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Description of weak measurements and weak values in the phase space representation of quantum mechanics

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Bryan Renard

Résumé

Les mesures faibles estiment la valeur d'un opérateur sur un système quantique tout en minimisant la perturbation de l'état, au contraire d'une mesure forte. Les observations tirées des mesures faibles avec post-sélection dépendent de nombres complexes appelés valeurs faibles. Lorsque très grandes ou complexes, elles sont appelées valeurs faibles anormales et indiquent un comportement quantique. Afin d'avoir plus d'intuition sur leur sens physique, nous les décrivons dans l'espace de phase quantique défini en utilisant la distribution de Wigner et la transformée de Weyl. Nous illustrons le formalisme par une mesure faible de deux oscillateurs harmoniques couplés. La valeur faible est interprétée comme la moyenne par rapport à un terme d'interférence entre la pré-sélection et la post-sélection. Le cas particulier de l'opérateur impulsion est étudié et une interprétation est proposée, par des notions de la littérature. Ensuite, nous transposons le modèle de mesure de von Neumann dans l'espace de phase pour des mesures fortes et faibles post-sélectionnées. En utilisant le noyau de Stratonovich-Weyl, nous généralisons par la suite le formalisme de l'espace de phase à des espaces de configuration courbes, utiles pour décrire des espaces contraints. Les résultats précédents sont étendus à cette situation.

MOTS-CLÉS - Mesure faible - Valeur faible - Espace de phase quantique - Distribution de Wigner - Espace courbe - Mesure de von Neumann

Abstract

Weak measurements estimate an operator's value of a quantum system while minimising the perturbation of the state, contrary to the usual strong measurement. The observations from weak measurements with post-selection depend on complex numbers called weak values. When very large or complex, they are called anomalous weak values and indicate a quantum behaviour. To get more intuition about their physical meaning, we describe them in the quantum phase space defined with the Wigner distribution and the Weyl transform. We illustrate the formalism by a weak measurement of two coupled harmonic oscillators. The weak value is interpreted as the average value over an interference between pre-selection and post-selection. The particular case of the momentum operator is studied and an interpretation is proposed, using notions from the literature. Then, we transpose the von Neumann model of measurement in phase space for strong and weak post-selected measurements. Using the Stratonovich-Weyl kernel, we then generalise the phase space formalism to curved configuration spaces, useful to describe constrained spaces. The previous results are extended to this situation.

KEYWORDS - Weak Measurement - Weak Value - Quantum Phase space - Wigner distribution - Curved space - von Neumann Measurement

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more.

— John von Neumann

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Contents

1 Introduction and Problem Statement

Quantum physics. These two words are quite recent in the scientific language, and are able to either scare or fascinate any person hearing these words. As Richard Feynman said so rightly, *"I think I can safely say that nobody really understands quantum mechanics"*. Indeed, the quantum behaviour of light and matter at the particle scale is so far away from our classical intuition that it brings to us many destabilising concepts or the so-called quantum paradoxes.

One of these concepts is the incompatibility. In quantum physics, measuring two non-commuting observables on the system gives different results depending which one is measured before the other. A common example is the measurement of the position x and momentum p of a particle. Another one is the contextuality [1] of quantum mechanics, meaning that a property of a particle is not intrinsic of the particle but depends of the act of measurement itself.

These problems can be resolved or studied using weak measurements [2, 3, 4, 5]. The idea of a weak measurement is to measure a property of a system with a very weak efficiency. This seems useless, since very little information is extracted from such a measurement. However, by performing a post-selection, i.e. by strongly measuring another property of the system, a conditional measurement is made. This gives additional information from the weak measurement.

Weak measurements depend on weak values [2, 3, 4, 5]. When measuring a given operator, the associated weak value can exceed the range of the eigenvalues of the operator. This allows for an amplification effect, experimentally useful to develop ultra-sensitive experiments [6, 7] or to detect minute phenomena [8]. Moreover, the weak value can be complex, a useful property to perform quantum tomography easily [9]. Non-commuting observables can also be measured, for example in the determination of trajectories in a two-slit interferometer [10]. Lastly, weak measurements are used to study quantum paradoxes [11, 12] and the foundations of quantum physics [13].

However, weak measurements originate from quantum effects and the physical interpretation of the experiments and the weak values is tricky. In a projective measurement, the eigenvalues of the operator are obtained. But in a weak post-selected measurement, it is a quantity related to the weak value. It can be very large and it is complex, so the question whether the weak value gives meaningful information about the property itself is open. Furthermore, the paradoxes involving weak measurements also evidence the problem of the interpretation of the weak values (for example in the Cheshire cat paradox [11] or the three-box paradox [12]).

A possible way to understand weak measurements is to bring their description closer to the classical intuition. Physical situations in classical physics are often described in phase space [14]. Each degree of freedom of the system gives an axis of the space, and one point of the phase space corresponds to one particular state of the system. The most common phase space is (x, p), to describe for example the trajectory of the pendulum. It is also often used in chaos theory, since it allows to observe the evolution of a system as a trajectory in phase space [15].

There exists a phase space description of quantum physics, using the notions of Wigner distribution and Weyl transform [16, 17, 18]. It is constructed by analogy with concepts of statistical physics [19]. For a quantum state $|\psi\rangle$, it gives a complete description of the (x, p) distribution instead of $\psi(x)$ and $\psi(p)$ separately. The Wigner function is a quasi-probability distribution, meaning that it can take negative values [16, 17]. The origin of the negativity is purely quantum since a negative point in phase space doesn't make sense in classical physics [20].

The objective is to better understand the process of weak measurement and the concept of weak values, by describing them in phase space. This might help to understand how the quantum system evolves in the process of measurement. The concept of phase space is also generalised to describe weak values and weak measurement in other phase spaces than (x, p), and this is more specifically applied to a curved space.

More precisely, the weak value is first described in the phase space (x, p), as a function of the pre-selection and the post-selection. This result is then extended to describe the weak value in a generalised phase space and in a curved space. In particular, the weak value of the momentum post-selected on the position is studied. We show that the real part of this weak value is linked to the notion of probability current and the imaginary part to the osmotic velocity [21, 22].

The von Neumann model of measurement is the framework in which the weak measurement is usually presented. It gives a continuous description of the measurement process. We describe this process in phase space, to have the possibility to know the phase space representation of the states at each step of the measurement. It is done for strong and weak measurements, as well as post-selected weak measurements. The results are then generalised to curved configuration spaces, to describe a measurement in constrained spaces. This can be useful in describing the phase space of a rotation problem or in curved space-time.

The thesis is organised as follows. The two main concepts of the work are first introduced, in Section 2 for weak measurements and Section 3 for the phase space formalism. In Section 4, the Wigner distribution is generalised and constructed on curved space. Section 5 is then devoted to the study of weak measurements and weak values in phase space.

2 Weak Measurements

2.1 Projective measurement

The notion of measurement in quantum physics is very different from its classical counterpart. Indeed, since the birth of quantum physics, it is known that a measurement of any physical property on a quantum system instantly modifies (projects) the system state. This was observed experimentally and the scientists carved this property as a postulate of quantum mechanics.

Definition 2.1: Quantum Postulate [23]

Consider a quantum system $\mathcal{S}$, initially in the state $|s\rangle$, and a physical quantity $\mathcal{A}$ to measure on the system. This quantity is described by a Hermitian operator (observable) $\hat{A}$, with (non-degenerate) eigenvalues a_i and eigenvectors $|a_i\rangle$ for $i = 1, ..., n$.

The measurement of $\hat{A}$ gives one of the eigenvalues a_i with probability $P_i = |\langle s|a_i\rangle|^2$.

The measurement *instantly projects* the system state on the eigenvector $|a_i\rangle$.

In that scenario, the initial state is completely modified and the measurement is complete, meaning that no additional information about the initial state can be obtained. The measurement has indeed destroyed it.

This understanding of a quantum measurement can be generalised to a quantum state that is initially mixed and not pure, using the reduced density matrix $\hat{\rho}$. The eigenvalue a_i is then obtained with probability $P_i = \mathrm{Tr}\left(\hat{\Pi}_i\hat{\rho}\right)$ with $\hat{\Pi}_i = |a_i\rangle\langle a_i|$ the projection operator on the corresponding eigenvector. The system state after the measurement is

$$\hat{\rho}_{proj} = \frac{\hat{\Pi}_i\hat{\rho}\hat{\Pi}_i}{\mathrm{Tr}\left(\hat{\Pi}_i\hat{\rho}\right)}. \tag{2.1}$$

It is the initial state, projected onto the eigenvector and normalised with respect to the probability. This process can be generalised for any possible measurement described by positive-operator valued measures (POVM) [24]. A POVM is usually incomplete, meaning that the resulting state might still depend on the initial state. An example is a weak measurement.

2.2 Motivation of weak measurements

A measurement in the usual quantum theory is a discontinuous process, the projection being instantaneous. In classical physics, it is rare to have truly discontinuous phenomena, if described as such, it usually means that an underlying continuous description exists. For example, switching on and off a generator might be considered as instantaneous in the time scale chosen. The charge and electric field in a pn junction are also not as discontinuous as described in textbooks.

The same reasoning might putatively be applied to the projection occurring in a quantum measurement. We could ask if the discontinuity comes from the use of a measurement device treated classically. The measurement apparatus is not taken into account in the usual description, but it would make sense to add it in the formalism with a given interaction with the system. Moreover, it is also assumed that the measurement is made with maximum accuracy.

A better description of the measurement process comes from the von Neumann model [4, 5]. This scheme takes into account the measurement device and its coupling with the initial state. It depends on the way the meter and the system interact and the strength of their interaction. The time during which the two parts evolve together is also taken into account. This quantum treatment gives an entanglement between the two subsystems but a projection is still necessary, in the end, to select one of the possibilities of the superposition.

Any usual projective measurement can be described in this scheme. Moreover, a more interesting behaviour appears when considering a weak coupling between the system and the meter. In this case, the information extracted from the system is small but its state is only slightly perturbed instead of the projection onto the corresponding eigenvector. This is the basis of a weak measurement [2, 3, 4, 5]. Performing it in a clever way, with pre- and post-selection, allows to gather meaningful information about the system while keeping it largely undisturbed (before post-selection). The advantage of a slight perturbation of the state is that it leads to the possibility to measure non-commuting observables (usually not possible in a strong measurement). This property is for example used to determine the average trajectories of photons in a two-slit interferometer [10].

More precisely, a post-selected weak measurement of an operator $\hat{A}$ depends on a weak value [2, 3, 4, 5]. As we shall see, this quantity can exceed the range of the eigenvalues of the operator or be complex; it is then called an anomalous weak value. The amplification effect, appearing for large weak values, is used in experiments to detect minute phenomena such as the spin Hall effect [8] or to design very sensitive experiments [6, 7]. Complex weak values allow to perform quantum tomography of states [9]. Paradoxical behaviours can also be studied (three-box paradox [12], Cheshire cat paradox [11], pigeonhole paradox [25]). Weak measurements provide insight into the foundations of quantum physics. Anomalous weak values manifest for example the contextuality of quantum mechanics [13].

2.3 von Neumann model of measurement

The von Neumann model inserts the influence of the measurement apparatus in the quantum formalism [4, 5]. Consider the system $\mathcal{S}$ and the meter $\mathcal{M}$. The objective is to entangle the two in order to correlate the measure of the operator $\hat{A}$ on the system and the readout performed on the meter. We assume that the system and the meter interact from $t = 0$ to $t = T$.

Proposition 2.1: von Neumann measurement [3, 4, 5]

Consider the initial system state $|\psi_i\rangle$ and the initial meter state $|\phi\rangle$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p},$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total coupling strength is $\gamma = \int_0^T g(t)dt$ and $|\psi_i\rangle = \sum_i \alpha_i |a_i\rangle$ so that the final joint state is

$$|\Psi\rangle = \int dx\, |x\rangle \sum_i \alpha_i \phi(x - \gamma a_i) |a_i\rangle$$

Proof. The total Hamiltonian describing the evolution of the state and the meter is

$$\hat{H} = \hat{H}_S + \hat{H}_M + \hat{H}_{int} \tag{2.2}$$

The Hamiltonians $\hat{H}_S$ and $\hat{H}_M$ give the free evolution of the system and the meter, respectively. After the weak measurement, the readout of the meter is done. The free evolution of the system is therefore not important, as long as the property $\hat{A}$ of the system doesn't change with time. The Hamiltonian $\hat{H}_S$ is consequently neglected. The evolution of the meter is also not taken into account. It could be considered through time-dependant operators in the Heisenberg representation but in the limit of a short interaction time, it is negligible [4, 5].

The evolution of the system is therefore directed by the unitary operator

$$\hat{U} = e^{-\frac{i}{\hbar}\int_0^T \hat{H} dt} = e^{-\frac{i}{\hbar}\int_0^T \hat{H}_{int}(t) dt} = e^{-\frac{i}{\hbar}\int_0^T g(t)dt \hat{A}\otimes\hat{p}} = e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}}. \tag{2.3}$$

The joint state $|\Psi\rangle$ after the interaction of the system $|\psi_i\rangle$ and the meter $|\phi\rangle$ is

$$|\Psi\rangle = \hat{U}\left(|\psi_i\rangle \otimes |\phi\rangle\right). \tag{2.4}$$

The eigenvectors $|a_i\rangle$ of the operator $\hat{A}$ form a basis of the system state. The initial state $|\psi_i\rangle$ is written in this basis,

$$|\psi_i\rangle = \sum_i \alpha_i |a_i\rangle . \tag{2.5}$$

Therefore, the joint state after the application of the evolution operator is

$$|\Psi\rangle = e^{-\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}} \left(\sum_i \alpha_i |a_i\rangle\right) \otimes |\phi\rangle = \sum_i \alpha_i e^{-\frac{i}{\hbar}\gamma a_i \hat{p}} |a_i\rangle \otimes |\phi\rangle . \tag{2.6}$$

In the x basis of the meter, we get a shift of the pointer state,

$$|\Psi\rangle = \int dx\, |x\rangle \langle x|\Psi\rangle = \int dx\, |x\rangle \sum_i \alpha_i e^{-\frac{i}{\hbar}\gamma a_i \hat{p}} \phi(x) |a_i\rangle = \int dx\, |x\rangle \sum_i \alpha_i \phi(x-\gamma a_i) |a_i\rangle \tag{2.7}$$

by the definition of the translation operator. □

This theorem shows that the joint state after the interaction is in a quantum superposition. Thus the meter and system are entangled. Coupling the meter and the system yields a shift in the x coordinate of the meter state, directly proportional to the eigenvalues of $\hat{A}$.

