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Beyond complex Networks.
Hypergraphs

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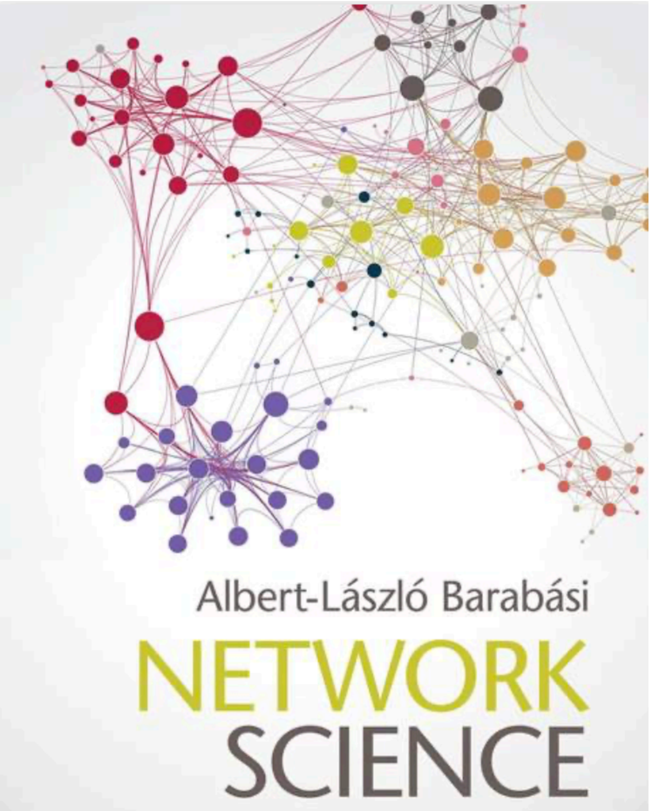
Motivation : We live in an interconnected world ...



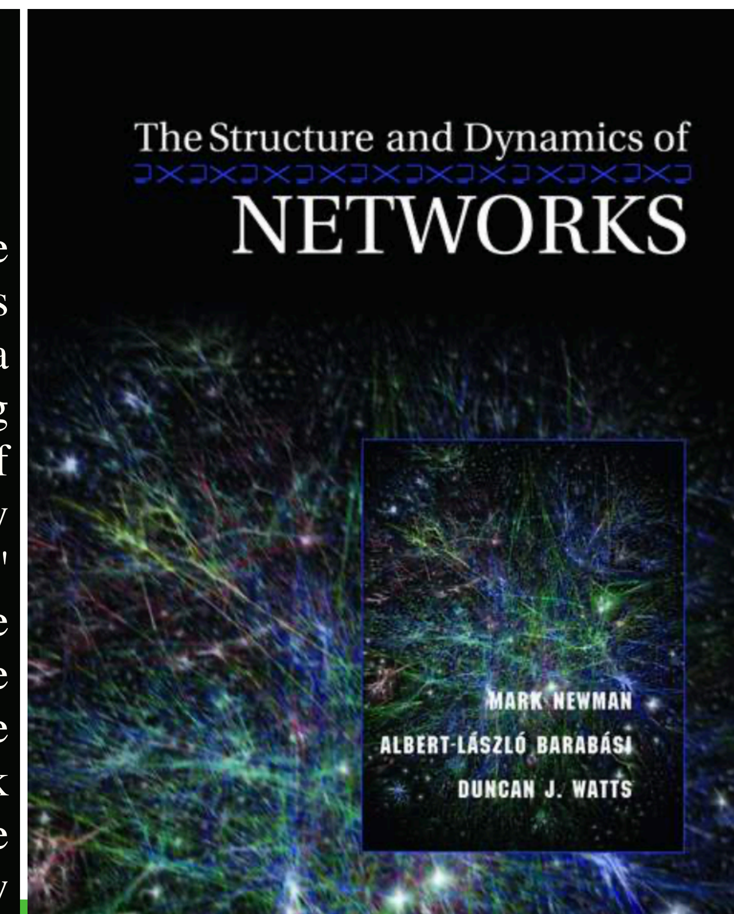
Network Science A.-L. Barabási

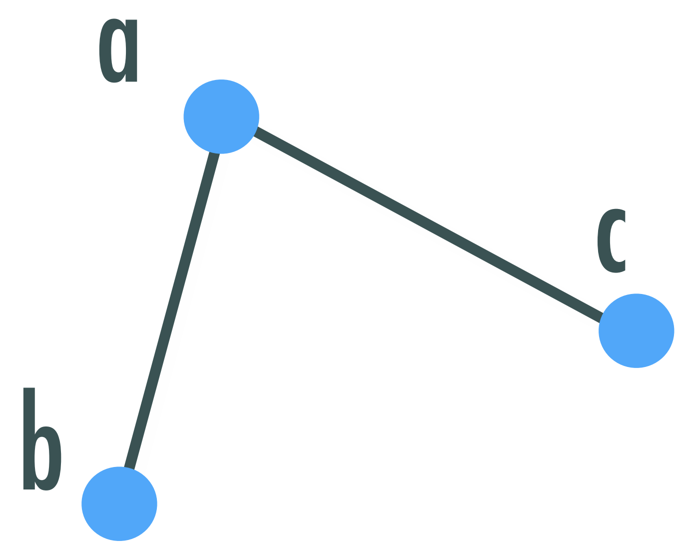
Networks are everywhere, from the Internet, to social networks, and the genetic networks that determine our biological existence. Illustrated throughout in full colour, this pioneering textbook, spanning a wide range of topics from physics to computer science, engineering, economics and the social sciences, introduces network science to an interdisciplinary audience. From the origins of the six degrees of separation to explaining why networks are robust to random failures, the author explores how viruses like Ebola and H1N1 spread, and why it is that our friends have more friends than we do. Using numerous real-world examples, this innovatively designed text includes clear delineation between undergraduate and graduate level material. The mathematical formulas and derivations are included within Advanced Topics sections, enabling use at a range of levels. Extensive online resources, including films and software for network analysis, make this a multifaceted companion for anyone with an interest in network science.

