Phase transitions in non-linear random walks on lattices

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Model and analytical results

Diffusion process

$$\dot{\rho}_i = \sum_{j=1}^{\Omega}\left[-\frac{A_{ij}}{k_i}\rho_i g(\rho_j) + \frac{A_{ji}}{k_j}\rho_j g(\rho_i)\right] = \sum_{j=1}^{\Omega} L_{ij}\left[\rho_j g(\rho_i) - \frac{k_j}{k_i}\rho_i g(\rho_j)\right] \quad (1)$$

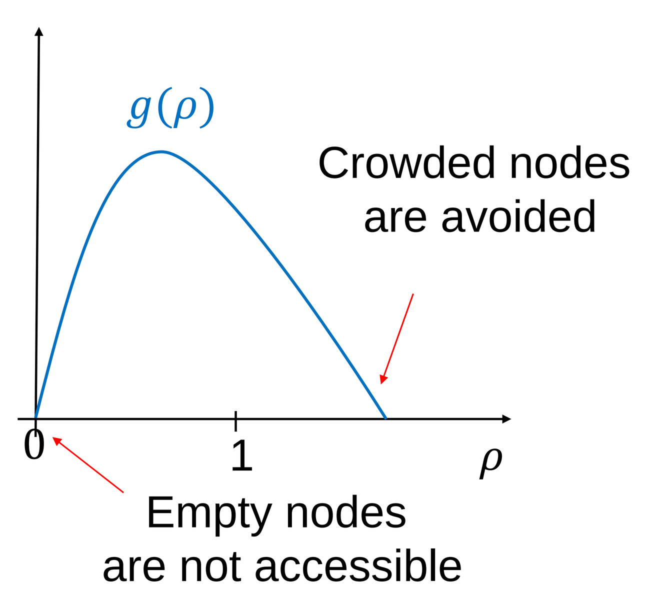
[1,2]

Random-walk Laplacian

$$L_{ij} = \frac{A_{ij}}{k_j} - \delta_{ij}$$

Mass conservation

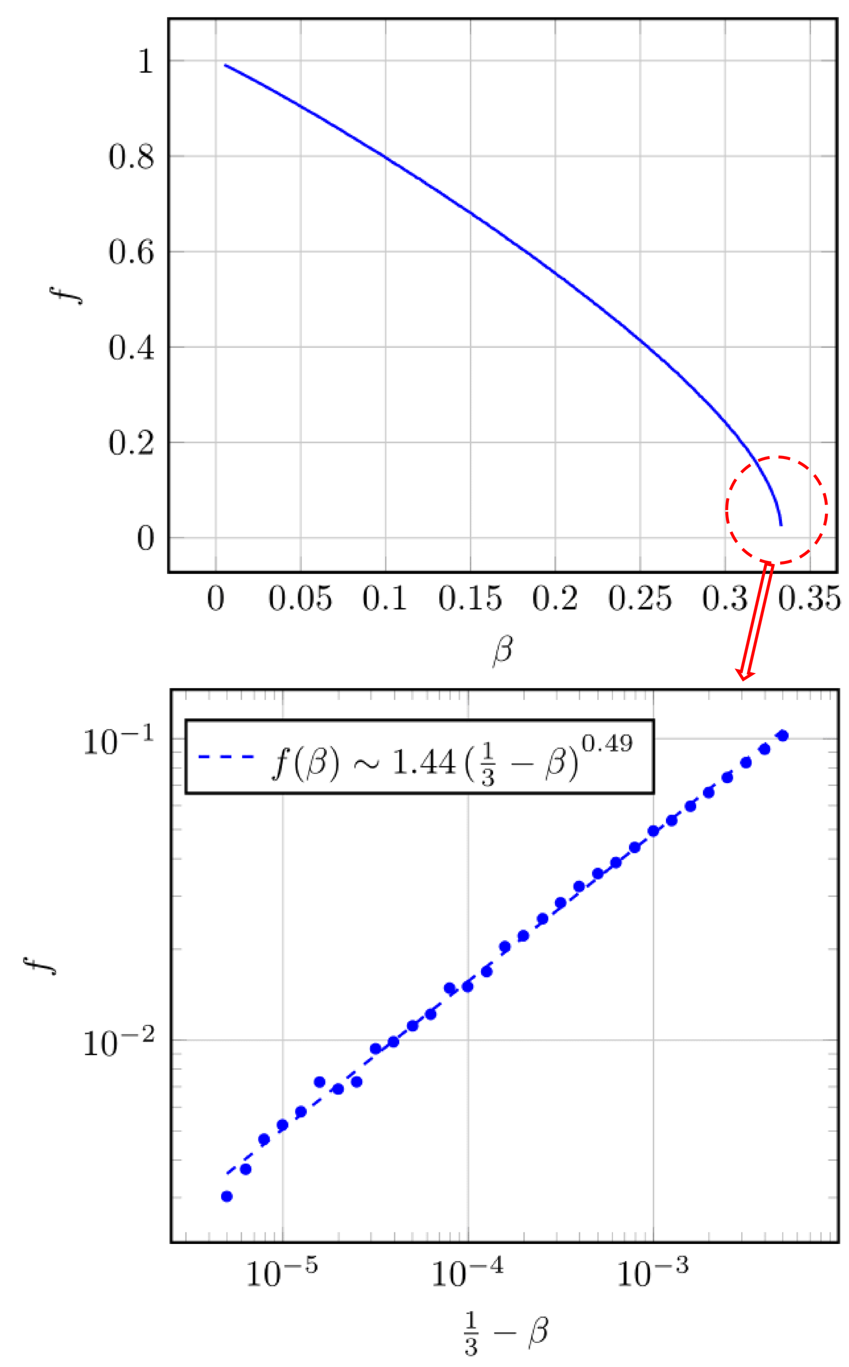
$$\sum_i \rho_i(t) = \Omega\beta, \beta \in [0,1]$$