Consider that the meter $\phi(x)$ is initially in the ground state of a harmonic oscillator,

$$\phi(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-\frac{x^2}{2\sigma^2}}, \tag{2.8}$$

It is the square root of a Gaussian distribution centered in $x = 0$ and of deviation $\frac{1}{2}\sigma$. If γ is large with respect to the deviation, as illustrated in Figure 1(a), the Gaussian distributions do not overlap. This figure shows the distribution of probability of the meter measurement, depending on the position x. The experimenter can clearly identify each one of the distributions. Since the link between the Gaussian distributions and the eigenvalues is clear, the experimenter can know the eigenvalue obtained. The system state is projected on the corresponding eigenvector, it is a strong measurement. However, if γ is small, there is overlap between the distributions, as illustrated in Figure 1(b). The distribution of probability is shown with the red line. The experimenter cannot identify the different Gaussians. The system state is not projected on a single eigenvector but on a mixture of the eigenvectors corresponding to the possible eigenvalues. We have what is called a weak measurement (without post-selection). Note that in practice, a weak measurement is such that the Gaussians of Figure 1(b) are completely merged and only the average value can be obtained. The representation corresponds to an intermediate measurement but is useful to show the idea.

If the coupling strength goes to zero, then the meter is nearly unmodified and the system is also barely disturbed. But as the coupling grows, the perturbation grows as well. A good condition to decide that a measurement is strong is, for a Gaussian distribution,

$$|\gamma|\Delta a \gg \sigma^2, \tag{2.9}$$

with Δa the minimal distance between the eigenvalues of $\hat{A}$ [4].

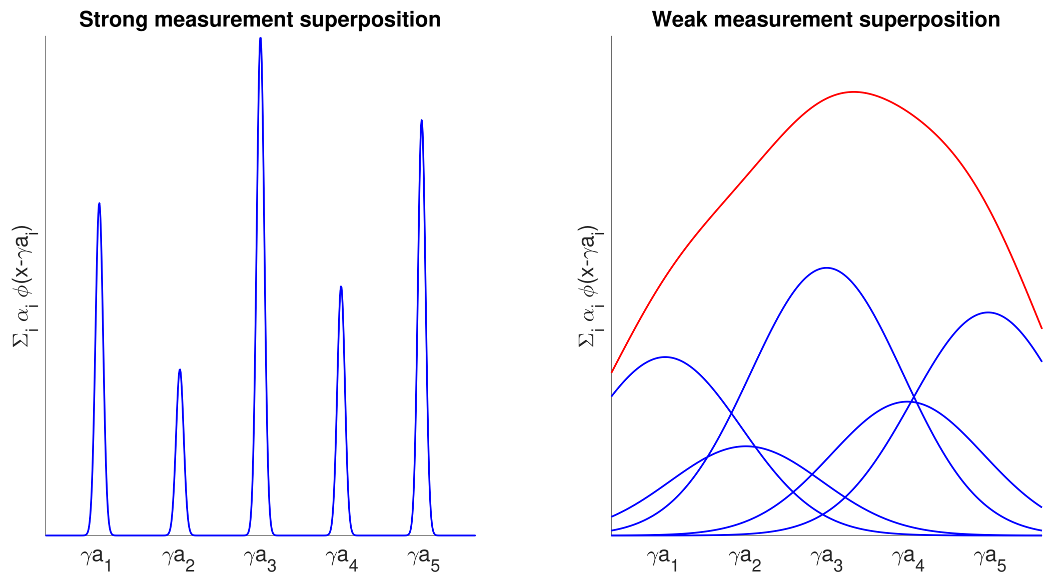


Figure 1: Overlap of Gaussian distributions. (*a*) In the strong measurement case. (*b*) In the weak measurement case. The red line is the sum of the individual Gaussians.

2.3.1 Example: Stern-Gerlach Experiment

The Stern-Gerlach experiment is an experiment used to measure the spin. A particle is sent through a magnet creating a magnetic field B oriented in the z direction. The interaction Hamiltonian is [26]

$$\hat{H}_{int} = -\mu \frac{\partial B_z}{\partial z} \hat{\sigma}_z \otimes \hat{z}. \tag{2.10}$$

with μ the magnetic moment. The system is the spin, the meter is the profile of the beam and the observable is $\hat{A} = \hat{\sigma_z}$. For a sufficiently strongly varying magnetic field, the two eigenvalues of the spin operator (± 1) are clearly separated and two beams are detected on the screen.

2.3.2 Example: Coupled Harmonic Oscillators

Another example is a bi-dimensional harmonic oscillator. Contrary to the previous example, the two Hilbert spaces (system and meter) are continuous. The corresponding Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m \omega_0^2 (\hat{x}^2 + C\hat{y}^2). \tag{2.11}$$

The two oscillators are decoupled. A magnetic field is applied to couple them through the orbital momentum operator $\hat{L}_z$. The Hamiltonian becomes

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m \omega_0^2 (\hat{x}^2 + C\hat{y}^2) - aB(t)\hat{L}_z. \tag{2.12}$$

The additional term is the Hamiltonian of interaction. The operator is $\hat{L}_z = \hat{p}_y \otimes \hat{x} - \hat{y} \otimes \hat{p}_x$, so the interaction Hamiltonian is

$$\hat{H}_{int} = -aB(t)\hat{L}_z = -aB(t)(\hat{p}_y \otimes \hat{x} - \hat{y} \otimes \hat{p}_x) = aB(t)\hat{y} \otimes \hat{p}_x - aB(t)\hat{p}_y \otimes \hat{x}. \tag{2.13}$$

The total coupling strength is $\gamma = \int_0^T aB(t)dt$. For a constant magnetic field during a finite time T, the coupling strength is $\gamma = aBT$. We consider that the oscillator in the x direction is the meter and the oscillator in the y direction is the system to probe. The coupling is slightly different than the one used in Proposition 2.1 because the observable $\hat{x}$ is also coupled with the system. This part of the coupling will give a shift of the p_x coordinate of the meter. This example will be used throughout the thesis to illustrate the concepts.

2.4 Weak post-selected measurement and weak value

Performing a weak measurement doesn't allow to extract much interesting information except the average value. However, there is a way to extract much more from a weak measurement, by performing a post-selection on the system after the weak interaction with the meter [2, 3, 4, 5]. This post-selection consists of making a final projective measurement of another operator and consider only one of the results. The others are discarded. The different steps of the measurement are summarized on Figure 2. The pre-selection is the choice of the initial state of the system, as is often done experimentally. The interaction, assumed to be weak, evolves the states according to the Hamiltonian. The post-selection is then performed. The correlated readout of the meter is done in parallel to measure the result of the experiment.

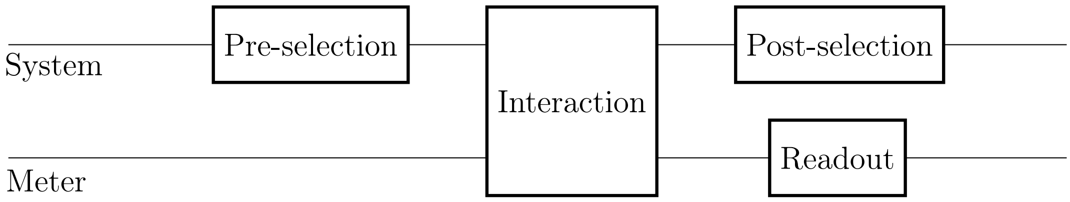


Figure 2: Steps of a weak measurement with pre- and post-selection.

The observations in a post-selected weak experiment depend on the weak value.

Definition 2.2: Weak Value [3, 4, 5]

Let $|\psi_i\rangle$ and $|\psi_f\rangle$ be the pre- and post-selected states of the weak measurement, respectively. The system's observable to measure is $\hat{A}$. The weak value is defined as

$$A_w = \frac{\langle\psi_f|\,\hat{A}\,|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}.$$

It is a complex number. The meter in a post-selected weak measurement is shifted by a quantity determined from this value, according to the von Neumann model.

Proposition 2.2: Weak post-selected von Neumann measurement [3, 4, 5]

Consider the initial system state $|\psi_i\rangle$ and the initial meter state $|\phi\rangle$. The post-selected system state is $|\psi_f\rangle$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p},$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. The final post-selected joint state is

$$|\Psi_j\rangle = |\psi_f\rangle \langle\psi_f|\Psi\rangle = |\psi_f\rangle \langle\psi_f|\psi_i\rangle\, e^{\gamma \operatorname{Im}(A_w)\hat{p}} \int dx\, |x\rangle\, \phi(x - \gamma \operatorname{Re}(A_w)).$$

Proof. The beginning of the proof is similar to Proposition 2.1. The post-selected joint state is

$$|\Psi_j\rangle = |\psi_f\rangle \langle\psi_f|\Psi\rangle = |\psi_f\rangle \langle\psi_f|\, e^{-\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}} \left(|\psi_i\rangle \otimes |\phi\rangle\right). \tag{2.14}$$

The coupling strength is small so the evolution operator is approximated to the first order,

$$e^{-\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}} \approx 1 - \frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}, \tag{2.15}$$

so that

$$\langle\psi_f| \left(1 - \frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}\right) |\psi_i\rangle = \langle\psi_f|\psi_i\rangle - \frac{i}{\hbar}\gamma \langle\psi_f|\, \hat{A}\, |\psi_i\rangle\, \hat{p} = \langle\psi_f|\psi_i\rangle \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}\right). \tag{2.16}$$

The weak value appears. By applying again the first-order approximation of the exponential in reverse, we get

$$|\Psi_j\rangle \approx |\psi_f\rangle \langle\psi_f|\psi_i\rangle \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}\right) |\phi\rangle \approx |\psi_f\rangle \langle\psi_f|\psi_i\rangle\, e^{-\frac{i}{\hbar}\gamma A_w \hat{p}}\, |\phi\rangle \tag{2.17}$$

$$= |\psi_f\rangle \langle\psi_f|\psi_i\rangle\, e^{-\frac{i}{\hbar}\gamma A_w \hat{p}} \int dx\, |x\rangle\, \phi(x) \tag{2.18}$$

$$= |\psi_f\rangle \langle\psi_f|\psi_i\rangle\, e^{\frac{1}{\hbar}\gamma \operatorname{Im}(A_w)\hat{p}} \int dx\, |x\rangle\, \phi(x - \gamma \operatorname{Re}(A_w)). \tag{2.19}$$

□

This means that applying the post-selection after the weak measurement shifts the x coordinate of the apparatus function in proportion to the real part of the weak value. A shift in the momentum p of the meter is also produced, depending on the real and imaginary parts, as is shown later in Proposition 2.3. For some applications, a weak measurement is used because an amplification effect can appear. Indeed, choosing $\operatorname{Re}(A_w)$ large allows to get a greater shift of the meter. To achieve this, the pre- and post-selections are typically chosen nearly orthogonal. Indeed, the denominator of the weak value becomes very small in that case. This creates the amplification effect, allowing to see a shift of the meter much larger than the range of eigenvalues of the operator $\hat{A}$. Some tiny effects, usually too small to be detected, can therefore be detected using weak measurements [6, 7, 8].

However, the probability of observing a result is approximately

$$P = |\langle\psi_f|\psi_i\rangle|^2. \tag{2.20}$$

For an amplification experiment, with nearly orthogonal initial and final states, it is a very small number. In order to determine the weak value in an experiment, it is therefore necessary to have enough data to observe the Gaussian distribution $\phi(x - \gamma \operatorname{Re}(A_w))$, so the experiment needs to be reproduced identically a large number of times. The distribution is then averaged

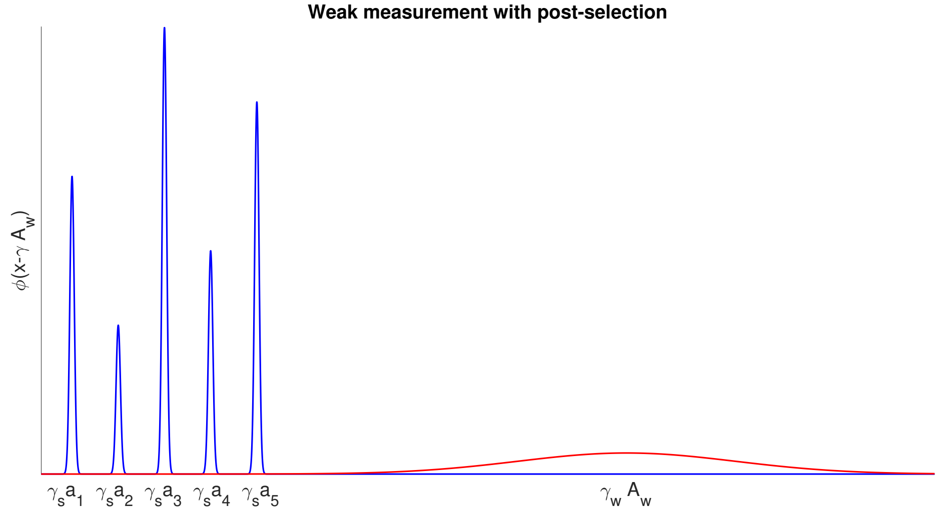


Figure 3: Shift of the Gaussian distribution for a post-selected weak measurement (coupling strength γ_w) (red). It is compared to a strong measurement (coupling strength $\gamma_s \gg \gamma_w$) (blue).

to obtain the weak value. A representation is given in Figure 3. Compared to the range of eigenvalues, the shift in a weak measurement can be much greater. However, the height of the Gaussian is very small and to detect it, the experiment is repeated numerous times.

As said before, for a complex weak value, a shift appears in both the x and p coordinates. To prove this, the following lemma is useful.