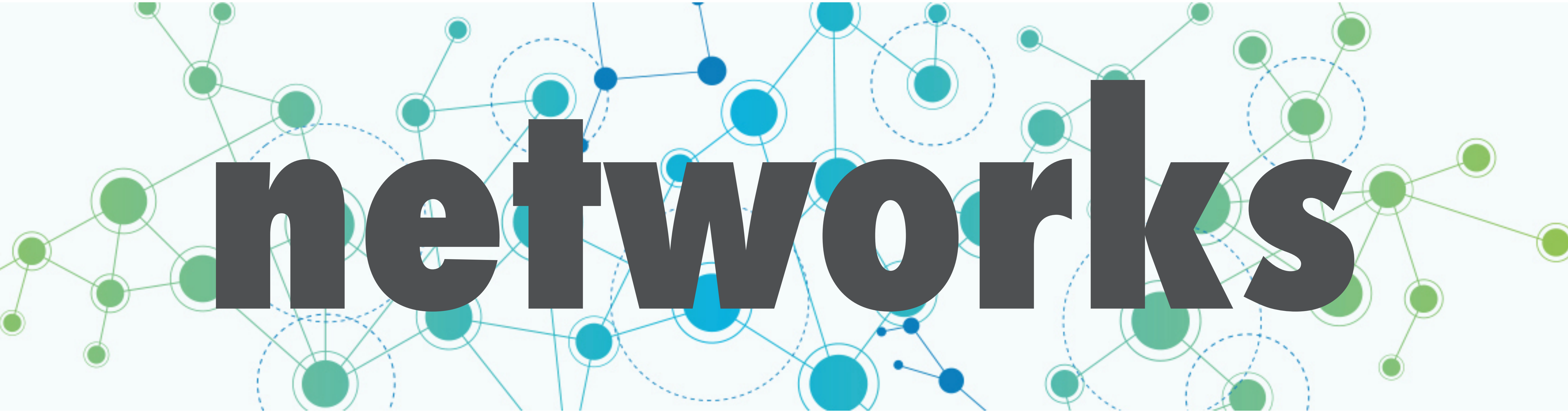
The Structure and Dynamics of Networks

A.-L. Barabási, M. Newman, D.J.Watts

From the Internet to networks of friendship, disease transmission, and even terrorism, the concept-and the reality-of networks has come to pervade modern society. But what exactly is a network? What different types of networks are there? Why are they interesting, and what can they tell us? In recent years, scientists from a range of fields-including mathematics, physics, computer science, sociology, and biology-have been pursuing these questions and building a new "science of networks." This book brings together for the first time a set of seminal articles representing research from across these disciplines. It is an ideal sourcebook for the key research in this fast-growing field. The book is organized into four sections, each preceded by an editors' introduction summarizing its contents and general theme. The first section sets the stage by discussing some of the historical antecedents of contemporary research in the area. From there the book moves to the empirical side of the science of networks before turning to the foundational modeling ideas that have been the focus of much subsequent activity. The book closes by taking the reader to the cutting edge of network science--the relationship between network structure and system dynamics. From network robustness to the spread of disease, this section offers a potpourri of topics on this rapidly expanding frontier of the new science.







Network = finite set of nodes pairwise connected, i.e., there is a link (edge) among the two nodes if there is some interaction among them

… where "basic" units interact each others

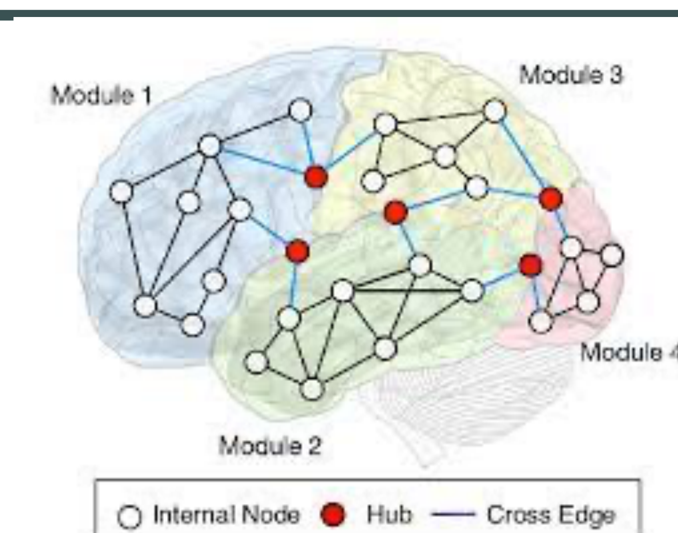
At larger scale

support for the spread of goods

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

At smaller scale

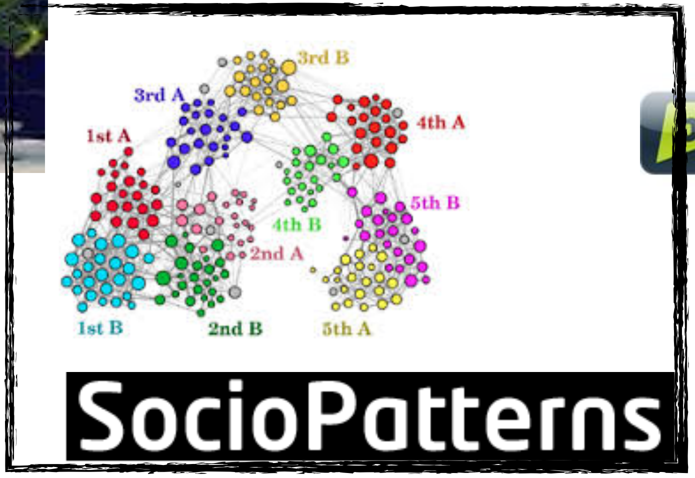


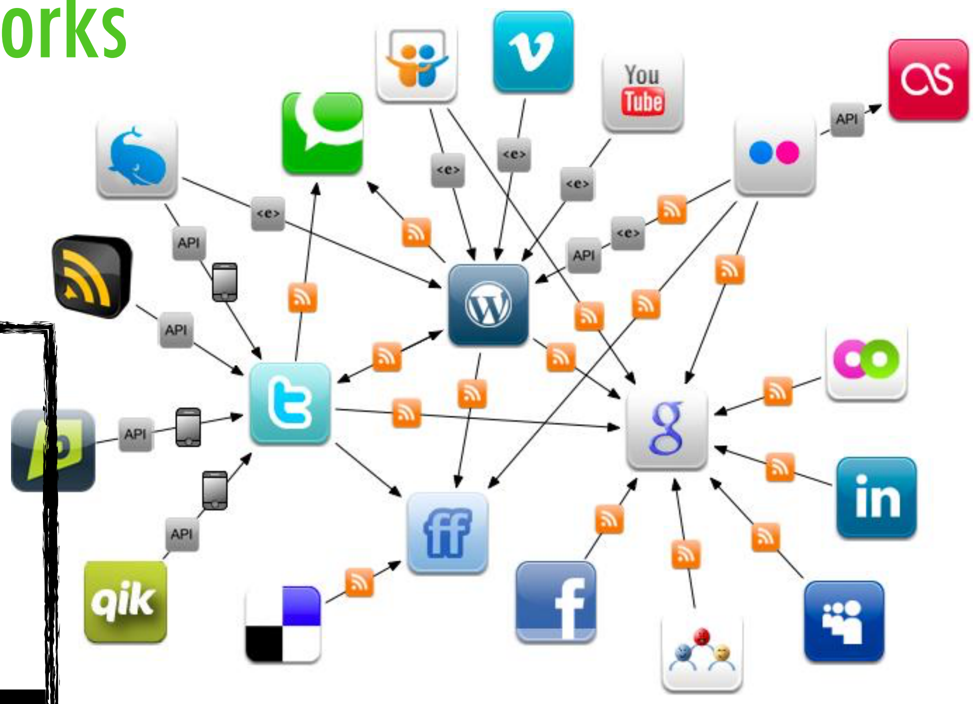
support for the spread of signals (memory, actions, thoughts, …)

