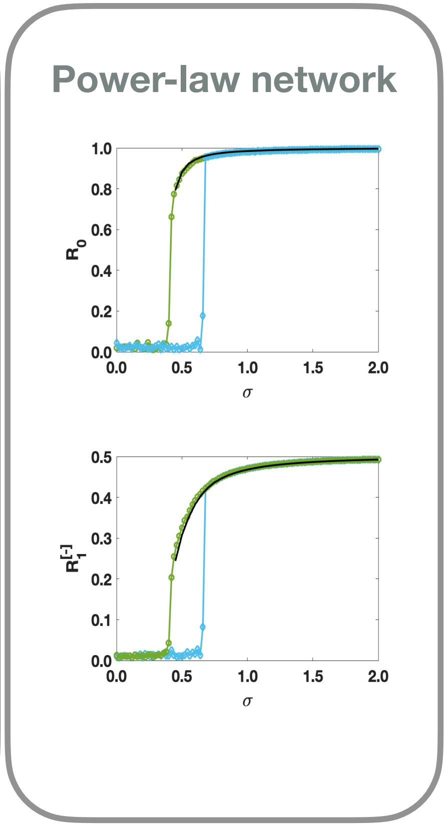
# Annealed solution on random networks

**The annealed solution captures the backward transition**

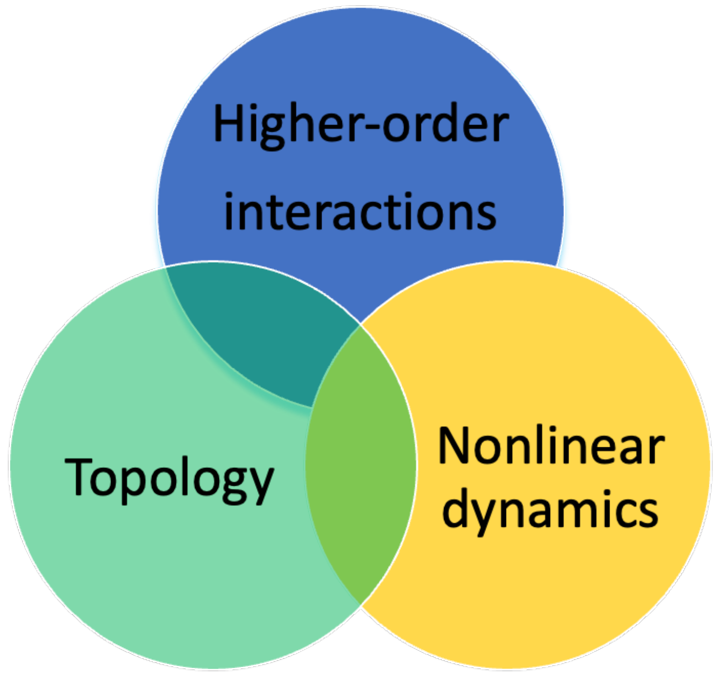
**Reveals that the transition is discontinuous**

**Gives very reliable results for connected networks that are not too sparse**



# Complexity challenge



# Lesson II:
# Topological Kuramoto model

- **Spectral properties of the Laplacians**

- **Topological Kuramoto model**
  - **The Kuramoto model on graphs**
  - **The Topological Kuramoto model**

# References

**Higher-order topological Kuramoto model**
**Millán, A.P., Torres, J.J. and Bianconi, G., 2020. Explosive higher-order Kuramoto dynamics on simplicial complexes. *Physical Review Letters*, *124*(21), p.218301.**

**Globally Coupled dynamics of nodes and links**
**Ghorbanchian, Reza, Juan G. Restrepo, Joaquín J. Torres, and Ginestra Bianconi. "Higher-order simplicial synchronization of coupled topological signals." *Communications Physics* 4, no. 1 (2021): 1-13.**