Lemma 2.1: Average value of the meter [3, 27]

Let $|\psi_i\rangle$ and $|\psi_f\rangle$ be the pre- and post-selected states of the weak measurement, respectively. The initial meter state is $|\phi\rangle$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p},$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. The post-selected joint state is

$$|\Psi_j\rangle = |\psi_f\rangle \langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}} \left(|\psi_i\rangle \otimes |\phi\rangle\right).$$

For any observable $\hat{M}$ on the meter space, the average value in the post-selected state is

$$\frac{\langle\Psi_j| \hat{\mathbb{1}}_s \otimes \hat{M} |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi| \hat{M} |\phi\rangle + \frac{i}{\hbar}\gamma \operatorname{Re}(A_w) \langle\phi| [\hat{p}, \hat{M}] |\phi\rangle + \frac{1}{\hbar}\gamma \operatorname{Im}(A_w) \langle\phi| \{\hat{p}, \hat{M}\} |\phi\rangle$$
$$- \frac{2}{\hbar}\gamma \operatorname{Im}(A_w) \langle\phi| \hat{M} |\phi\rangle \langle\phi| \hat{p} |\phi\rangle .$$

Proof. The objective is to evaluate the normalised average value $\frac{\langle\Psi_j|\hat{\mathbb{1}}_s\otimes\hat{M}|\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle}$ of any operator $\hat{M}$ acting on the meter Hilbert space. From the proof of Proposition 2.2, the post-selected joint state $|\Psi_j\rangle$, for a weak coupling strength γ, is approximately equal to

$$|\Psi_j\rangle = |\psi_f\rangle \langle\psi_f|\psi_i\rangle \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}\right) |\phi\rangle . \tag{2.21}$$

Therefore, the denominator to the first order in γ is

$$\langle\Psi_j|\Psi_j\rangle = \langle\psi_f|\psi_f\rangle \, |\langle\psi_f|\psi_i\rangle|^2 \langle\phi| \left(1 + \frac{i}{\hbar}\gamma \overline{A}_w \hat{p}\right) \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}\right) |\phi\rangle \tag{2.22}$$

$$= \langle\psi_f|\psi_f\rangle \, |\langle\psi_f|\psi_i\rangle|^2 \left(\langle\phi|\phi\rangle - \frac{i}{\hbar}\gamma A_w \langle\phi|\,\hat{p}\,|\phi\rangle + \frac{i}{\hbar}\gamma \overline{A}_w \langle\phi|\,\hat{p}\,|\phi\rangle\right), \tag{2.23}$$

and the numerator is

$$\langle\Psi_j|\, \hat{\mathbb{1}}_s \otimes \hat{M}\, |\Psi_j\rangle = \langle\psi_f|\psi_f\rangle \, |\langle\psi_f|\psi_i\rangle|^2 \langle\phi| \left(1 + \frac{i}{\hbar}\gamma \overline{A}_w \hat{p}\right) \hat{M} \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}\right) |\phi\rangle \tag{2.24}$$

$$= \langle\psi_f|\psi_f\rangle \, |\langle\psi_f|\psi_i\rangle|^2 \left(\langle\phi|\,\hat{M}\,|\phi\rangle - \frac{i}{\hbar}\gamma A_w \langle\phi|\,\hat{M}\hat{p}\,|\phi\rangle + \frac{i}{\hbar}\gamma \overline{A}_w \langle\phi|\,\hat{p}\hat{M}\,|\phi\rangle\right), \tag{2.25}$$

so that

$$\frac{\langle\Psi_j|\, \hat{\mathbb{1}}_s \otimes \hat{M}\, |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \frac{\langle\phi|\,\hat{M}\,|\phi\rangle - \frac{i}{\hbar}\gamma A_w \langle\phi|\,\hat{M}\hat{p}\,|\phi\rangle + \frac{i}{\hbar}\gamma \overline{A}_w \langle\phi|\,\hat{p}\hat{M}\,|\phi\rangle}{\langle\phi|\phi\rangle - \frac{i}{\hbar}\gamma A_w \langle\phi|\,\hat{p}\,|\phi\rangle + \frac{i}{\hbar}\gamma \overline{A}_w \langle\phi|\,\hat{p}\,|\phi\rangle}. \tag{2.26}$$

We assume that the initial meter state is normalised, $\langle\phi|\phi\rangle = 1$. The approximation $\frac{1}{1-x} \approx 1 + x$ is used to obtain

$$\frac{\langle\Psi_j|\, \hat{\mathbb{1}}_s \otimes \hat{M}\, |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi|\,\hat{M}\,|\phi\rangle - \frac{i}{\hbar}\gamma A_w \langle\phi|\,\hat{M}\hat{p}\,|\phi\rangle + \frac{i}{\hbar}\gamma \overline{A}_w \langle\phi|\,\hat{p}\hat{M}\,|\phi\rangle$$

$$+ \frac{i}{\hbar}\gamma (A_w - \overline{A}_w) \langle\phi|\,\hat{M}\,|\phi\rangle \langle\phi|\,\hat{p}\,|\phi\rangle . \tag{2.27}$$

The weak value is decomposed as $A_w = a + ib$ to get

$$\frac{\langle\Psi_j|\, \hat{\mathbb{1}}_s \otimes \hat{M}\, |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi|\,\hat{M}\,|\phi\rangle + \frac{i}{\hbar}\gamma a \langle\phi|\,[\hat{p}, \hat{M}]\,|\phi\rangle + \frac{1}{\hbar}\gamma b \langle\phi|\,\{\hat{p}, \hat{M}\}\,|\phi\rangle - \frac{2}{\hbar}\gamma b \langle\phi|\,\hat{M}\,|\phi\rangle \langle\phi|\,\hat{p}\,|\phi\rangle . \tag{2.28}$$

Using the definition of the commutator and the anti-commutator gives the result. □

The lemma points out that the average value of the meter observable $\hat{M}$ (such as $\hat{x}$ or $\hat{p}$) after interaction and post-selection is the initial average value shifted by a quantity proportional to both the real and imaginary part of the weak value and to the commutator and anti-commutator of $\hat{M}$ with $\hat{p}$. More particularly, the shift for $\hat{M} = \hat{x}$ and $\hat{M} = \hat{p}$ is studied.

Proposition 2.3: Shifts for a complex weak value [3, 27]

Let $|\psi_i\rangle$ and $|\psi_f\rangle$ be the pre- and post-selected states of the weak measurement, respectively. The initial meter state is $|\phi\rangle$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p},$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. The post-selected joint state is

$$|\Psi_j\rangle = |\psi_f\rangle \langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}} (|\psi_i\rangle \otimes |\phi\rangle) .$$

The average value of the meter observable $\hat{x}$ is

$$\frac{\langle\Psi_j| \hat{\mathbb{1}}_s \otimes \hat{x} |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi| \hat{x} |\phi\rangle + \gamma \operatorname{Re}(A_w) + \frac{m}{\hbar}\gamma \operatorname{Im}(A_w)\frac{d \operatorname{Var}_\phi(\hat{x})}{dt}$$

and the shift for the meter observable $\hat{p}$ is

$$\frac{\langle\Psi_j| \hat{\mathbb{1}}_s \otimes \hat{p} |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi| \hat{p} |\phi\rangle + \frac{2}{\hbar}\gamma \operatorname{Im}(A_w) \operatorname{Var}_\phi(\hat{p}).$$

Proof. First, the shift in $\hat{p}$ is obtained by using Lemma 2.1 directly, writing $A_w = a + ib$,

$$\frac{\langle\Psi_j| \hat{\mathbb{1}}_s \otimes \hat{p} |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi| \hat{p} |\phi\rangle + \frac{2}{\hbar}\gamma b \langle\phi| \hat{p}^2 |\phi\rangle - \frac{2}{\hbar}\gamma b \left(\langle\phi| \hat{p} |\phi\rangle\right)^2 = \langle\phi| \hat{p} |\phi\rangle + \frac{2}{\hbar}\gamma b \operatorname{Var}_\phi(\hat{p}) \quad (2.29)$$

The shift in $\hat{x}$ is obtained in the same way, using the Lemma 2.1. The commutator $[\hat{p}, \hat{x}] = -i\hbar$ appears in the formula, as well as the anti-commutator $\{\hat{p}, \hat{x}\}$. The average value of $\hat{p}$ is also present. To obtain a shift depending only on $\hat{x}$, these two quantities need to be calculated. This is done using the Heisenberg equation of motion to get

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m}, \qquad \frac{d\hat{x}^2}{dt} = \frac{\hat{p}\hat{x} + \hat{x}\hat{p}}{m}, \quad (2.30)$$

so the average value of the position operator is

$$\frac{\langle\Psi_j| \hat{\mathbb{1}}_s \otimes \hat{x} |\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} = \langle\phi| \hat{x} |\phi\rangle + \frac{i}{\hbar}\gamma a \langle\phi| [\hat{p}, \hat{x}] |\phi\rangle + \frac{1}{\hbar}\gamma b \left(\langle\phi| \{\hat{p}, \hat{x}\} |\phi\rangle - 2\langle\phi| \hat{x} |\phi\rangle \langle\phi| \hat{p} |\phi\rangle\right) \quad (2.31)$$

$$= \langle\phi| \hat{x} |\phi\rangle + \gamma a + \frac{1}{\hbar}\gamma b \left(m\frac{d\langle\phi| \hat{x}^2 |\phi\rangle}{dt} - 2m \langle\phi| \hat{x} |\phi\rangle \frac{d\langle\phi| \hat{x} |\phi\rangle}{dt} \right) \quad (2.32)$$

$$= \langle\phi| \hat{x} |\phi\rangle + \gamma a + \frac{m}{\hbar}\gamma b \frac{d \operatorname{Var}_\phi(\hat{x})}{dt}. \quad (2.33)$$

□

The shift in the position of the meter is proportional to both the real and the imaginary parts of the weak value. Moreover, it also depends on the variance of the initial distribution in $\hat{x}$. The shift in the momentum of the meter is proportional to the imaginary part of the weak value as well as to the variance of the distribution of $\hat{p}$.

2.4.1 Conclusion - Importance of weak measurements

Weak measurements have four important advantages:

1. They produce an amplification of the shift of the meter compared to a strong measurement. This allows the detection of very tiny effects, such as the spin Hall effect of light [8] (see example below).

2. They keep the original system state mostly unperturbed. This allows to measure incompatible observables, or physical properties impossible to measure with a strong measurement, such as the trajectories of photons in a two-slit interferometer [10] (see example below).

3. They allow to probe paradoxical effects such as the three-box paradox [12] or the Cheshire cat paradox [11]. This gives insights into the foundations of quantum physics.

4. They are complex, and both the real and imaginary parts can be measured. This allows to perform direct tomography of quantum states [9] and determining their real and imaginary parts effectively by measuring an adequate weak value.

2.4.2 Example: Stern-Gerlach Experiment

The Stern-Gerlach experiment is the original example of weak measurement proposed by Aharonov in [26]. The experiment is the same as in Section 2.3.1, with a weak magnetic field and additional pre-selection and post-selection. The beam is prepared with its spin oriented $|\uparrow_\xi\rangle$ in the direction ξ at an angle α with respect to the x axis (pre-selection). The weak measurement is performed in the z direction by a first magnet and a second one does a strong measurement of the spin in the x direction, $\hat{\sigma}_x$. Only the eigenvalue $+1$ is kept (post-selection). The experiment is represented in Figure 4. The weak value is

$$\sigma_{zw} = \frac{\langle\uparrow_x| \hat{\sigma}_z |\uparrow_\xi\rangle}{\langle\uparrow_x|\uparrow_\xi\rangle} = \tan\frac{\alpha}{2}. \tag{2.34}$$

The coupling is made with the $\hat{z}$ operator of the beam, so a shift of p_z is observed after the post-selection,

$$\delta p_z = \mu \frac{\partial B_z}{\partial z} \tan\frac{\alpha}{2}. \tag{2.35}$$

This translates by a shift of the z coordinate after the free evolution up to the screen.

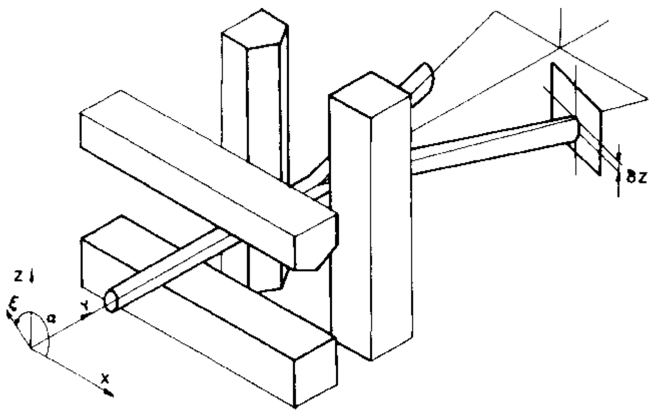


Figure 4: Experimental device for a post-selected weak measurement of the spin along the z direction. [26]

2.4.3 Example: Weakly Coupled Harmonic Oscillators

Consider the previous example of Section 2.3.2. The interaction Hamiltonian is

$$\hat{H}_{int} = aB\hat{y} \otimes \hat{p}_x - aB\hat{p}_y \otimes \hat{x}. \tag{2.36}$$

To the first order in $\gamma = aBT$, the two coupling terms are applied successively and no superposition of the two couplings is taken into account. The first term of the coupling gives the weak value of the operator y and the second term gives the weak value of the operator p_y,

$$y_w = \frac{\langle\psi_f|\,\hat{y}\,|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}, \qquad p_{yw} = \frac{\langle\psi_f|\,\hat{p}_y\,|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{2.37}$$

Consider the initial meter state ϕ as the ground state of the harmonic oscillator in the x direction and its Fourier transform (see Appendix B),

$$\phi(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma_m}} e^{-\frac{x^2}{2\sigma_m^2}}, \qquad \phi(p_x) = \frac{\sqrt{\sigma_m}}{\sqrt{\hbar}\sqrt[4]{\pi}} e^{-\frac{p_x^2\sigma_m^2}{2\hbar^2}}. \tag{2.38}$$

with $\sigma_m^2 = \frac{\hbar}{m\omega_0}$. It is the square root of a Gaussian distribution of deviation $\frac{\hbar}{2\sigma_m}$. From Proposition 2.3, a shift is present in both meter distributions, depending on the two weak values. To the first order, the shift corresponding to the two couplings is the sum of the two individual shifts. Therefore, the shift of the x distribution considering the two weak values is

$$\phi\left(x - \gamma\,\mathrm{Re}(y_w) - \frac{m}{4\hbar}\gamma\,\mathrm{Im}(y_w)\frac{d\sigma_m^2}{dt} + \frac{1}{2\hbar}\gamma\,\mathrm{Im}(p_{yw})\sigma_m^2\right), \tag{2.39}$$

and for the p distribution,

$$\phi\left(p_x - \frac{1}{2\hbar}\gamma\,\mathrm{Im}(y_w)\frac{\hbar^2}{\sigma_m^2} + \gamma\,\mathrm{Re}(p_{yw}) + \frac{m}{2\hbar}\gamma\,\mathrm{Im}(p_{yw})\frac{d}{dt}\frac{\hbar^2}{\sigma_m^2}\right). \tag{2.40}$$