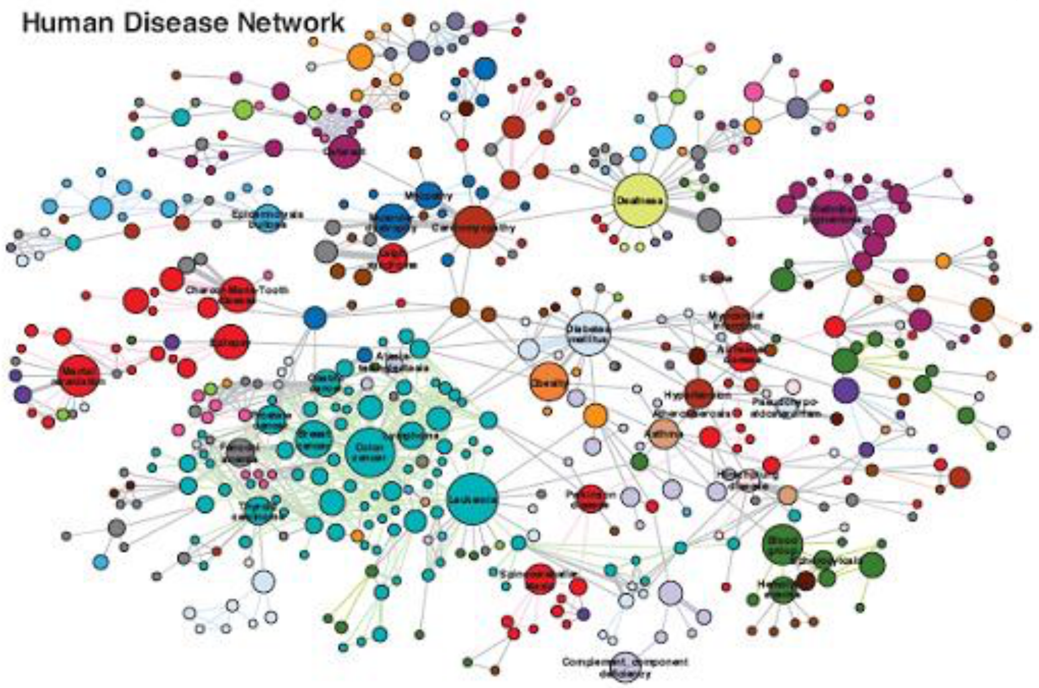
Networks are everywhere

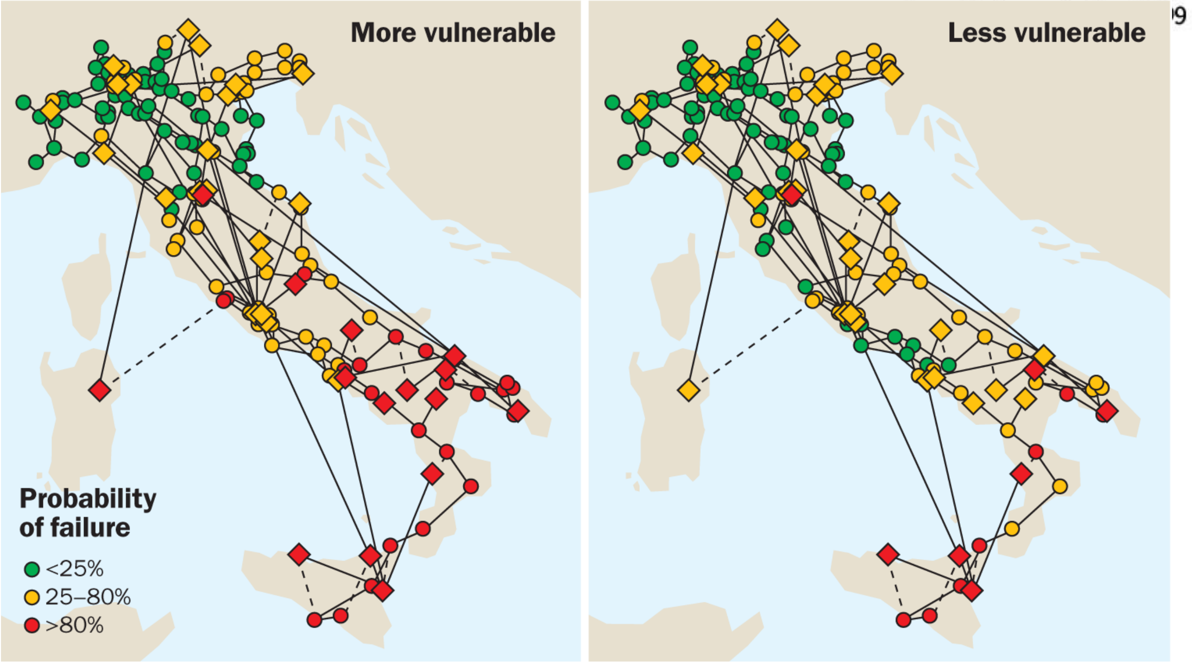
world flights map

social networks









proteins networks

technological networks

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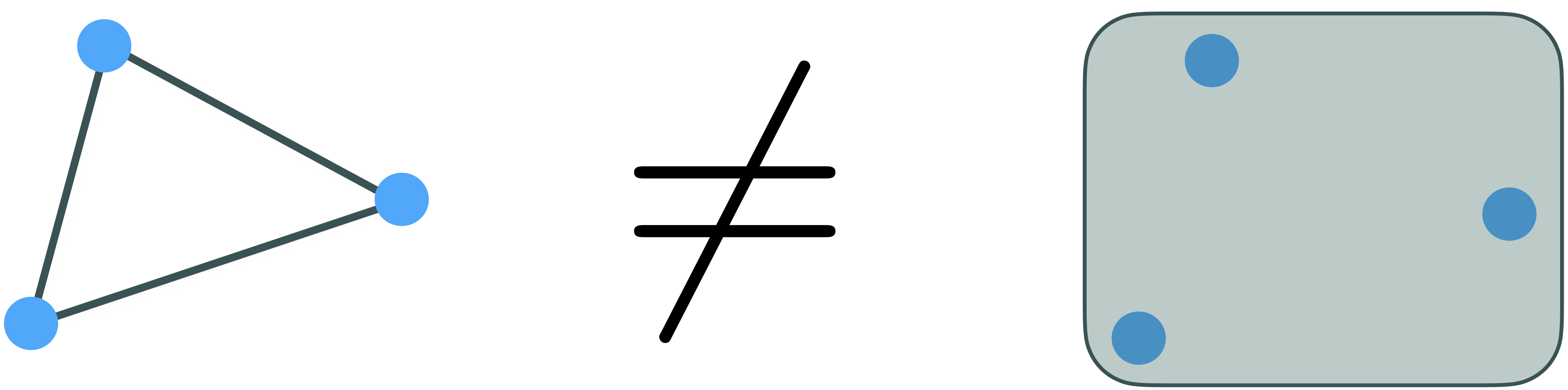
Dynamics