For regular networks and $g(x) = x^a(1-x)^a$, the homogeneous state ($\rho_i = \beta\ \forall i = 1,\cdots,\Omega$) of the dynamical system (1) is unstable when $\beta < \frac{a-1}{2a-1}$ and $a > 1$.

Empty nodes (nodes for which $\lim_{t\to+\infty}\rho_i(t) = 0$) can emerge.

Phase transition in a complete graph



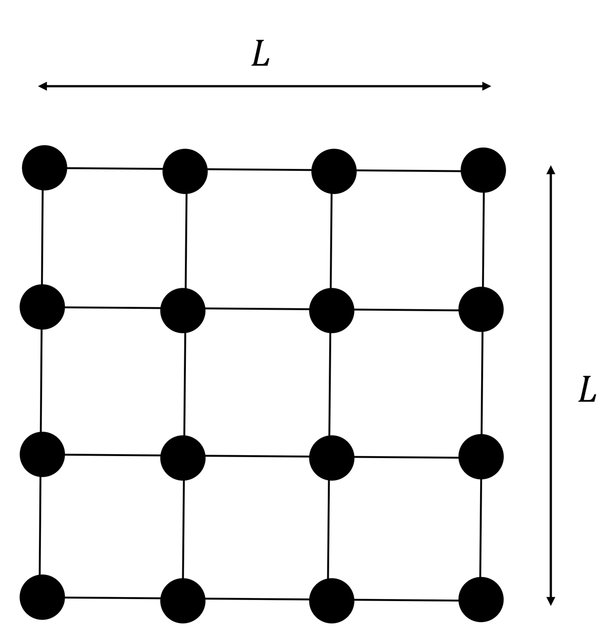
Fraction f of empty nodes in the steady state as a function of the average node density β, for a complete graph made of 5000 nodes and $g(x) = x^2(1-x)^2$. Data were averaged over 20 configurations.

For each configuration, nodes densities were initialized randomly in the interval $[\beta - \frac{\delta}{2}, \beta + \frac{\delta}{2}]$, with $\delta = 10^{-4}$.