The parameter σ_m is constant, so the two derivatives are equal to zero. To calculate the weak values, assume that the pre-selected state is the ground state ψ_0 of the harmonic oscillator in the y direction,

$$\psi_i(y) = \psi_0(y) = \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma_s}} e^{-\frac{y^2}{2\sigma_s^2}}. \tag{2.41}$$

with $\sigma_s^2 = \frac{\hbar}{Cm\omega_0}$ The post-selected state is taken as a superposition of the ground state and the first excited state of the harmonic oscillator,

$$\psi_f(y) = \alpha\psi_0(y) + (1-\alpha)\psi_1(y), \tag{2.42}$$

depending on a parameter $\alpha \in \mathbb{R}$ (Figure 5). It is not normalised, but the normalisation factor cancel in the weak value so it is not taken into account. The first excited state $\psi_1(y)$ is

$$\psi_1(y) = \frac{1}{\sqrt[4]{\pi}}\sqrt{\frac{2}{\sigma_s}}\frac{y}{\sigma_s} e^{-\frac{y^2}{2\sigma_s^2}}. \tag{2.43}$$

Both states are normalised, $\langle\psi_0|\psi_0\rangle = \langle\psi_1|\psi_1\rangle = 1$ and they are orthogonal to each other, $\langle\psi_1|\psi_0\rangle = 0$. The parameter α controls the orthogonality between the pre-selection and the post-selection,

$$\langle\psi_f|\psi_i\rangle = (\alpha\,\langle\psi_0| + (1-\alpha)\,\langle\psi_1|)\,|\psi_0\rangle = \alpha. \tag{2.44}$$

The weak values y_w and p_{yw} are calculated in Appendix C and are equal to

$$y_w = \frac{(1-\alpha)\sigma_s}{\alpha\sqrt{2}}, \qquad p_{yw} = \frac{i(1-\alpha)\hbar}{\alpha\sqrt{2}\sigma_s}. \tag{2.45}$$

The weak value y_w is real while p_{yw} is purely imaginary. Assuming that σ_s and σ_m are constant in time, the shift of the meter is

$$\phi\left(x - \gamma\frac{(1-\alpha)\sigma_s}{\alpha\sqrt{2}} + \gamma\frac{(1-\alpha)}{2\alpha\sqrt{2}\sigma_s}\sigma_m^2\right) \tag{2.46}$$

and represented in Figure 6. For α close to 1, the distribution is slightly shifted. However, for α close to 0, the distribution is largely shifted. Indeed, a small α gives pre- and post-selected states more orthogonal to each other, leading to the amplification effect of weak values. The p coordinate of the meter (equation 2.40) depends on the imaginary part of y_w and the real part of p_{yw}. They are zero, so the distribution is not shifted. Note that the experimenter can isolate one part of the total shift or the other by selecting the parameters σ_m and σ_s. Indeed, if $\sigma_s \gg \sigma_m$, the first shift is predominant and if $\sigma_s \ll \sigma_m$, then the second shift is measured.

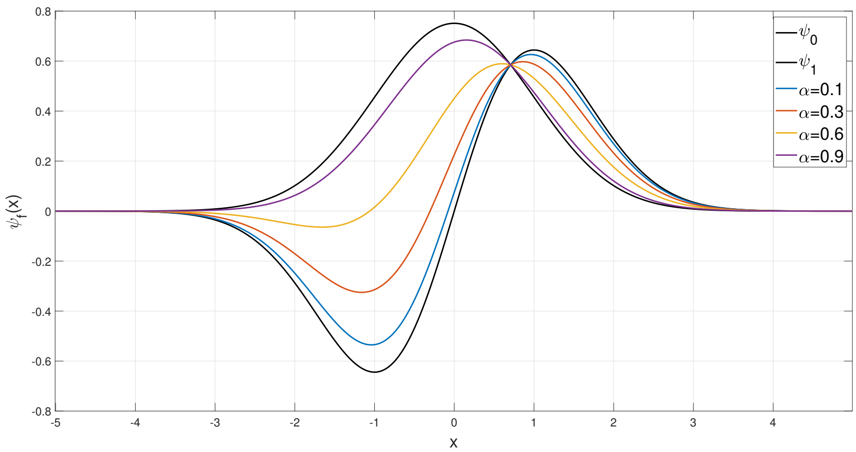


Figure 5: Representation of the final state ψ_f (non normalised) as a function of the parameter α. The parameter varies the proximity to the ground state ψ_0 or the first excited state ψ_1.

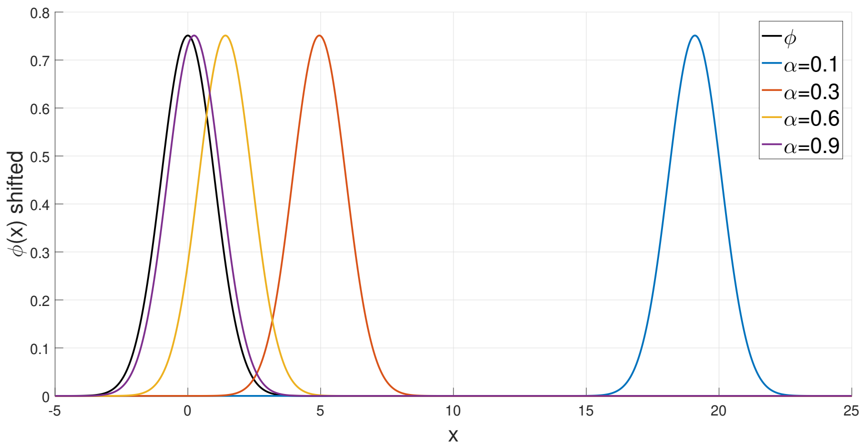


Figure 6: Shift of the meter state for a weak post-selected measurement. Each color is associated to a different parameter α, controlling the orthogonality between the pre- and the post-selection.

2.5 Applications

Weak measurements, more than a theoretical phenomenon, have been observed experimentally. Even if their interpretation is still subject to caution, their existence and usefulness is undeniable. Experimenters are now able to use them at their advantage, and the two following examples illustrate the experimental use of weak measurements.

2.5.1 Spin Hall effect of light

The spin Hall effect of light arises when a beam passes through an interface, typically air-glass for example. The initial beam, assumed to be linearly polarised, is split into two slightly displaced beams. They are respectively left- and right-circularly polarised. The phenomenon originates from the spin-orbit interaction and is illustrated in Figure 7a. The shift is only of a fraction of the wavelength. For a He-Ne laser beam, for example, the shift can be of the order of 10 nm. Experimentally, such a small shift is very complex to detect.

Weak measurements, by their amplification properties, allow to put into evidence the shift [8]. Indeed, the spin Hall effect amounts to a weak measurement of the spin component along the z direction, $\hat{\sigma}_z$. The evolution is given by $e^{-i\delta\hat{\sigma}_z\hat{k}_y}$, with $\hat{k}_y$ the transverse momentum and δ the shift of the beams from the spin Hall effect of light.

The meter is the transverse mode and the system is the polarisation of the light beam. The simplified experimental setup is shown in Figure 7b, with polarisers performing the pre- and post-selections. The pre-selection is the horizontal polarisation state $|H\rangle$, and the post-selection is a nearly vertical state $|V \pm \Delta\rangle$, with Δ the angular distance to the exact vertical state and the perpendicularity of the pre- and post-selection. Expressed in the terms of the basis states $|+\rangle = \frac{|H\rangle + i|V\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|H\rangle - i|V\rangle}{\sqrt{2}}$ of $\hat{\sigma}_z$, it is

$$|\psi_i\rangle = |H\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \tag{2.47}$$

$$|\psi_f\rangle = |V \pm \Delta\rangle = -ie^{\mp i\Delta}|+\rangle + ie^{\pm i\Delta}|-\rangle\,, \tag{2.48}$$

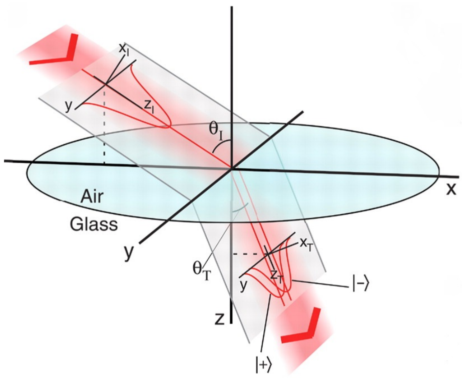
They are nearly orthogonal, to have a great amplification effect. Indeed, the weak value is

$$(\sigma_z)_w = \frac{\langle\psi_f|\,\hat{\sigma}_z\,|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = -\frac{e^{\mp i\Delta} + e^{\pm i\Delta}}{e^{\pm i\Delta} - e^{\mp i\Delta}} = \pm i\cot\Delta. \tag{2.49}$$

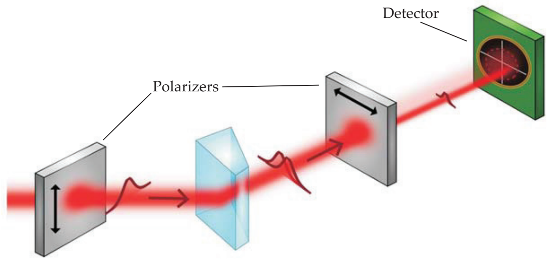
For Δ close to zero, the weak value is big. The shift in the meter is given by $\delta(\sigma_z)_w$ so measuring this shift, big enough to be detected because of the weak value amplification, allows to recover the strength δ of the coupling. The shift coming from the spin Hall effect of light is therefore measured. This experiment increases the effect by an order of 10^4 [8]. The experimental results are quite close to the theoretical expectations.

2.5.2 Trajectories in a two-slit interferometer

The two-slit interferometer is a well-known experiment evidencing the wave propagation of light as well as its particle behaviour. The principle of the experiment is simple. A beam is sent through a plate with two parallel slits. The light goes through and an interference pattern is observed on a screen behind the slits. The interference shows that the light propagates like a wave. However, the experiment can be made photon by photon, detected individually on the screen, evidencing the particle aspect of light. The experiment therefore shows the wave-particle duality of light. Moreover, if one put a sensor at the slit exits, then the interference pattern disappears. Indeed, the measurement projects the state of the photon.



(a) Spin Hall effect at an air-glass interface. The initial beam is splitted into two slightly shifted ones with orthogonal polarisation. [8]



(b) Experimental setup to highlight the spin Hall effect of light [2]. Polarisers perform the pre- and post-selections, and the effect takes place in the prism. [2]

Figure 7: Quantum spin Hall effect of light

However, weak measurements only slightly disturb the initial state, so they could be used to determine the average trajectory of a photon from the slits to the screen [10]. The system is the transverse mode of the beam and the meter is the polarisation. In this way, the weak value of the transverse momentum $\hat{k}_x$ is measured at a given distance from the slits. The average trajectory is then extrapolated, according to the weak value, up to a slightly further distance from the slits where the measurement is performed again. By this process, the average trajectories of the photons can be obtained.

The Hamiltonian of interaction is $\hat{H}_{int} = \gamma \hat{k}_x \hat{S}$ with $\hat{S} = \frac{\hbar}{2}(|H\rangle\langle H| - |V\rangle\langle V|) = \frac{\hbar}{2}\hat{\sigma}_z$ acting on the polarisation. The shift will appear on the phase of the state. The initial system state is the wavefunction itself, $|\psi_i\rangle = |\psi_{path}\rangle$, and the post-selection is performed on $|\psi_f\rangle = |x_f\rangle$, a given position in the experiment. The phase is shifted by an amount proportional to $(k_x)_w$ and this can be measured to reconstruct the trajectories.

The result of the experiment is given in Figure 8. The computed average trajectories are shown. The interference pattern is clearly visible at the end, and it seems that the upper (bottom) part of the photons on the screen only comes from the upper (bottom) slit. This is interesting because the trajectories observed correspond to those expected from the de Broglie-Bohm interpretation of quantum physics. This interpretation gives back a localised position to any quantum particle but the movement of the particle is dictated by a pilot wave. This is a non-local deterministic (but statistical) theory, still subject to a lot of debate.

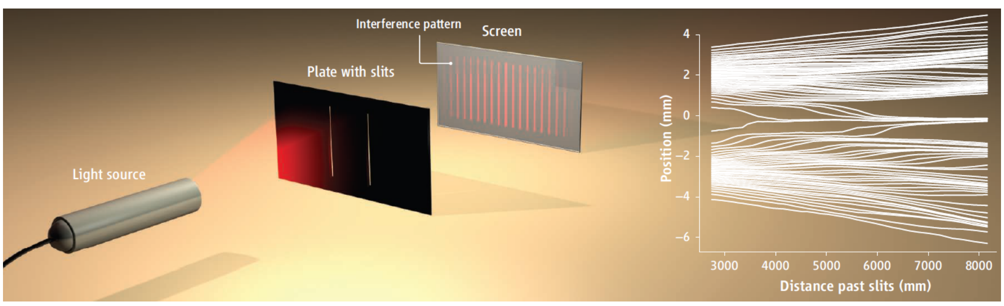


Figure 8: Two-slit interferometer and the trajectories measured using weak measurements. [28]

3 Phase Space Formalism

3.1 Motivation

The Heisenberg and Schrodinger formulations of quantum mechanics are very efficient and are widely used since the beginning of quantum physics to describe the mathematics surrounding the theory. However, physically understanding the meaning of the different objects of the theory is complex. Indeed, the theory describes *wavefunctions* defined in a *Hilbert space*, on which act *Hermitian operators* [23]. These notions are far away from our classical $\mathbb{R}^3$ world.

One of the main problems comes from the uncertainty principle for x and p,

$$\Delta x \Delta p \sim \hbar. \tag{3.1}$$

This relation states that no one can know with infinite precision the two variables simultaneously [23]. This translates in the mathematical formalism, where for a given state $|\psi\rangle$, the x representation $\psi(x)$ or the p representation $\psi(p)$ can be known, but never both simultaneously.

On another side, classical mechanics is generally described in phase space.

Definition 3.1: Phase space [15]

Consider a physical system presenting n degrees of freedom. The associated phase space is a space of dimension n, where each degree of freedom is represented as an axis. Every possible state of the system is a point in the phase space.

For example, chaotic systems, planetary orbits or oscillators are usually represented and studied using phase spaces [15]. Indeed, these examples often only have the position and momentum as degrees of freedom, giving the phase space (x, p).