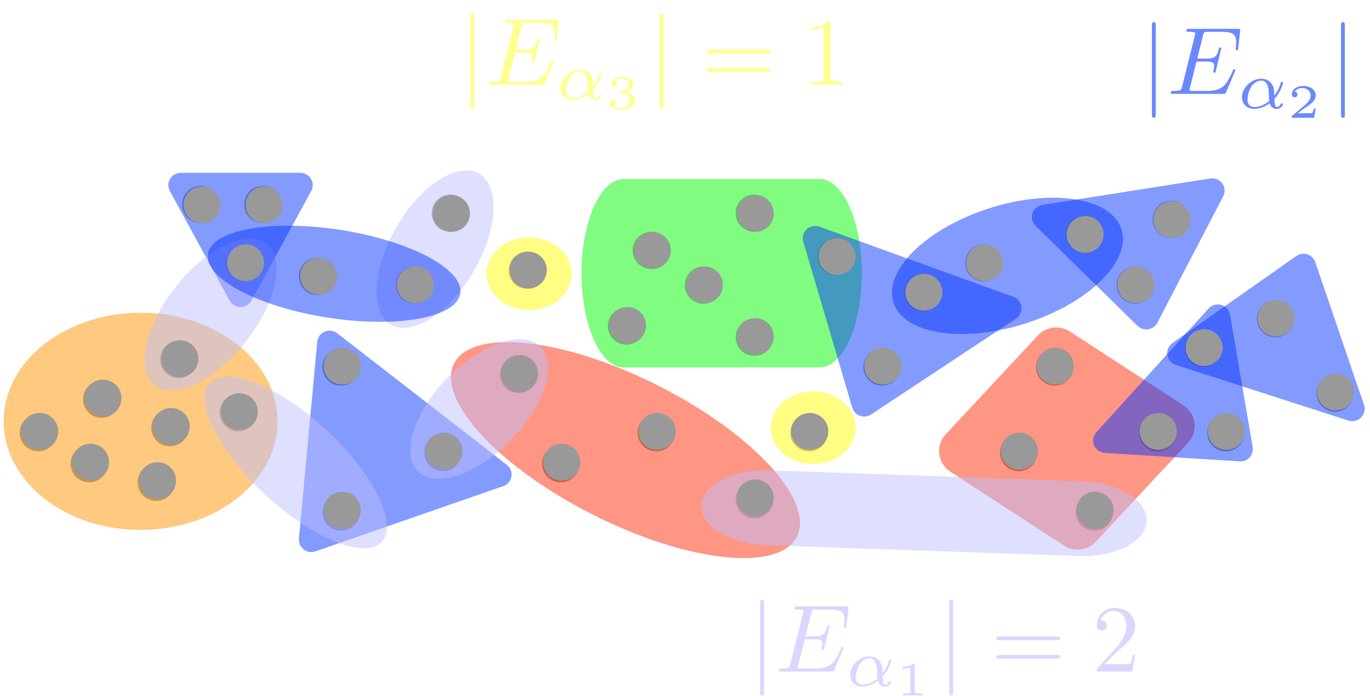
Structure

limitation



Hypergraphs

Hypergraphs



ensemble of nodes

hyperedges

Incidence matrix

$$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$$

Hyperadjacency matrix

$$A = ee^{\mathrm{T}}$$

Hyperedge matrix

$$C = e^{\mathrm{T}}e$$

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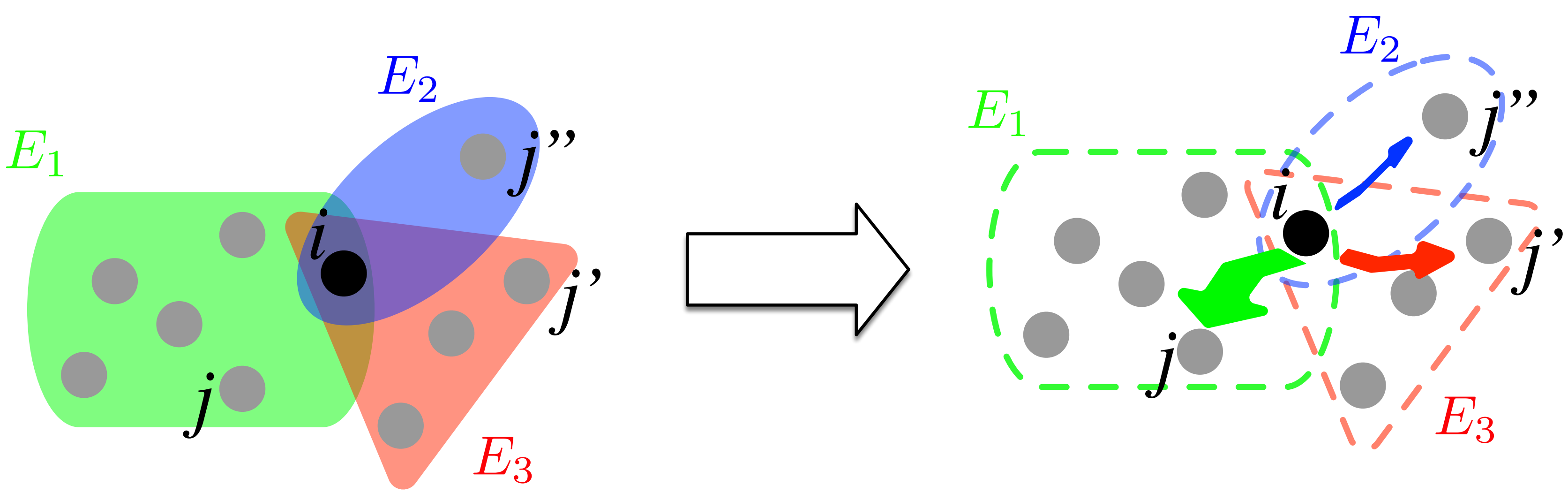
Hypergraphs Random walks



$$L^H_{ij} = \delta_{ij} - \frac{k^H_{ij}}{\sum_{\ell \neq i} k^H_{i\ell}}$$

$$k^H_{ij} = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

$$p_i^{(\infty)} = \frac{k^H_i}{\sum_\ell k^H_\ell}$$

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Random walks on hypergraphs

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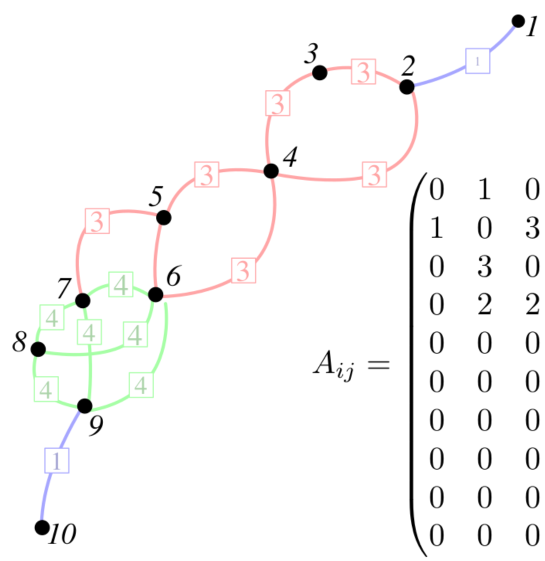
Hypergraphs & projections

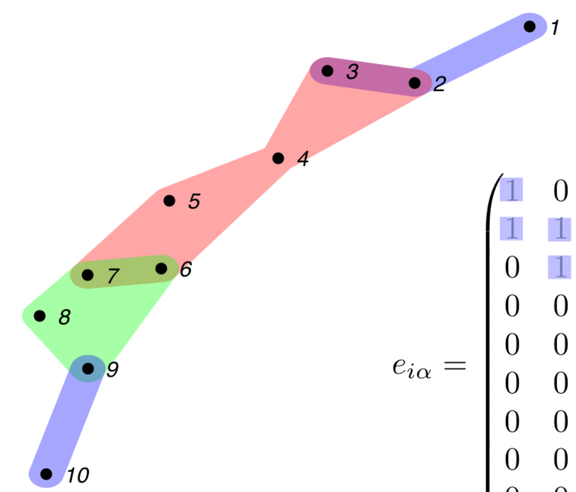
clique reduced multigraph



$$A_{ij} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 1&0&3&2&0&0&0&0&0&0\\ 0&3&0&2&0&0&0&0&0&0\\ 0&2&2&0&2&2&0&0&0&0\\ 0&0&0&2&0&4&2&0&0&0\\ 0&0&0&2&4&0&5&3&3&0\\ 0&0&0&0&2&5&0&3&3&0\\ 0&0&0&0&0&3&3&0&3&0\\ 0&0&0&0&0&3&3&3&0&1\\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