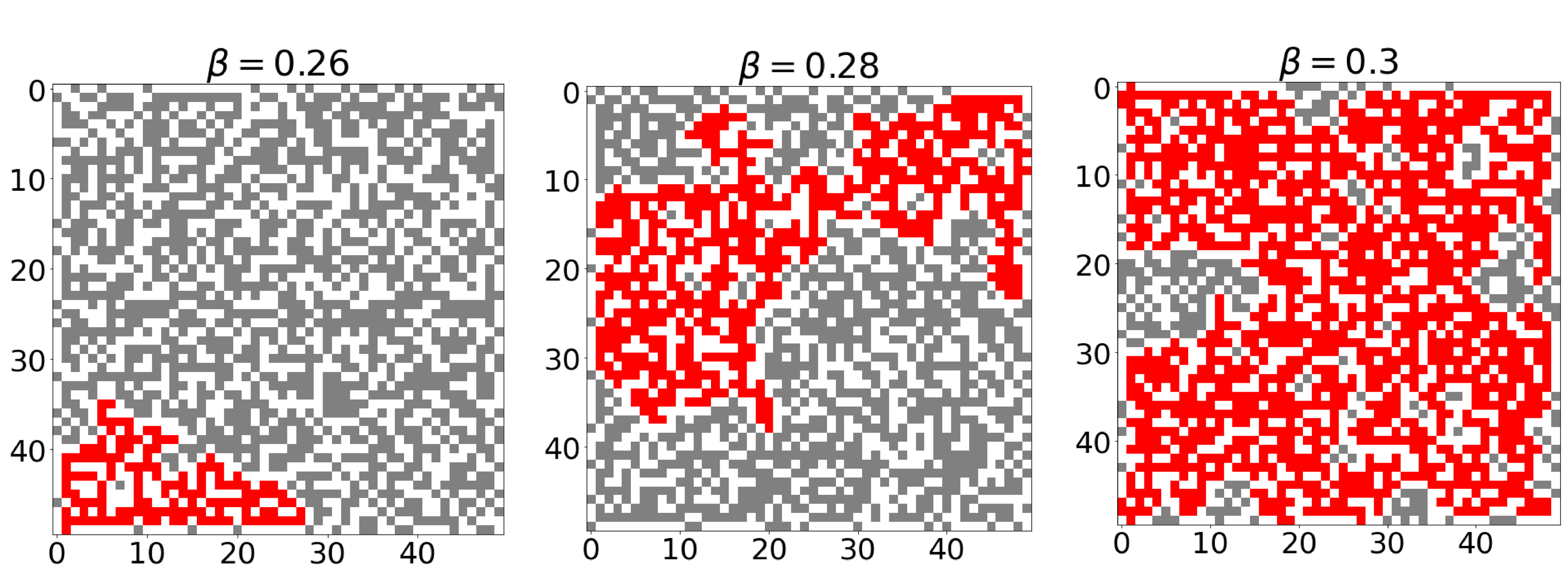
Close to $\beta = \frac{1}{3}$, the fraction of empty nodes exhibits a power-law behavior.