Quantum mechanics exposes many puzzling behaviours, and is complex to understand. Therefore, having a quantum phase space description would get us closer to the common classical sense. This can bring new insights in the theory. To this aim, physicists developed such a quantum phase space using quasi-probability distributions [29].

The idea lying behind their definition comes from statistical physics and the notion of ensemble averages [19]. A statistical ensemble is defined by a probability density, $P_{cl}(x, p)$, acting on the phase space of the system. It is normalised, $\iint P_{cl}(x, p) = 1$, and always positive. The average value of a property $A_{cl}(x, p)$ is computed as

$$\langle A \rangle = \iint A_{cl}(x, p) P_{cl}(x, p) dx dp. \tag{3.2}$$

The property is evaluated at every point of the phase space, depending on the probability that the system is in the given (x, p) state, to get the average value. We want a similar construction to hold in quantum physics. This means that we have to define some "probability distribution" $P_{quant}(x, p)$ such that, for an observable $\hat{A}$ on the system,

$$\langle A \rangle = \iint A_{quant}(x, p) P_{quant}(x, p) dx dp. \tag{3.3}$$

The factor $A_{quant}(x,p)$ needs to be carefully defined and the distribution as well. The probability distribution describes the system in the phase space.

A great advantage is that it is much easier to compare classical and quantum physics, as well as the transition between the two regimes. However, the quantum behaviour comes out at some point and some properties of the probability distributions have to be relaxed. The passage from quantum observables to classical functions is also not straightforward and different ways of making it exist, giving different yet perfectly valid phase space descriptions of the system. In this work, we will more particularly focus on the most common, the Wigner distribution.

3.2 Wigner distribution and Weyl transform

The Wigner distribution, for a given state $\hat{\rho}$, represents the state in the phase space (x,p).

Definition 3.2: Wigner distribution [16, 17, 18, 19]

For a quantum state $\hat{\rho}$, the Wigner distribution is

$$W(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\middle|\,\hat{\rho}\,\middle|x-\frac{y}{2}\right\rangle dy.$$

For a pure state, the density operator is $\hat{\rho} = |\psi\rangle\langle\psi|$ so the Wigner function becomes

$$W(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)dy. \tag{3.4}$$

On the contrary, a mixed state is written as

$$\hat{\rho} = \sum_j P_j\,|\psi_j\rangle\langle\psi_j|\,, \tag{3.5}$$

with P_j being the probability linked to the pure state $|\psi_j\rangle$. By linearity, the Wigner function is

$$W(x,p) = \sum_j P_j W_j(x,p), \tag{3.6}$$

and is therefore the weighted sum of the Wigner distribution of each of the states $|\psi_j\rangle$ composing the mixed state. An equivalent writing of the distribution depending on the p basis exist,

$$W(x,p) = \frac{1}{h}\int e^{\frac{i}{\hbar}xu}\left\langle p+\frac{u}{2}\middle|\,\hat{\rho}\,\middle|p-\frac{u}{2}\right\rangle du. \tag{3.7}$$

It can also be noted that the wavefunction is found back from the Wigner distribution by

$$\psi(x) = \frac{1}{\psi^*(0)}\int W\left(\frac{x}{2},p\right)e^{\frac{i}{\hbar}px}dp. \tag{3.8}$$

where the prefactor $\frac{1}{\psi^*(0)}$ is determined by the normalisation condition.

We have a description of a quantum state in phase space. To get a complete phase space description of quantum physics, we also need to define the equivalent of an operator in phase space. This is done using the Weyl transform.

Definition 3.3: Weyl Transform [16, 18, 19]

For any operator $\hat{A}$ acting on the Hilbert space, the phase space equivalent $\tilde{A}(x,p)$ is the Weyl transform of $\hat{A}$,

$$\tilde{A}(x,p) = \int e^{-\frac{i}{\hbar}py} \left\langle x + \frac{y}{2} \right| \hat{A} \left| x - \frac{y}{2} \right\rangle dy.$$

The Wigner function can be rewritten using the Weyl transform. Indeed, it is proportional to the Weyl transform of the density operator $\hat{\rho}$,

$$W(x,p) = \frac{\tilde{\rho}}{h}. \tag{3.9}$$

For any operator $\hat{A}$, the average value can be calculated as in statistical physics, with the Wigner distribution $W(x,p)$ acting like a probability distribution and $\tilde{A}(x,p)$ as the representation of the operator $\hat{A}$ in phase space. To show this, the following lemma is necessary.

Lemma 3.1: Trace property of the Weyl transform [16, 17, 18]

For any operators $\hat{A}$ and $\hat{B}$, the following relation holds:

$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \frac{1}{h} \iint \tilde{A}(x,p)\tilde{B}(x,p)dxdp.$$

Proof. It is straightforward to show it using the definition of the Weyl transform. The relation $\frac{1}{h}\int e^{ipy/\hbar}dp = \delta(y)$ is used to go from the first to the second line [1].

$$\frac{1}{h}\iint \tilde{A}(x,p)\tilde{B}(x,p)dxdp = \frac{1}{h}\iiiint e^{-\frac{i}{\hbar}py}e^{-\frac{i}{\hbar}py'} \left\langle x+\frac{y}{2}\right|\hat{A}\left|x-\frac{y}{2}\right\rangle \left\langle x+\frac{y'}{2}\right|\hat{B}\left|x-\frac{y'}{2}\right\rangle dxdpdydy' \tag{3.10}$$

$$= \iiint \delta(y+y') \left\langle x+\frac{y}{2}\right|\hat{A}\left|x-\frac{y}{2}\right\rangle \left\langle x+\frac{y'}{2}\right|\hat{B}\left|x-\frac{y'}{2}\right\rangle dxdydy' \tag{3.11}$$

$$= \iint \left\langle x+\frac{y}{2}\right|\hat{A}\left|x-\frac{y}{2}\right\rangle \left\langle x-\frac{y}{2}\right|\hat{B}\left|x+\frac{y}{2}\right\rangle dxdy \tag{3.12}$$

$$= \iint \langle u+y|\hat{A}|u\rangle \langle u|\hat{B}|u+y\rangle\, dudy \qquad u = x-\frac{y}{2} \tag{3.13}$$

$$= \iint \langle v|\hat{A}|u\rangle \langle u|\hat{B}|v\rangle\, dudv \qquad v = u+y \tag{3.14}$$

$$= \int \langle v|\hat{A}\hat{B}|v\rangle\, dv = \mathrm{Tr}\left(\hat{A}\hat{B}\right). \tag{3.15}$$

□

This lemma is used to compute the average value of any operator $\hat{A}$, from its Weyl transform $\tilde{A}(x,p)$ in phase space.

[1] It will be used in multiple proofs of the thesis

Proposition 3.1: Average value [16, 17, 18]

For any operator $\hat{A}$, the average value of $\hat{A}$ in phase space is

$$\langle A \rangle = \iint W(x,p)\tilde{A}(x,p)dxdp.$$

Proof. From Lemma 3.1 and the writing of the Wigner distribution of the Weyl transform of $\hat{\rho}$ (Equation (3.9)), we find

$$\iint W(x,p)\tilde{A}(x,p)dxdp = \iint \frac{\tilde{\rho}}{h}\tilde{A}(x,p)dxdp = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right) = \langle A \rangle . \tag{3.16}$$

□

As will be detailed later, the Wigner distribution is not the only distribution able to describe the quantum state in phase space. However, it has interesting properties that distinguish it from the other distributions and make it the most studied one in literature. More precisely, it always gives the correct marginals over x and p and it is real. The other distributions do not necessarily have these properties.

Proposition 3.2: Properties of the Wigner distribution [16, 17, 18]

The Wigner distribution

1. gives the correct marginal distributions of x and p,

$$\int W(x,p)dp = \langle x|\, \hat{\rho}\, |x\rangle = |\psi(x)|^2, \qquad \int W(x,p)dx = \langle p|\, \hat{\rho}\, |p\rangle = |\psi(p)|^2;$$

2. is normalised,

$$\iint W(x,p)dxdp = 1;$$

3. is real;

4. can take negative values.

The Wigner distribution is therefore a quasi-probability, differing from a classical probability because of the negative values that it can take.

The proof is given in Appendix D. Each point is proven using the definition of the Wigner distribution.

An important feature of the Wigner distribution is its translation property. The displacement of the quantum state $\psi(x)$ to $\psi(x-b)$ changes $W(x,p)$ to $W(x-b,p)$ in phase space, and equivalently for the momentum coordinate.

Lastly, an equivalent of the Schrödinger equation is defined. It gives the time evolution of the Wigner distribution in phase space.

Proposition 3.3: Time evolution in phase space [16, 17, 19, 30]

The evolution of the Wigner distribution in phase space is given by the differential equation

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W(x,p)}{\partial x} + \sum_{s=0}^{\infty}(-\hbar^2)^s \frac{1}{(2s+1)!}\left(\frac{1}{2}\right)^{2s} \frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}} \left(\frac{\partial}{\partial p}\right)^{2s+1} W(x,p),$$

for a potential $U(x)$ acting on the system.

This equation describes how the Wigner distribution evolves in time in the phase space. This is a quite complex differential equation, it is therefore often more practical to study the evolution using the Schrödinger equation and then compute the Wigner distribution of the resulting state.

3.2.1 Example: Harmonic Oscillator

Consider a harmonic oscillator. The ground state and the first excited states are

$$\psi_0(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-\frac{x^2}{2\sigma^2}}, \qquad \psi_1(x) = \frac{1}{\sqrt[4]{\pi}}\sqrt{\frac{2}{\sigma}}\frac{x}{\sigma} e^{-\frac{x^2}{2\sigma^2}}, \tag{3.17}$$

with $\sigma = \frac{\hbar}{m\omega}$. The Wigner distribution of the ground state is, using the Appendix A,

$$W_0(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py} \left\langle x+\frac{y}{2}\middle|\psi_0\right\rangle \left\langle \psi_0\middle|x-\frac{y}{2}\right\rangle dy \tag{3.18}$$

$$= \frac{1}{h}\int e^{-\frac{i}{\hbar}py} \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-\frac{(x+\frac{y}{2})^2}{2\sigma^2}} \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-\frac{(x-\frac{y}{2})^2}{2\sigma^2}} dy \tag{3.19}$$

$$= \frac{1}{h\sigma\sqrt{\pi}}\int e^{-\frac{i}{\hbar}py} e^{-\frac{x^2+\frac{y^2}{4}}{\sigma^2}} dy = \frac{1}{h\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} \int e^{-\frac{i}{\hbar}py} e^{-\frac{y^2}{4\sigma^2}} dy \tag{3.20}$$

$$= \frac{1}{h\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} 2\sigma\sqrt{\pi} e^{-\frac{p^2 4\sigma^2}{4\hbar^2}} = \frac{2}{h} e^{-\frac{x^2}{\sigma^2}} e^{-\frac{p^2\sigma^2}{\hbar^2}}. \tag{3.21}$$

The Wigner distribution of the first excited state can be calculated in the same way and is

$$W_1(x,p) = \frac{2}{h}\left(-1 + 2\frac{\sigma^2 p^2}{\hbar^2} + 2\frac{x^2}{\sigma^2}\right) e^{-\frac{x^2}{\sigma^2}} e^{-\frac{p^2\sigma^2}{\hbar^2}}. \tag{3.22}$$

The two distributions are represented in the upper part of Figure 9. The ground state of the harmonic oscillator, on the left, is of minimal uncertainty. It is therefore the "most classical" state we can have, called a coherent state. The first excited state, on the right, is not a coherent state. There is clearly some quantum behaviour visible from the negative part.

The two lower figures are pure and mixed states constructed from two displaced ground states of the harmonic oscillator. For the mixed state $W_{mixed}(x,p)$, the Wigner distribution of the two shifted ground states of the harmonic oscillator are summed,

$$W_{mixed}(x,p) = \frac{1}{2}\left(W_0(x-a,p) + W_0(x+a,p)\right), \tag{3.23}$$

with a the shift considered. The resulting state is mixed. Indeed, it is the Wigner distribution of the sum of two density operators, resulting in a mixed state in most cases. The two parts

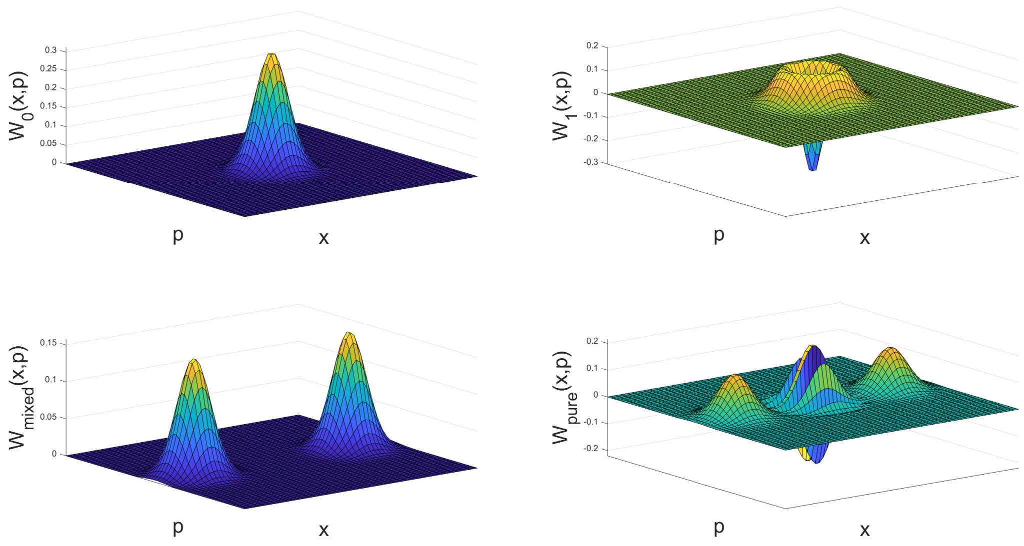


Figure 9: Representation of Wigner functions. On the up left is the Wigner function of the ground state of the harmonic oscillator. On the up right is the Wigner function of the first excited state of the harmonic oscillator. On the bottom left is the Wigner function of a mixed state (a classical statistical distribution of the two states) and the one from a pure state is given on the bottom right. The pure state corresponds to the superposition of the two parts of the mixed state.

visible correspond to two coherent states, constructed from a shift of the initial ground state. They are both of minimal uncertainty. They evolve during time by describing an ellipse in phase space. This corresponds to the movement of a classical harmonic oscillator with the minimal quantum uncertainty.