A

hypergraph

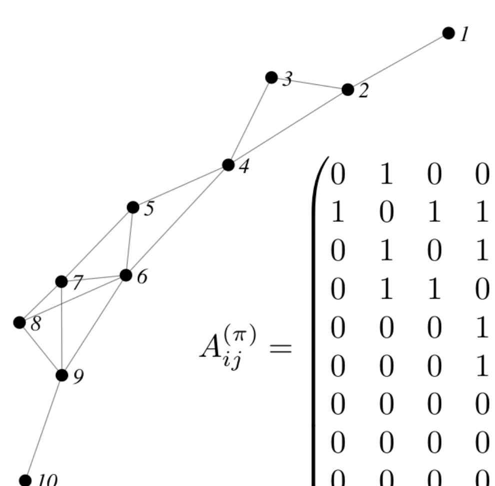


$$e_{i\alpha} = \begin{pmatrix} 1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

B

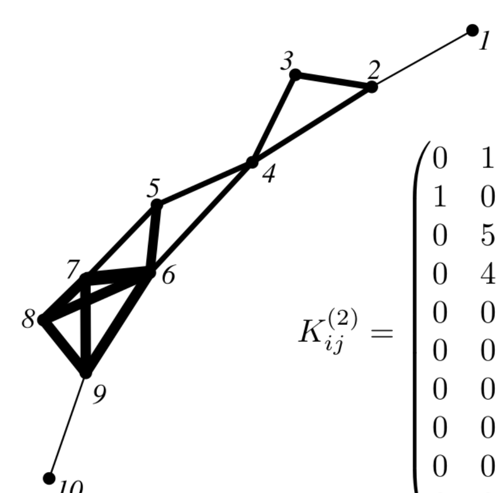
C

projected network



$$A^{(\pi)}_{ij} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 1&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0\\ 0&1&1&0&1&1&0&0&0&0\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&1&1&0&1&1&1&0\\ 0&0&0&0&1&1&0&1&1&0\\ 0&0&0&0&0&1&1&0&1&0\\ 0&0&0&0&0&1&1&1&0&1\\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

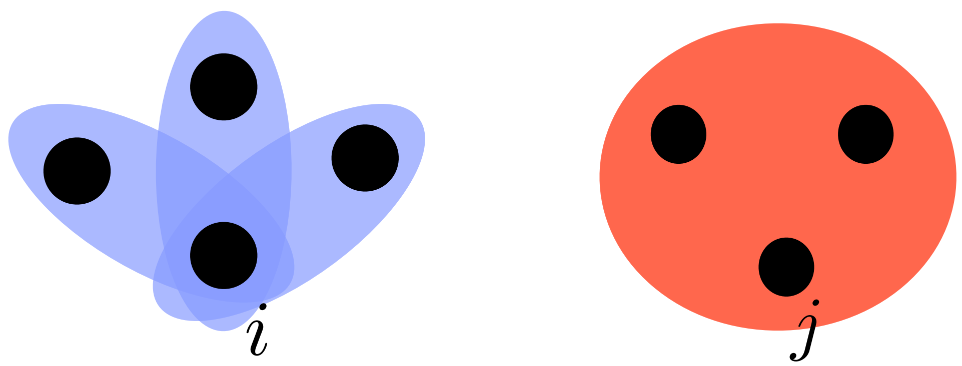
equivalent weighted network



$$K^{(2)}_{ij} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 1&0&5&4&0&0&0&0&0&0\\ 0&5&0&4&0&0&0&0&0&0\\ 0&4&4&0&4&4&0&0&0&0\\ 0&0&0&4&0&8&4&0&0&0\\ 0&0&0&4&8&0&13&9&9&0\\ 0&0&0&0&4&13&0&9&9&0\\ 0&0&0&0&0&9&9&0&9&0\\ 0&0&0&0&0&9&9&9&0&1\\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

Ranking inversion (I)

$$k_i^H = 3 < k_j^H = 4$$



$$\text{rank}(j) > \text{rank}(i)$$

clique projection

$$\text{rank}(j) < \text{rank}(i)$$

$$k_i = 3 > k_j = 2$$

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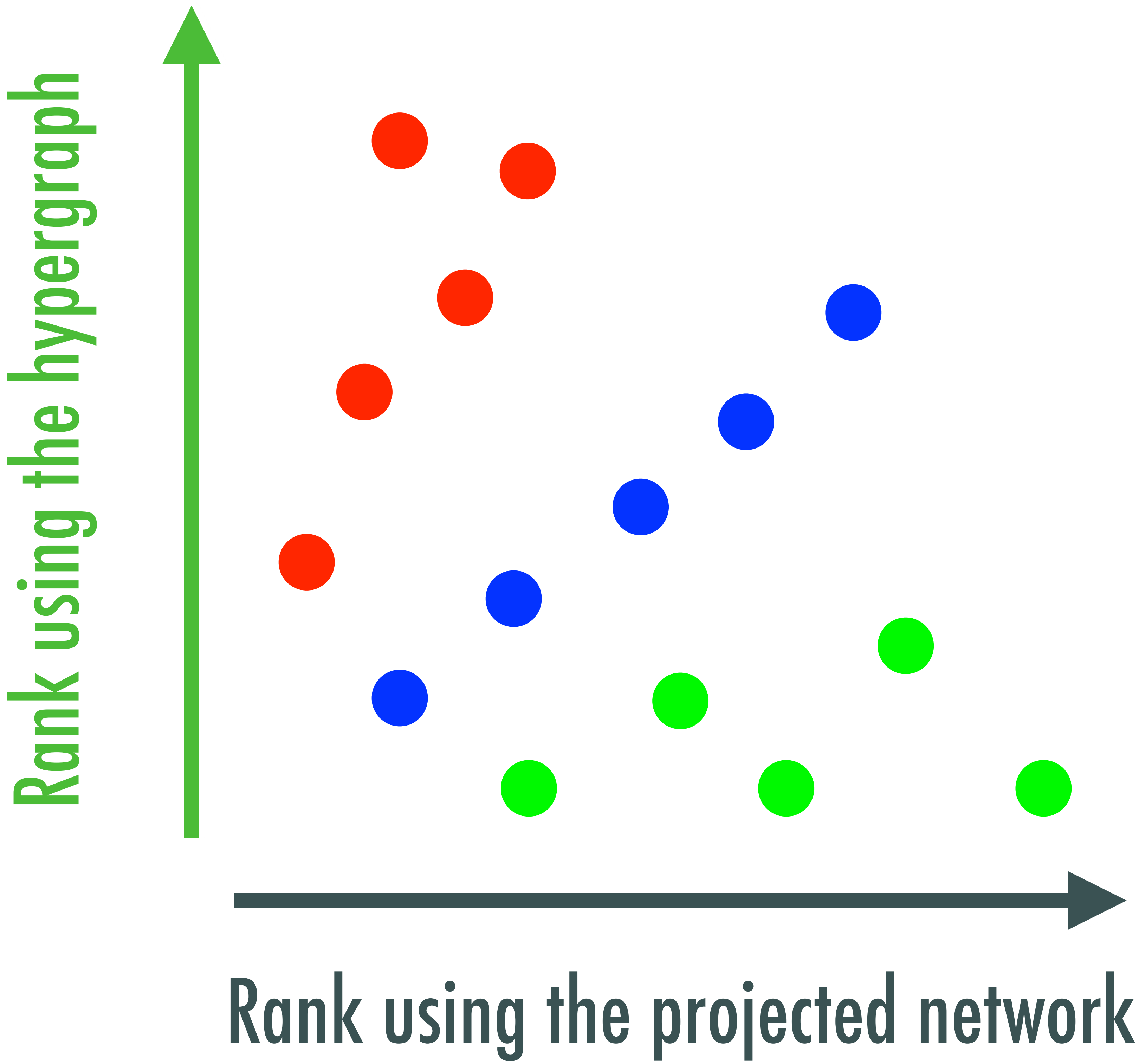
Ranking on hypergraphs and on the projected network