β plays the role of a **control parameter**

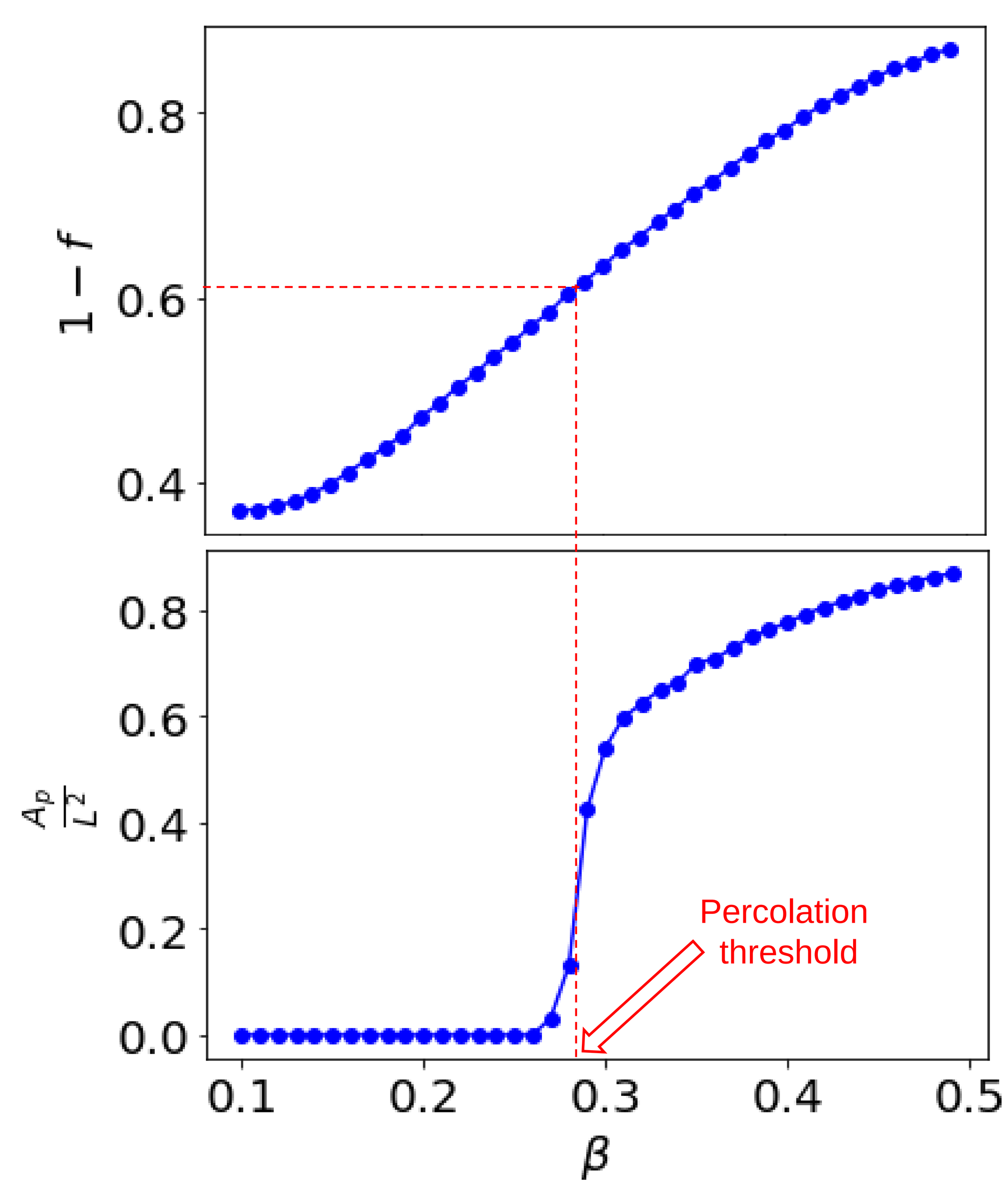
Distribution of mass on top of a square lattice



Site percolation: empty nodes play the role of removed nodes since they no longer participate in the dynamics.
In the steady state, a **cluster** is defined as a set of nearest-neighbor filled nodes.



Empty nodes are represented in white. Non-empty nodes are represented in grey or in red if they belong to the largest cluster. The critical threshold happens when $\beta \approx 0{,}28$. Above the critical threshold, the largest cluster almost surely percolates. Nodes densities were initialized randomly in $[0,1]$ and normalized such that $\sum_i \rho_i(0) = \Omega\beta$, with $\Omega = L^2$ and $L = 50$. We considered $g(x) = x^2(1-x)^2$.



Top : Fraction of non-empty nodes, $1-f$, as a function of the average node density β, for a square lattice with $\Omega = L^2$ nodes and $L = 50$.

Bottom : Size of the percolating cluster, $\frac{A_p}{L^2}$, as a function of β.

In both panels, data were averaged over 10 configurations, with nodes densities initialized randomly in $[0,1]$ and normalized such that $\sum_i \rho_i(0) = \Omega\beta$.

References

[1] Asllani, M., Carletti, T., Di Patti, F., Fanelli, D., & Piazza, F. (2018). Hopping in the crowd to unveil network topology. *Physical review letters*, *120*(15), 158301.

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Conclusion

We considered a non-linear diffusion process on top of lattices. When the average node density decreases below a given threshold, the homogeneous state can become unstable and empty nodes can emerge. Numerical simulations suggest that the distribution of empty nodes bears similarities with the site percolation process.

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