In the pure case $W_{pure}(x,p)$, the two shifted wavefunctions $\psi_0(x-a)$ and $\psi_0(x+a)$ are added, normalised and the associated Wigner function is shown,

$$W_{pure}(x,p) = W_{\frac{1}{N}(\psi_0(x-a)+\psi_0(x+a))}(x,p). \tag{3.24}$$

The total wavefunction is pure. The resulting Wigner distribution is very different from the mixed one and exhibits non-classical behaviour, it is not a coherent state. It is clearly visible in the middle of the distribution, resulting from the coupling between the two states. The Wigner distribution takes negative values and this feature is essential in describing quantum superposition [20]. Indeed, in a classical situation, the probability distribution is always positive, like for the mixed state. However, this example shows that the Wigner distribution is a quasi-probability and the negative values come from the quantum behaviour.

3.3 Other phase space distributions

In the previous sections, were introduced the Wigner distribution and the Weyl transform. Other definitions of the phase space distribution and the operator correspondence exist. This non-uniqueness is closely related to what is called the ordering problem [17, 18, 19, 30]. This problem appears when considering the way to link classical functions to quantum observables (or quantum observables to classical functions).

For example, the quantum observable $\hat{x}$ is simply given by x in phase space. It is the same for $\hat{p}$, that is given by p. For any observable $\hat{A}(\hat{x},\hat{p})$ linear in $\hat{x}$ and $\hat{p}$, the corresponding classical function is directly $A(x,p)$ by "removing the hat". However, for more complex situations, the ordering problem appears. If we have $\hat{A}(\hat{x},\hat{p}) = \hat{x}\hat{p}$, the instinctive mapping is $A(x,p) = xp$ (standard ordering) [17, 18]. However, we can write

$$\hat{A}(\hat{x},\hat{p}) = \hat{x}\hat{p} = \hat{p}\hat{x} - [\hat{p},\hat{x}] = \hat{p}\hat{x} + i\hbar. \tag{3.25}$$

By replacing the operators by phase space variables, we get $A(x,p) = px + i\hbar$ (anti-standard ordering). A third way to write it is

$$\hat{A}(\hat{x},\hat{p}) = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{1}{2}i\hbar, \tag{3.26}$$

that gives a third different mapping to phase space, $A(x,p) = xp + \frac{1}{2}i\hbar$. It is called symmetric ordering. We see that the mappings are different, because the operators do not commute (unlike their classical equivalents in classical physics). This is called the ordering problem. The ordering chosen is important to determine the phase space distribution used. To understand this, we evaluate the Weyl transform of the operator $\hat{A} = \hat{x}\hat{p}$ [17]. Using that $\hat{p} = \int p' \left|p'\right\rangle \left\langle p'\right|$ and $\langle x|p\rangle = \frac{1}{\sqrt{\hbar}} e^{\frac{i}{\hbar}xp}$, we get

$$\begin{aligned}
\tilde{A}(x,p) &= \int e^{-\frac{i}{\hbar}py} \left\langle x + \frac{y}{2}\right| \hat{x}\hat{p} \left|x - \frac{y}{2}\right\rangle dy && (3.27)\\
&= \int e^{-\frac{i}{\hbar}py} \left(x + \frac{y}{2}\right) \left\langle x + \frac{y}{2}\right| \hat{p} \left|x - \frac{y}{2}\right\rangle dy && (3.28)\\
&= \iint e^{-\frac{i}{\hbar}py} \left(x + \frac{y}{2}\right) p' \left\langle x + \frac{y}{2}\middle|p'\right\rangle \left\langle p'\middle|x - \frac{y}{2}\right\rangle dydp' && (3.29)\\
&= \frac{1}{h} \iint e^{-\frac{i}{\hbar}py} \left(x + \frac{y}{2}\right) p' e^{\frac{i}{\hbar}(x+\frac{y}{2})p'} e^{-\frac{i}{\hbar}(x-\frac{y}{2})p'} dydp' && (3.30)\\
&= \frac{1}{h} \iint p' e^{-\frac{i}{\hbar}(p-p')y} \left(x + \frac{y}{2}\right) dydp' && (3.31)\\
&= \frac{x}{h} \iint p' e^{-\frac{i}{\hbar}(p-p')y} dydp' + \frac{1}{2h} \iint yp' e^{-\frac{i}{\hbar}(p-p')y} dydp' && (3.32)\\
&= x \int p'\delta(p-p')dp' + \frac{i}{4\pi} \iint p' \frac{\partial}{\partial p}\left(e^{-\frac{i}{\hbar}(p-p')y}\right) dydp' && (3.33)\\
&= xp + \frac{i}{4\pi}\frac{\partial}{\partial p} \iint p' e^{-\frac{i}{\hbar}(p-p')y} dydp' = xp + \frac{i\hbar}{2}\frac{\partial}{\partial p} \int p'\delta(p-p')dp' && (3.34)\\
&= xp + \frac{i\hbar}{2}\frac{\partial p}{\partial p} = xp + \frac{i\hbar}{2}. && (3.35)
\end{aligned}$$

We obtain a correspondence with the symmetric ordering. The Weyl transform, and consequently the Wigner distribution, correspond to this ordering. More formally, let's take an arbitrary classical function $A(x,p)$ [19]. Its Fourier transform $\alpha(\sigma,\tau)$ on phase space is

$$A(x,p) = \iint \alpha(\sigma,\tau) e^{\frac{i}{\hbar}(\sigma x + \tau p)} d\sigma d\tau. \tag{3.36}$$

We want to find its operator equivalent. We can choose

$$\hat{A}(\hat{x},\hat{p}) = \iint \alpha(\sigma,\tau) e^{\frac{i}{\hbar}(\sigma\hat{x} + \tau\hat{p})} d\sigma d\tau. \tag{3.37}$$

This choice leads to the Wigner distribution. The Proposition 3.1 for $e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})}$ is rewritten in the following way, replacing the Wigner distribution by an unknown distribution $P(x,p)$,

$$\mathrm{Tr}\Big(\hat{\rho}e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})}\Big) = \iint P(x,p)e^{\frac{i}{\hbar}(\sigma x+\tau p)}dxdp. \tag{3.38}$$

The evaluation of the average value of the exponential operator is important to know the average value of the operator $\hat{A}$. Moreover, this operator defines displacement operators from which are constructed coherent states by translations in phase space. We will show that the expression works only if $P(x,p) = W(x,p)$. The exponential of commutators is evaluated using the BCH (Baker-Campbell-Hausdorff) formula,

$$e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})} = e^{\frac{i}{\hbar}\sigma\hat{x}}e^{\frac{i}{\hbar}\tau\hat{p}}e^{\frac{i}{2\hbar}\sigma\tau}. \tag{3.39}$$

Therefore, spotting the translation operator $e^{\frac{i}{\hbar}\tau\hat{p}}\left|\xi\right\rangle = \left|\xi-\tau\right\rangle$, we get

$$\begin{aligned}
\mathrm{Tr}\Big(\hat{\rho}e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})}\Big) &= e^{\frac{i}{2\hbar}\sigma\tau}\,\mathrm{Tr}\Big(\hat{\rho}e^{\frac{i}{\hbar}\sigma\hat{x}}e^{\frac{i}{\hbar}\tau\hat{p}}\Big) = e^{\frac{i}{2\hbar}\sigma\tau}\int \left\langle\xi\right|\hat{\rho}e^{\frac{i}{\hbar}\sigma\hat{x}}e^{\frac{i}{\hbar}\tau\hat{p}}\left|\xi\right\rangle d\xi && (3.40)\\
&= e^{\frac{i}{2\hbar}\sigma\tau}\int \left\langle\xi\right|\hat{\rho}e^{\frac{i}{\hbar}\sigma(\xi-\tau)}\left|\xi-\tau\right\rangle d\xi = e^{-\frac{i}{2\hbar}\sigma\tau}\int e^{\frac{i}{\hbar}\sigma\xi}\left\langle\xi\right|\hat{\rho}\left|\xi-\tau\right\rangle d\xi && (3.41)\\
&= \int e^{\frac{i}{\hbar}\sigma x}\left\langle x+\frac{\tau}{2}\right|\hat{\rho}\left|x-\frac{\tau}{2}\right\rangle dx \qquad x=\xi-\frac{\tau}{2} && (3.42)\\
&= \iint e^{\frac{i}{\hbar}\sigma x}\delta(y-\tau)\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle dxdy && (3.43)\\
&= \frac{1}{h}\iiint e^{\frac{i}{\hbar}p(\tau-y)}e^{\frac{i}{\hbar}\sigma x}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle dxdpdy && (3.44)\\
&= \iint W(x,p)e^{\frac{i}{\hbar}(\sigma x+\tau p)}dxdp, && (3.45)
\end{aligned}$$

by setting

$$W(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle dy \tag{3.46}$$

as the Wigner function. So if we identify the operator $e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})}$ to the classical function $e^{\frac{i}{\hbar}(\sigma x+\tau p)}$, the Wigner distribution has to be used to compute the expectation value of any observable. For different identifications of the exponential, we obtain other distributions with different properties (Table 1). A more general result gives the distribution depending on a function $f(\sigma,\tau)$ [30],

$$\mathrm{Tr}\Big(\hat{\rho}e^{\frac{i}{\hbar}(\sigma\hat{x}+\tau\hat{p})}f(\sigma,\tau)\Big) = \iint P^f(x,p)e^{\frac{i}{\hbar}(\sigma x+\tau p)}dxdp, \tag{3.47}$$

with

$$P^f(x,p) = \frac{1}{h^2}\iiint \left\langle x'+\frac{\tau}{2}\right|\hat{\rho}\left|x'-\frac{\tau}{2}\right\rangle f(\sigma,\tau)e^{\frac{i}{\hbar}\sigma(x'-x)}e^{-\frac{i}{\hbar}\tau p}d\sigma d\tau dx'. \tag{3.48}$$

If we choose $f(\sigma,\tau) = 1$, we get back the Wigner distribution. Other choices lead to other distributions, summarised in Table 1. Each one is specific in the properties it exhibits (Table 2). For example, the standard-ordered and the anti-standard (Kirkwood) distributions are not real, contrary to the Wigner distribution. The Husimi distribution is always real and positive, an advantage compared to the Wigner distribution, but doesn't give the proper marginal distributions.

Distribution functions	Rules of association	f
Wigner	Weyl ($e^{i\xi q+i\eta p} \leftrightarrow e^{i\xi\hat{q}+i\eta\hat{p}}$)	1
standard-ordered	standard ($e^{i\xi q+i\eta p} \leftrightarrow e^{i\xi\hat{q}}e^{i\eta\hat{p}}$)	$e^{-i\hbar\xi\eta/2}$
antistandard-ordered (Kirkwood)	antistandard ($e^{i\xi q+i\eta p} \leftrightarrow e^{i\eta\hat{p}}e^{i\xi\hat{q}}$)	$e^{i\hbar\xi\eta/2}$
normal-ordered (Glauber-Sudarshan)	normal	$e^{\frac{\hbar\xi^2}{4m\omega}+\frac{\hbar m\omega\eta^2}{4}}$
antinormal-ordered	antinormal	$e^{-\frac{\hbar\xi^2}{4m\omega}-\frac{\hbar m\omega\eta^2}{4}}$
generalised antinormal-ordered (Husimi)	generalised antinormal	$e^{-\frac{\hbar\xi^2}{4m\kappa}-\frac{\hbar m\kappa\eta^2}{4}}$

Table 1: Quasi-probability distributions, rules of association and f functions. [30]

Distribution function	Properties				
	Bilinear	Real	Nonnegative	Marginal distributions	Complete, Orthonormal
Wigner	yes	yes	no	yes	yes
standard-ordered	yes	no	no	yes	yes
antistandard-ordered	yes	no	no	yes	yes
normal-ordered	yes	yes	no	no	no
antinormal-ordered	yes	yes	yes	no	no
Husimi	yes	yes	yes	no	no

Table 2: Properties of the different quasi-probability distributions. [30]

3.4 Cross-Wigner distribution

The Weyl transform of an operator of the form $\hat{\rho}_{\psi,\phi} = |\psi\rangle\langle\phi|$ is called the cross-Wigner distribution between the two pure states. It is useful when describing interferences between the states.

Definition 3.4: Cross-Wigner Distribution [31, 32]

For two pure states $|\psi\rangle$ and $|\phi\rangle$, the cross-Wigner distribution is

$$W_{\psi,\phi}(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy.$$

The cross-Wigner distribution appears when computing the Wigner distribution of the sum of two wavefunctions ψ and ϕ. Starting from now, we will specify by a subscript the wavefunction to which the Wigner function is linked.

Proposition 3.4: Wigner distribution of a sum of pure states [31, 32]

The Wigner distribution of $\frac{1}{\sqrt{N}}(|\psi\rangle + |\phi\rangle)$ is

$$W_{\psi+\phi}(x,p) = \frac{1}{N}W_\psi(x,p) + \frac{1}{N}W_\phi(x,p) + \frac{2}{N}\operatorname{Re}(W_{\psi,\phi}(x,p)).$$

with $N = 2 + \langle\psi|\phi\rangle + \langle\phi|\psi\rangle$ a normalisation factor.

Proof.

$$
\begin{aligned}
W_{\psi+\phi}(x,p) &= \frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\right|(|\psi\rangle+|\phi\rangle)(\langle\psi|+\langle\phi|)\left|x-\frac{y}{2}\right\rangle dy && (3.49)\\
&= \frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\left(\psi\left(x+\frac{y}{2}\right)+\phi\left(x+\frac{y}{2}\right)\right)\left(\psi^*\left(x-\frac{y}{2}\right)+\phi^*\left(x-\frac{y}{2}\right)\right)dy && (3.50)\\
&= \frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\left(\psi\left(x+\frac{y}{2}\right)+\phi\left(x+\frac{y}{2}\right)\right)\left(\psi^*\left(x-\frac{y}{2}\right)+\phi^*\left(x-\frac{y}{2}\right)\right)dy && (3.51)\\
&= \frac{1}{N}W_\psi(x,p)+\frac{1}{N}W_\phi(x,p)+\frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy\\
&\quad+\frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\phi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)dy && (3.52)\\
&= \frac{1}{N}W_\psi(x,p)+\frac{1}{N}W_\phi(x,p)+\frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy\\
&\quad+\frac{1}{Nh}\int e^{\frac{i}{\hbar}py}\phi\left(x-\frac{y}{2}\right)\psi^*\left(x+\frac{y}{2}\right)dy \qquad (y\to -y) && (3.53)\\
&= \frac{1}{N}W_\psi(x,p)+\frac{1}{N}W_\phi(x,p)+\frac{1}{Nh}\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy\\
&\quad+\frac{1}{Nh}\left(\int e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy\right)^* && (3.54)\\
&= \frac{1}{N}W_\psi(x,p)+\frac{1}{N}W_\phi(x,p)+\frac{2}{N}\mathrm{Re}(W_{\psi,\phi}(x,p)). && (3.55)
\end{aligned}
$$

□

The cross-Wigner distribution encodes an interference between the two states. In the example of Figure 9 on the bottom left, the interference pattern appearing in the center is due to the cross-Wigner distribution between the two Gaussian states. If $\psi = \phi$, we get back the usual Wigner function. The cross-Wigner distribution is however not normalised over the whole phase space.