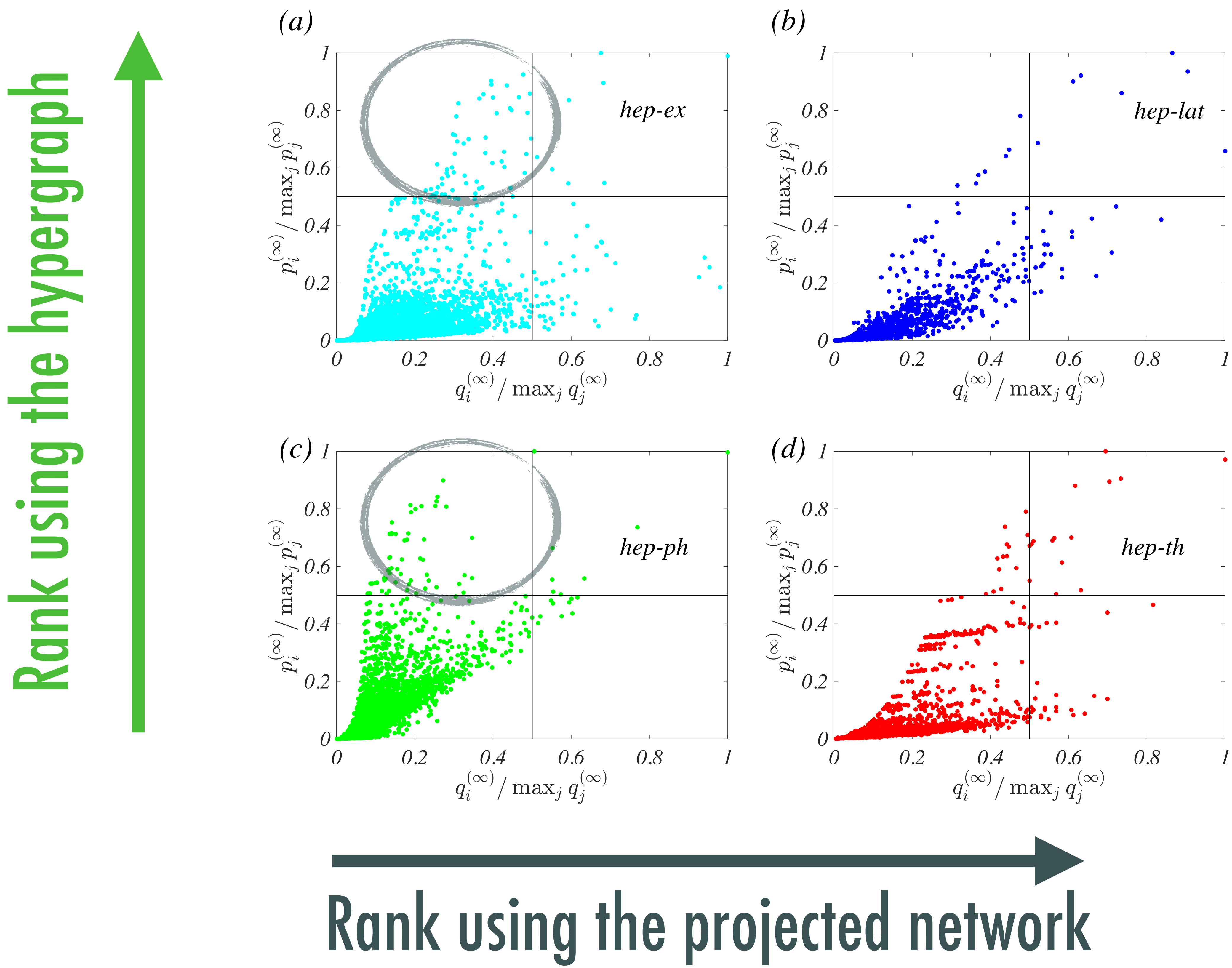


High-order structures do matter

Number of neighbours do matter

Same relevance

Ranking inversion arXiv.



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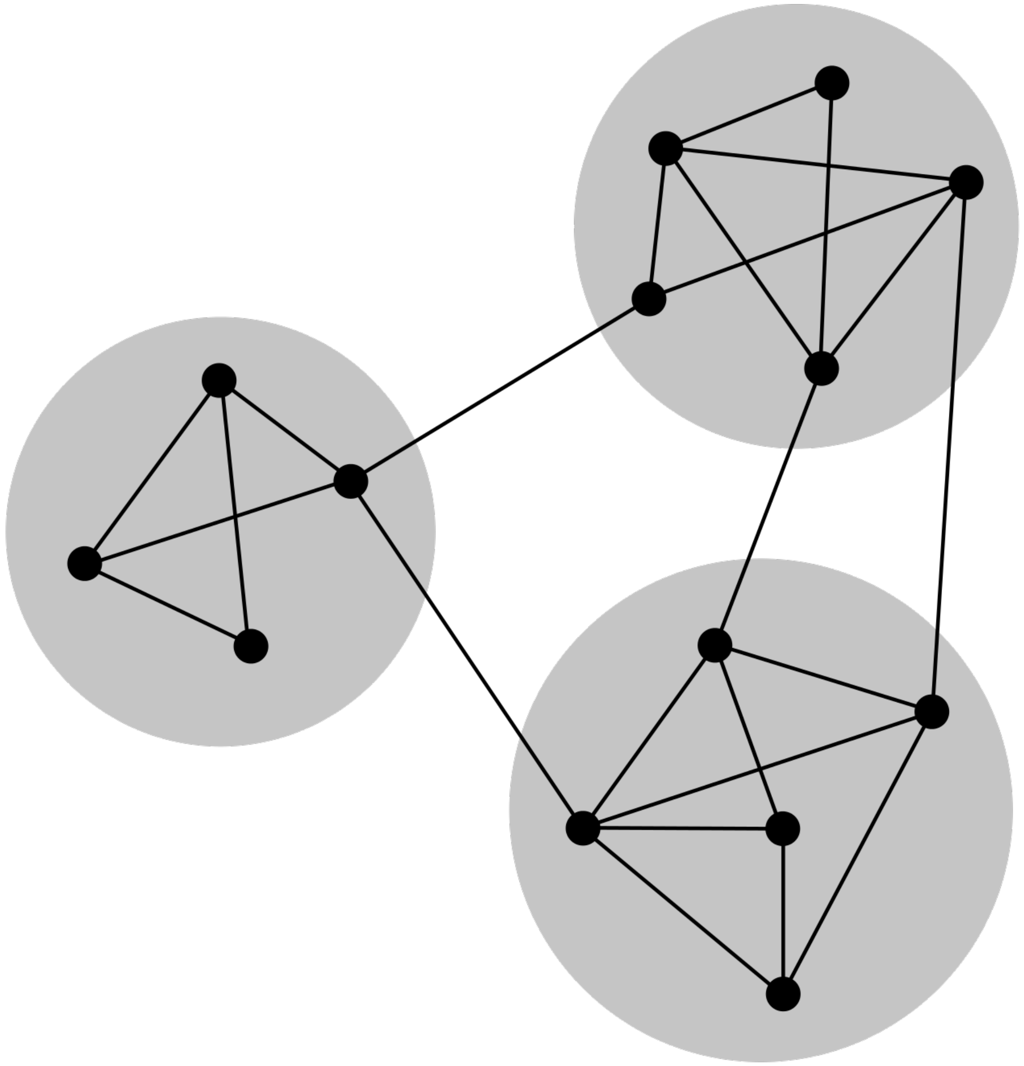
Network community

Community detection

Group of node tightly connected among them and weakly connected with the rest of the network.

Communities allow to “simplify” the network structure



Communities allow to “better” understand the network dynamics

Community detection in hypergraphs

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Random walks and community detection in hypergraphs

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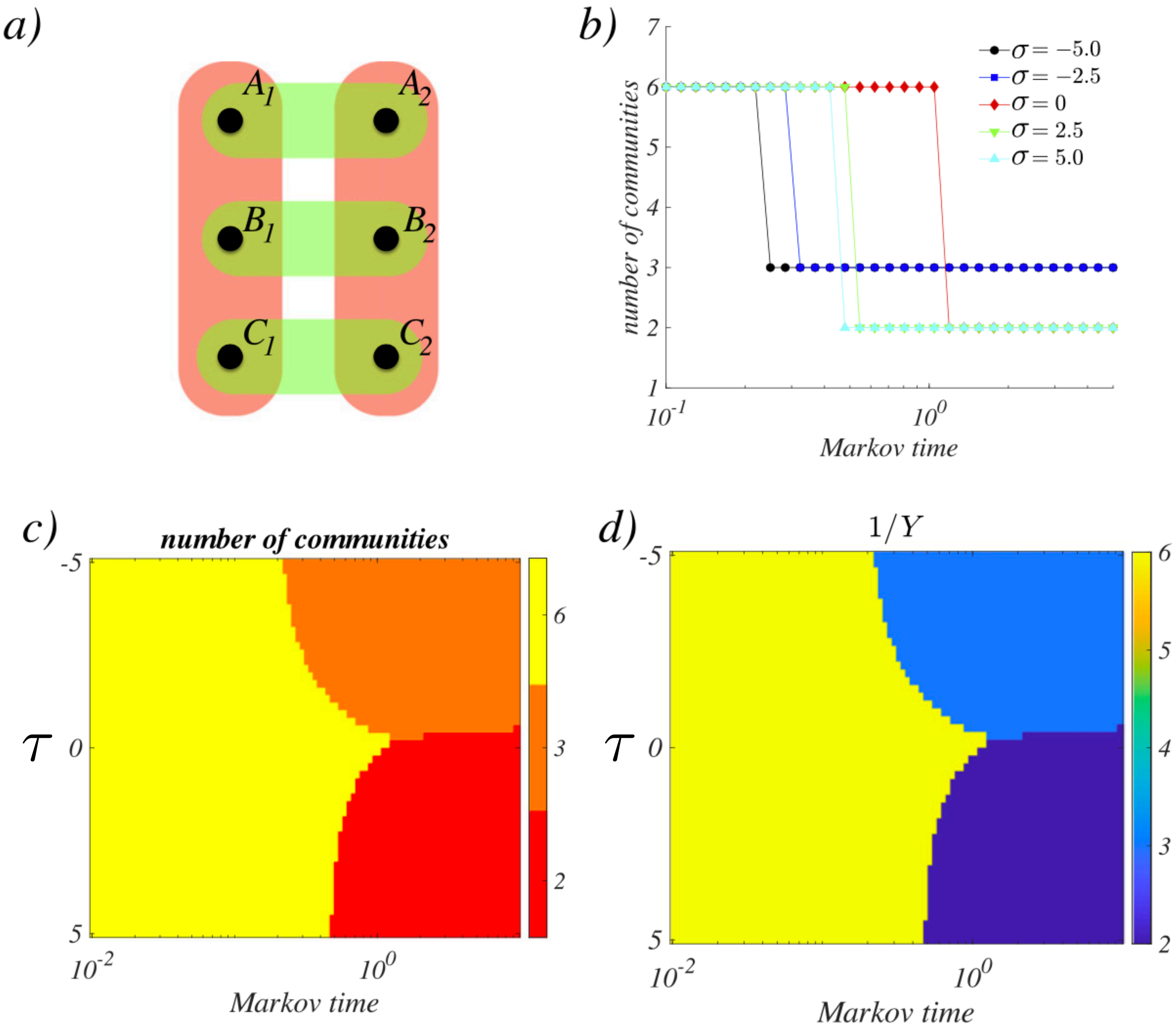
Keywords: hypergraphs, random walks, higher-order networks

Abstract

We propose a one-parameter family of random walk processes on hypergraphs, where a parameter biases the dynamics of the walker towards hyperedges of low or high cardinality. We show that for each value of the parameter, the resulting process defines its own hypergraph projection on a weighted network. We then explore the differences between them by considering the community structure associated to each random walk process. To do so, we adapt the Markov stability framework to hypergraphs and test it on artificial and real-world hypergraphs.

Community detection

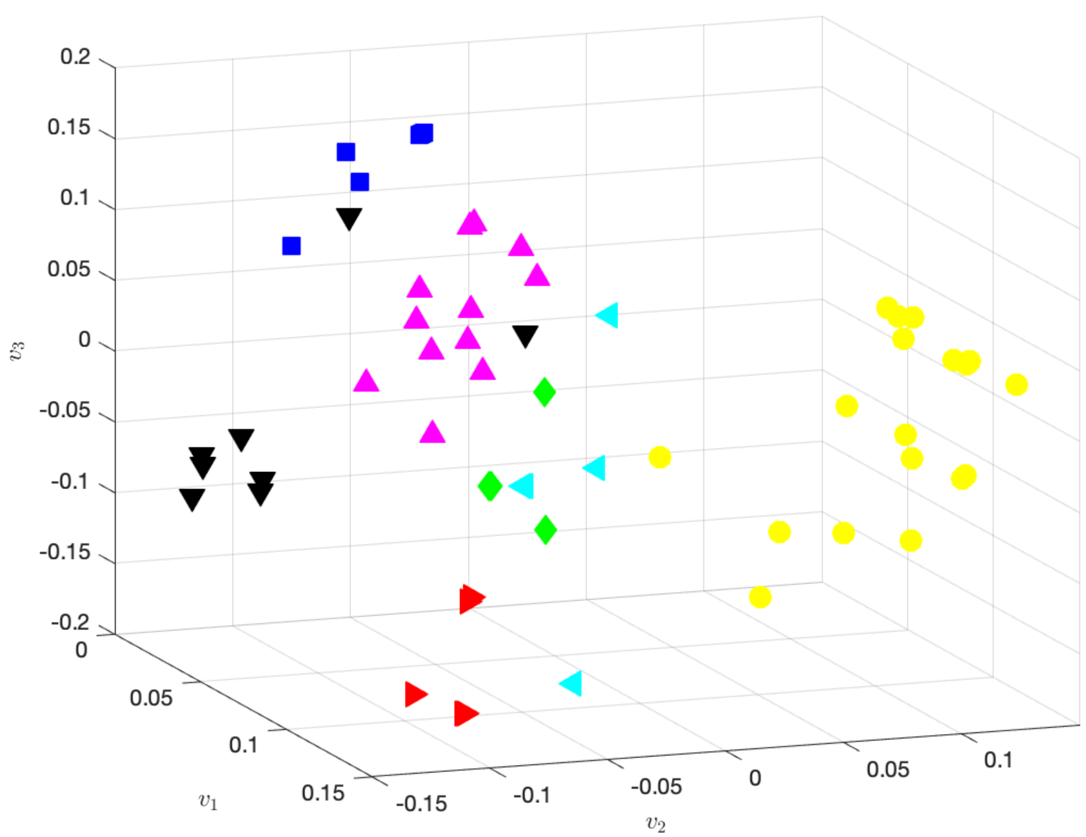
$$k_{ij}^H = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$