Proposition 3.5: Norm of the cross-Wigner distribution [31, 32]

The norm of the cross-Wigner distribution is the scalar product between the two states,

$$\iint W_{\psi,\phi}(x,p)dpdx = \langle\phi|\psi\rangle\,.$$

Proof.

$$
\begin{aligned}
\iint W_{\psi,\phi}(x,p)dpdx &= \frac{1}{h}\iiint e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dydpdx && (3.56)\\
&= \iint \delta(y)\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dydx && (3.57)\\
&= \int \psi(x)\phi^*(x)dx = \langle\phi|\psi\rangle\,. && (3.58)
\end{aligned}
$$

□

If $|\psi\rangle = |\phi\rangle$, the norm equals to one as wanted for the Wigner function. Another interesting property is that the cross-Wigner distribution corresponds to the Weyl transform of the operator $\hat{\rho}_{\psi,\phi} = |\psi\rangle\langle\phi|$,

$$\tilde{\rho}_{\psi,\phi}(x,p) = \int e^{-\frac{i}{\hbar}py} \left\langle x + \frac{y}{2} \middle| \psi \right\rangle \left\langle \phi \middle| x - \frac{y}{2} \right\rangle dy = \int e^{-\frac{i}{\hbar}py} \psi\left(x + \frac{y}{2}\right) \phi^*\left(x - \frac{y}{2}\right) dy = hW_{\psi,\phi}(x,p). \tag{3.59}$$

Contrary to the Wigner distribution, the cross-Wigner distribution is not real, it can take complex values. It is interesting to note that the cross-Wigner distribution is also used in the classical theory of signal processing, as it is the Fourier transform of the ambiguity function.

3.4.1 Example: Harmonic Oscillator

Consider the ground state of the harmonic oscillator, $\psi_0(x)$. In the example of Section 3.2.1, the Wigner distribution of the sum of two shifted wavefunctions, $\psi_0(x-a)$ and $\psi_0(x+a)$, is evaluated. The representation of this state exhibits non-classical behaviour. This comes from the cross-Wigner distribution of the two states, following Proposition 3.4. To show it, we calculate the cross-Wigner distribution,

$$\begin{aligned}
W_{\psi_0(x-a),\psi_0(x+a)}(x,p) &= \frac{1}{h}\int e^{-\frac{i}{\hbar}py} \psi_0\left(x - a + \frac{y}{2}\right) \psi_0\left(x + a - \frac{y}{2}\right) dy && (3.60)\\
&= \frac{1}{h\sigma\sqrt{\pi}} \int e^{-\frac{i}{\hbar}py} e^{-\frac{(x-a+\frac{y}{2})^2}{2\sigma^2}} e^{-\frac{(x+a-\frac{y}{2})^2}{2\sigma^2}} dy && (3.61)\\
&= \frac{2}{h\sigma\sqrt{\pi}} e^{-\frac{i}{\hbar}2pa} \int e^{-\frac{i}{\hbar}2py'} e^{-\frac{(x+y')^2}{2\sigma^2}} e^{-\frac{(x-y')^2}{2\sigma^2}} dy' && (3.62)\\
&= \frac{2}{h\sigma\sqrt{\pi}} e^{-\frac{i}{\hbar}2pa} e^{-\frac{x^2}{\sigma^2}} \int e^{-\frac{i}{\hbar}2py'} e^{-\frac{y'^2}{\sigma^2}} dy' && (3.63)\\
&= \frac{2}{h\sigma\sqrt{\pi}} e^{-\frac{i}{\hbar}2pa} e^{-\frac{x^2}{\sigma^2}} \sigma\sqrt{\pi} e^{-\frac{p^2\sigma^2}{\hbar^2}} && (3.64)\\
&= \frac{2}{h} e^{-\frac{i}{\hbar}2pa} e^{-\frac{x^2}{\sigma^2}} e^{-\frac{p^2\sigma^2}{\hbar^2}}. && (3.65)
\end{aligned}$$

The real part of the distribution is

$$\mathrm{Re}\left(W_{\psi_0(x-a),\psi_0(x+a)}(x,p)\right) = \frac{2}{h}\cos\left(\frac{2pa}{\hbar}\right) e^{-\frac{x^2}{\sigma^2}} e^{-\frac{p^2\sigma^2}{\hbar^2}}. \tag{3.66}$$

It is represented in Figure 10. It is clear that it is exactly the middle of Figure 9, so the cross-Wigner distribution is the origin of the interference visible in Figure 9.

4 Wigner Distribution on Curved Space

4.1 Motivation

Until now, the phase space introduced was the (x,p) space, for example in the situation of a 1D harmonic oscillator (Section 3.2.1). The quasi-probability distribution can easily be extended to $\mathbb{R}^n \times \mathbb{R}^n$ for a motion in an n-dimensional continuous space, with coordinates $(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n)$. However, other scenarios might include different kinds of phase spaces. For example, the phase space description of the spin cannot be defined using the

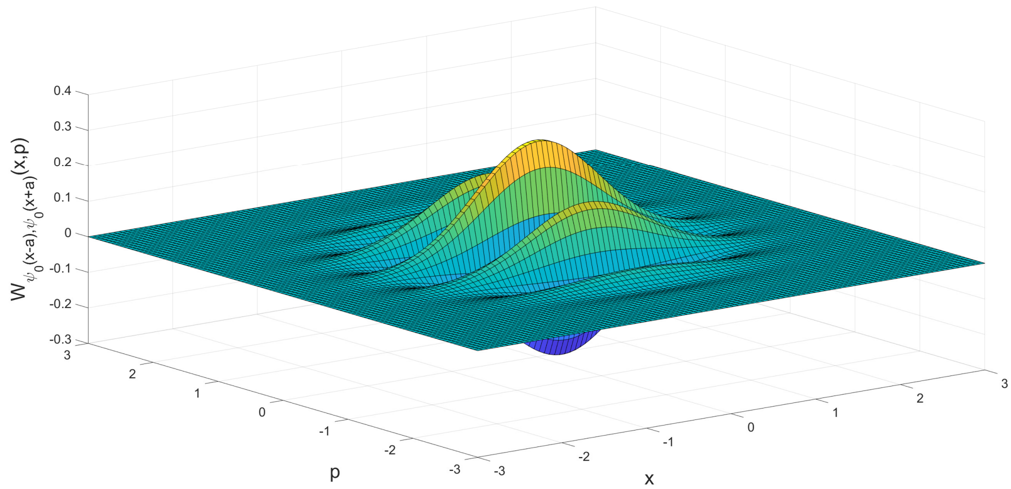


Figure 10: Cross-Wigner distribution of two shifted ground states of the harmonic oscillator.

Wigner distribution, in the way they were defined at least. Moreover, the spin degree of freedom is a discrete system and the phase space description can thus be discrete as well[2], in opposition to the continuous variables considered until now [33, 34, 35].

A generalisation of the notion of phase space is therefore needed and is defined using the concept of Lie groups [36, 37, 38]. In the context of this master's thesis, we will apply this generalisation process to curved configuration spaces [39]. Indeed, such spaces can appear when considering, for example, constrained experiments. An example is the motion of a particle on the surface of a sphere. It could describe the rotation or vibration of a molecule [40], or be useful in the framework of transformation optics.

4.2 Lie algebra and Lie groups

The generalised phase space is constructed using Lie algebras and Lie groups.

Definition 4.1: Lie algebra [36, 37, 38]

A Lie algebra $\mathfrak{g}$ is a finite-dimensional vector space on a field[a] $\mathbb{K}$ defined with a Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\forall x, y, z \in \mathfrak{g}, \forall a, b \in \mathbb{K}$,

1. The Lie bracket is bilinear,
$$[ax + by, z] = a[x, z] + b[y, z] \qquad [z, ax + by] = a[z, x] + b[z, y].$$
2. The Lie bracket is skew symmetric,
$$[x, y] = -[y, x].$$
3. The Lie bracket respects the Jacobi identity,
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

[a] for example, $\mathbb{C}$ or $\mathbb{R}$.

[2] It is not studied here because of a lack of time but the concepts presented can be applied to that case.

The skew symmetry implies that $[x, x] = 0, \forall x \in \mathfrak{g}$. If for two elements x and y of $\mathfrak{g}$, $[x, y] = 0$, then the two elements are said to commute. An interesting example is the Heisenberg-Weyl algebra used in quantum physics [36, 37, 38]. It is a Lie algebra generated by the commutator of $\hat{x}$ and $\hat{p}$ as the Lie bracket, together with the identity,

$$[i\hat{x}, -i\hat{p}] = i\hbar\hat{I}, \qquad [i\hat{x}, i\hbar\hat{I}] = 0, \qquad [-i\hat{p}, i\hbar\hat{I}] = 0. \tag{4.1}$$

This Lie algebra is noted $\mathfrak{h}$. Any element of the algebra is given as a combination of these three elements,

$$g = ia\hbar\hat{I} + ip_0\hat{x} - ix_0\hat{p} \in \mathfrak{g}. \tag{4.2}$$

The concept of Lie algebra is closely related to Lie groups.

Definition 4.2: Lie group [36, 37, 38]

A Lie group G is a smooth manifold such that

1. G, equipped with a multiplication operation $\bullet$, is a group (the multiplication is associative, has an identity and an inverse element).
2. The product operation $\bullet : G \times G \to G$ and the map sending an element to its inverse $\mathrm{Inv} : G \to G$ are smooth.

Any Lie group is associated to a Lie algebra. For a Lie algebra $\mathfrak{g}$, the associated matrix Lie group G is given by the exponential map $\exp : \mathfrak{g} \to G$. This map is defined for any element $g \in \mathfrak{g}$ and for any $t \in \mathbb{R}$ as $\exp(tg)$.

The example of the Heisenberg-Weyl algebra $\mathfrak{h}$ therefore translates to the Heisenberg-Weyl group H, through the exponential map [36, 37, 38]. For any $g \in \mathfrak{h}$, an element of the group is given by

$$e^g = e^{ia\hat{I} + ip_0\hat{x}/\hbar - ix_0\hat{p}/\hbar} = e^{ia}e^{i(p_0\hat{x} - x_0\hat{p})/\hbar} = e^{ia}\hat{D}(x_0, p_0), \tag{4.3}$$

with $\hat{D}(x_0, p_0)$ the displacement operator. This operator is generally used in quantum physics to apply a translation to a given state. Applied on the ground state of the harmonic oscillator, it gives a coherent state. Any element of the Heisenberg-Weyl group is therefore described by the three parameters a, x_0 and p_0 and thus $H \approx \mathbb{R} \times \mathbb{R}^2$. The exponential e^{ia} encodes the information about the non-commutativity (BCH formula). We can see that the exponential of equation (3.38) reappears, motivating the interest to study it.

For position-momentum states, the Hilbert space is the set of square-integrable[3] functions L^2. The set of displacement operators is an irreducible representation[4] of the Heisenberg-Weyl group [36, 37, 38] because the only subspaces invariant under the application of the displacement operators are $\{0\}$ and L^2. Moreover, from the Stone-von Neumann theorem, it is the unique irreducible representation of H up to an unitary transformation [36, 37, 38].

4.3 Generalisation of quasi-probability distributions

The phase space is generalised for any Lie group G. This gives a mathematical and abstract phase space, that reduces to (x, p) in some cases.

[3] A function $f(x)$ is square integrable if $\int |f(x)|^2 dx$ converges.

Definition 4.3: Generalised phase space [41]

For a Lie group G of finite dimension and an isotropy subgroup K of G, the generalised phase space X is the quotient[a] group $X = G/K$ $(= \{gK : g \in G\})$.

[a]The quotient group is the equivalence class of all the elements of G equal up to the action of K.

Consider an irreducible representation[4] T of G. For any $g \in G$ and for a given reference state $|\psi_0\rangle$, it acts as

$$|\psi_g\rangle = T(g)\,|\psi_0\rangle\,. \tag{4.4}$$

The resulting state $|\psi_g\rangle$ is interpreted as a coherent state. The isotropy subgroup K is the set of all the elements k in G that leave any state $|\psi_0\rangle$ unchanged up to a phase factor,

$$T(k)\,|\psi_0\rangle = e^{i\phi(k)}\,|\psi_0\rangle\,. \tag{4.5}$$

The subgroup K therefore consists of all the global phase transformations. This is why G/K is used as the phase space, omitting the information about the global phase. From the definition of a quotient space, for any element $\Omega \in X$ we have $g = \Omega k$ with $k \in K$. Therefore,

$$T(g)\,|\psi_0\rangle = T(\Omega)T(k)\,|\psi_0\rangle = e^{i\phi(k)}T(\Omega)\,|\psi_0\rangle = e^{i\phi(k)}\,|\psi_\Omega\rangle\,. \tag{4.6}$$

The coherent state is uniquely defined with Ω, and two different elements of G with the same Ω will give the same coherent state, up to a global phase.

To illustrate these concepts, let's get back to the Heisenberg-Weyl group H. For any element of the group, described by the three parameters a, x_0, p_0, the irreducible representation is the displacement operator $e^{ia}\hat{D}(x_0, p_0)$. The isotropy subgroup is the multiplication by a phase factor, so it is all the elements in the group such that $x_0 = 0$ and $p_0 = 0$. These elements only apply a global phase, given by a. Therefore, the coherent states are described by $\Omega = (x_0, p_0)$ on the phase space $H/\mathbb{R} = (\mathbb{R} \times \mathbb{R}^2)/\mathbb{R} = \mathbb{R}^2$. The definition of the generalised phase space is therefore consistent with the usual phase space.

On this generalised phase space, a generalised Wigner distribution as well as a generalised Weyl transform can be defined, through the concept of Stratonovich-Weyl image.