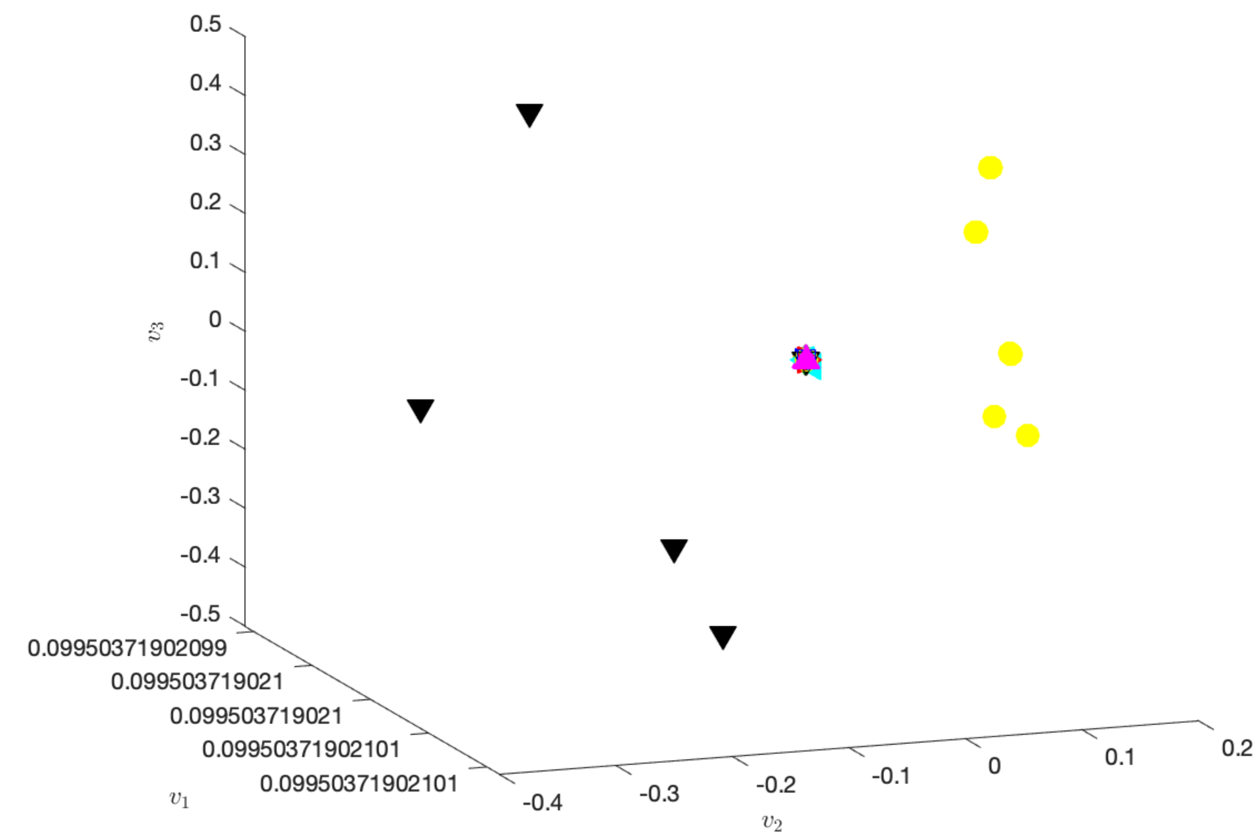
$$Y = \sum_{i=1}^{M} \frac{S_i^2}{N^2},$$

Classification

UCI zoo database, 101 animals, 16 features (hair, feathers, eggs, milk, airborne, aquatic, predator, toothed, backbone, breathes, venomous, fins, tail, domestic, number of legs, cat size), 7 classes.



Hypergraph



Clique projection

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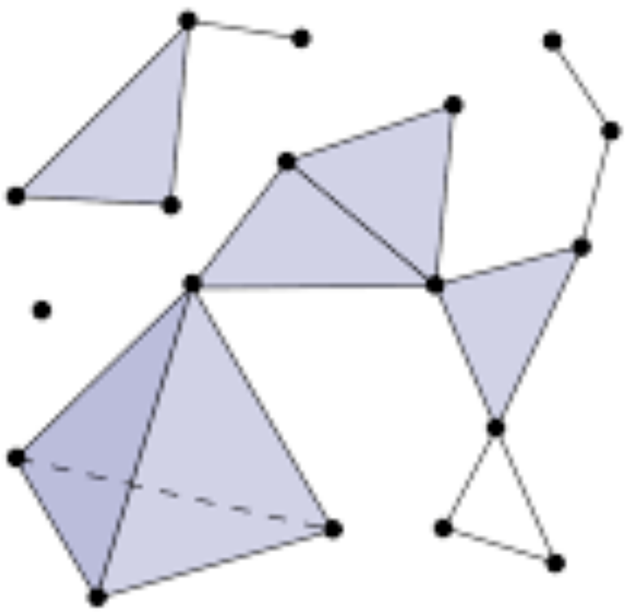
Simplicial complexes and Hypergraphs

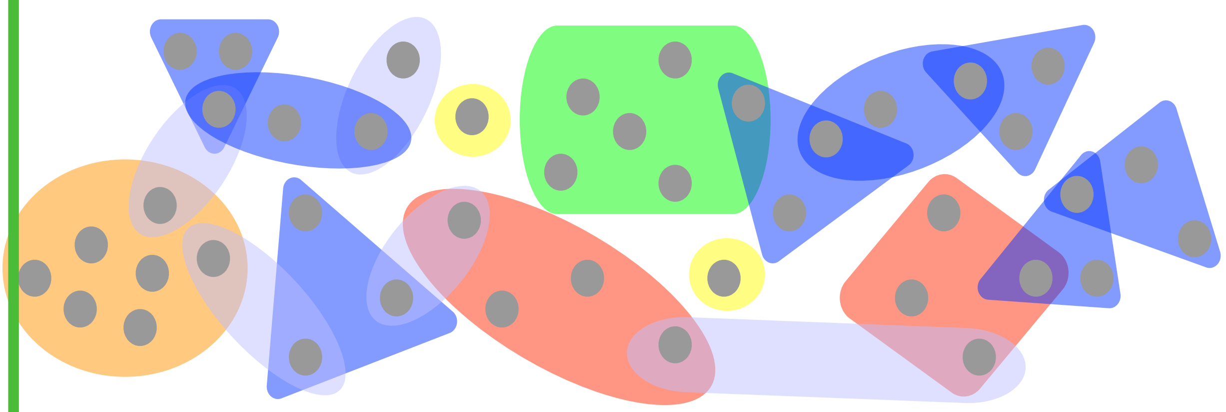
Simplicial complexes



d-simplex = d+1 nodes
(all linked together)

1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

Hypergraphs



hyperedge = set of nodes

Timoteo Carletti

Beyond complex Networks: Hypergraphs

Thank you

Any questions?

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