Definition 4.4: Generalised phase space transform [41]

Let X be the phase space, $\hat{A}$ an operator and $d\mu$ the invariant integration measure on X. The phase space description of the operator is the Stratonovich-Weyl image $F_A^{(s)}$ of $\hat{A}$. For any $\Omega \in X$, five conditions are fulfilled:

1. Linearity: The map from $\hat{A}$ to $F_A^{(s)}$ is linear.
2. Reality:
$$F_A^{(s)}(\Omega) = \left(F_{A^\dagger}^{(s)}(\Omega)\right)^*.$$
3. Normalisation:
$$\mathrm{Tr}\left(\hat{A}\right) = \int_X F_A^{(s)}(\Omega)d\mu(\Omega).$$

[4]A representation is irreducible if the only subspaces invariant under the application of T are $\{0\}$ and G.

4. Traciality:
$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \int_X F_A^{(s)}(\Omega)F_B^{(-s)}(\Omega)d\mu(\Omega).$$

5. Covariance: For any $g \in G$,
$$F_{g\cdot A}^{(s)}(\Omega) = F_A^{(s)}(g^{-1}\Omega), \qquad g \cdot A = T(g)\hat{A}T^{-1}(g).$$

The reality condition means that if $\hat{A}$ is Hermitian ($\hat{A} = \hat{A}^\dagger$), the distribution is real. The traciality condition ensures the statistical interpretation, so that the average value of $\hat{A}$ can be calculated with an equivalent of the ensemble average on phase space, by taking $\hat{B} = \hat{\rho}$. Lastly, the covariance expresses that the phase space must get the symmetry of the initial quantum system. For example, if $\hat{A} = |\psi_0\rangle\langle\psi_0|$, then $g \cdot A = T(g)|\psi_0\rangle\langle\psi_0| T^{-1}(g) = |\psi_g\rangle\langle\psi_g|$. The phase space description must therefore be the same as before, with an additional "translation" from $|\psi_0\rangle$ to $|\psi_g\rangle$.

A way to express the phase space transform is using the generalised Weyl rule, with the Stratonovich-Weyl kernel.

Definition 4.5: Generalised Weyl rule [41]

For any $\Omega \in X$ and any operator $\hat{A}$, the generalised Weyl rule gives a generalised phase space Weyl transform $F_A^{(s)}$ by
$$F_A^{(s)}(\Omega) = \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right),$$
with $\hat{\Delta}^{(s)}(\Omega)$ an operator called the Stratonovich-Weyl kernel.

The image is effectively linear, the first condition is therefore fulfilled by construction. The other conditions are translating to conditions on the kernel. The Weyl rule can be inverted, to find the operator $\hat{A}$ from $F_A^{(s)}$.

Proposition 4.1: Inverse of the generalised Weyl rule [41]

For any Stratonovich-Weyl image $F_A^{(s)}$ on the phase space X, the operator $\hat{A}$ is obtained using the inverse of the Weyl rule,
$$\hat{A} = \int_X F_A^{(s)}(\Omega)\hat{\Delta}^{(-s)}(\Omega)d\mu(\Omega).$$

Proof. Consider an arbitrary operator $\hat{A}$. From the traciality rule and Definition 4.5, we have
$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \int_X F_A^{(s)}(\Omega)F_B^{(-s)}(\Omega)d\mu(\Omega) \tag{4.7}$$
$$= \int_X F_A^{(s)}(\Omega)\,\mathrm{Tr}\left(\hat{B}\hat{\Delta}^{(-s)}(\Omega)\right)d\mu(\Omega) \tag{4.8}$$
$$= \mathrm{Tr}\left\{\left[\int_X F_A^{(s)}(\Omega)\hat{\Delta}^{(-s)}(\Omega)d\mu(\Omega)\right]\hat{B}\right\}. \tag{4.9}$$

Since $\hat{B}$ is arbitrary,

$$\hat{A} = \int_X F_A^{(s)}(\Omega)\hat{\Delta}^{(-s)}(\Omega)d\mu(\Omega). \tag{4.10}$$

□

The conditions 2-5 from Definition 4.4 define conditions on the Stratonovich-Weyl kernel.

Proposition 4.2: Conditions on the Stratonovich-Weyl kernel [41]

To describe a phase space representation in phase space through the Weyl rule, the Stratonovich-Weyl kernel $\hat{\Delta}^{(s)}$ has to respect the following conditions, for any $\Omega \in X$ (in relation to the corresponding conditions of Definition 4.4),

2. Hermiticity:
$$\hat{\Delta}^{(s)}(\Omega) = (\hat{\Delta}^{(s)}(\Omega))^\dagger.$$
3. Normalisation:
$$\int_X \hat{\Delta}^{(s)}(\Omega)d\mu(\Omega) = \hat{\mathbb{1}}.$$
4. Traciality:
$$\hat{\Delta}^{(s)}(\Omega) = \int_X \hat{\Delta}^{(s')}(\Omega')\,\mathrm{Tr}\Big(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\Big)d\mu(\Omega').$$
5. Covariance: For any $g \in G$,
$$\hat{\Delta}^{(s)}(g\Omega) = T(g)\hat{\Delta}^{(s)}(\Omega)T(g)^{-1}.$$

The proof is given in Appendix E. The choice of the kernel, through the s index, decides the choice of the distribution [41]. For $s = -1$, we can get the normal-ordered distribution. If $s = 1$, it is the anti-normal ordered distribution that is obtained. Lastly, the case $s = 0$ can give the Wigner distribution.

4.3.1 Example: Wigner distribution in 1D

As explained, the phase space is $X = \mathbb{R}^2$ and the irreducible representation is the set of displacement operators $\hat{D}(x_0, p_0)$. The objective is to define the kernels in this space [39]. We choose a starting point of the phase space, $(x_0, p_0) = (0, 0)$. From there, the kernels are defined by the application of the irreducible representation,

$$\hat{\Delta}(x_0, p_0) = \hat{D}(x_0, p_0)\hat{\Delta}(0, 0)\hat{D}^\dagger(x_0, p_0). \tag{4.11}$$

The initial value of the kernel, $\hat{\Delta}(0, 0)$, is still unknown. This is a choice and different possibilities exist while respecting the previous conditions. To find the Wigner distribution back, we choose

$$\hat{\Delta}(0,0) = \frac{1}{h}\iint \hat{D}(x', p')dx'dp' = \int \left(\frac{1}{h}\int \hat{D}(x', p')dp'\right)dx' = \int \left|\frac{x'}{2}\right\rangle\left\langle -\frac{x'}{2}\right| dx', \tag{4.12}$$

using the relation

$$\frac{1}{h}\int \hat{D}(x', p')dp' = \left|\frac{x'}{2}\right\rangle\left\langle -\frac{x'}{2}\right|. \tag{4.13}$$

Injecting this in equation (4.11), we get

$$\hat{\Delta}(x_0,p_0) = \int e^{\frac{i}{\hbar}(p_0\hat{x}-x_0\hat{p})} \left|\frac{x'}{2}\right\rangle \left\langle -\frac{x'}{2}\right| e^{-\frac{i}{\hbar}(p_0\hat{x}-x_0\hat{p})} dx' \tag{4.14}$$

$$= \int e^{-\frac{i}{2\hbar}p_0x_0} e^{\frac{i}{\hbar}p_0\hat{x}} e^{-\frac{i}{\hbar}x_0\hat{p}} \left|\frac{x'}{2}\right\rangle \left\langle -\frac{x'}{2}\right| e^{-\frac{i}{\hbar}p_0\hat{x}} e^{\frac{i}{\hbar}x_0\hat{p}} e^{-\frac{i}{2\hbar}p_0x_0} dx' \tag{4.15}$$

$$= \int e^{-\frac{i}{\hbar}p_0x_0} e^{\frac{i}{\hbar}p_0x_0} e^{\frac{i}{\hbar}p_0\frac{x'}{2}} \left|x_0+\frac{x'}{2}\right\rangle \left\langle x_0-\frac{x'}{2}\right| e^{\frac{i}{\hbar}p_0\frac{x'}{2}} dx' \tag{4.16}$$

$$= \int e^{\frac{i}{\hbar}p_0x'} \left|x_0+\frac{x'}{2}\right\rangle \left\langle x_0-\frac{x'}{2}\right| dx', \tag{4.17}$$

With this kernel, the Weyl transform $\tilde{A}$ of the operator $\hat{A}$ is obtained back,

$$\tilde{A}(x_0,p_0) = \mathrm{Tr}\Big(\hat{A}\hat{\Delta}(x_0,p_0)\Big) = \int e^{\frac{i}{\hbar}p_0x'} \left\langle x_0-\frac{x'}{2}\right| \hat{A} \left|x_0+\frac{x'}{2}\right\rangle dx' = \int e^{-\frac{i}{\hbar}p_0y} \left\langle x_0+\frac{y}{2}\right| \hat{A} \left|x_0-\frac{y}{2}\right\rangle dy. \tag{4.18}$$

For $\hat{A} = \frac{\hat{\rho}}{h}$, we get the Wigner distribution back. It respects all the required properties of the Definition 4.4.

4.4 Curved space

In some situations, a curved configuration space is considered [39]. The Wigner distribution must be defined on this space. For example, a particle might be constrained on a sphere or a hyperbolic plane. It can be useful in the determination of the motion of a rigid body describing quantumly the orientation of a molecule [40]. It could also be used in transformation optics, to study the trajectory of light in metamaterials.

Proposition 4.3: Phase space formalism in curved space [39]

Consider a Riemannian manifold of dimension n. Any vector is $x = (x^1, x^2, ..., x^n) \in \mathbb{R}^n$. The metric tensor is $g_{ij}(x)$ and the metric determinant is $g(x) = \det g_{ij}(x)$. We assume that the coordinates $x^i, i = 1, ..., n$ are not bounded. The Weyl transform of the operator $\hat{A}$ on curved space is

$$\tilde{A}^g = \int \sqrt[4]{g\left(x+\frac{y}{2}\right) g\left(x-\frac{y}{2}\right)} e^{-\frac{i}{\hbar}p_iy^i} \left\langle x+\frac{y}{2}\right| \hat{A} \left|x-\frac{y}{2}\right\rangle dy$$

and the Wigner function is

$$W^g(x,p) = \frac{1}{h^n} \int \sqrt[4]{g\left(x+\frac{y}{2}\right) g\left(x-\frac{y}{2}\right)} e^{-\frac{i}{\hbar}p_iy^i} \left\langle x+\frac{y}{2}\right| \hat{\rho} \left|x-\frac{y}{2}\right\rangle dy.$$

Proof. The manifold is parameterised by the coordinates x^i and the associated operators are $\hat{x}^i$ so that

$$\hat{x}^i \left|x\right\rangle = x^i \left|x\right\rangle. \tag{4.19}$$

The associated conjugate momentum operators are $\hat{p}_i$. They are continuous because the coordinates x^i are not bounded. To satisfy the usual commutation relations, the momentum is

$$\hat{p}_i = -i\hbar\left(\frac{\partial}{\partial x^i} + \frac{1}{2}\Gamma^j_{ji}(x)\right), \tag{4.20}$$

with $\Gamma^j_{ji}(x)$ the Christoffel symbols[5] at x. To define the Wigner function on the curved space, we apply the procedure using the Stratonovich-Weyl kernels (Definition 4.5). Let's start with the definition of the identity operator. For a curved space, it is given by [42]

$$\hat{\mathbb{1}} = \int \sqrt{g(x)}\, |x\rangle\langle x|\, dx = \int |p\rangle\langle p|\, dp. \tag{4.21}$$

The scalar product $\langle x|p\rangle$ can be shown to be [39]

$$\langle x|p\rangle = \frac{e^{\frac{i}{\hbar}p_i x^i}}{h^{n/2}\sqrt[4]{g(x)}}. \tag{4.22}$$

The action of the translation operators, in x and p, is

$$e^{-\frac{i}{\hbar}x^i\hat{p}_i}|x'\rangle = \int e^{-\frac{i}{\hbar}x^i p_i}|p\rangle\langle p|x'\rangle\, dp = \frac{1}{h^{n/2}\sqrt[4]{g(x')}}\int e^{-\frac{i}{\hbar}p_i(x^i+x'^i)}|p\rangle\, dp \tag{4.23}$$

$$= \frac{\sqrt[4]{g(x+x')}}{\sqrt[4]{g(x')}}\int \langle p|x+x'\rangle |p\rangle\, dp = \frac{\sqrt[4]{g(x+x')}}{\sqrt[4]{g(x')}}|x+x'\rangle, \tag{4.24}$$

$$e^{\frac{i}{\hbar}p_i\hat{x}^i}|p'\rangle = \int e^{\frac{i}{\hbar}p_i x^i}\sqrt{g(x)}\,|x\rangle\langle x|p'\rangle\, dx = \int \frac{\sqrt[4]{g(x)}}{h^{n/2}} e^{\frac{i}{\hbar}(p_i+p'_i)x^i}|x\rangle\, dx \tag{4.25}$$

$$= \int \sqrt{g(x)}\,\langle x|p+p'\rangle |x\rangle\, dx = |p+p'\rangle. \tag{4.26}$$

The translation operator acts in the same way as usual for the momentum but is modified for the position. Indeed, the translation acts as if it also translates the metric determinant. This is consistent because for calculations, what matters is the metric at the given point, so operating a translation must also translate the current metric. The metric g will now be used as an exponent to denote the operators in this metric. In the same way as in the example of section 4.3.1, the displacement operator is

$$\hat{D}^g(x,p) = e^{\frac{i}{\hbar}(p_i\hat{x}^i - x^i\hat{p}_i)}, \tag{4.27}$$

that gives the undisplaced kernel

$$\hat{\Delta}(0,0) = \frac{1}{h^n}\iint \hat{D}^g(x',p')dx'dp' = \int \sqrt[4]{g\left(-\frac{x'}{2}\right)g\left(\frac{x'}{2}\right)}\left|\frac{x'}{2}\right\rangle\left\langle -\frac{x'}{2}\right| dx', \tag{4.28}$$

[5] The Christoffel symbols are defined by $\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$.

and the displaced kernel,

$$\hat{\Delta}(x,p) = \int \sqrt[4]{g\left(-\frac{x'}{2}\right) g\left(\frac{x'}{2}\right)} e^{\frac{i}{\hbar}(p_i \hat{x}^i - x^i \hat{p}_i)} \left|\frac{x'}{2}\right\rangle \left\langle -\frac{x'}{2}\right| e^{-\frac{i}{\hbar}(p_i \hat{x}^i - x^i \hat{p}_i)} dx' \tag{4.29}$$

$$= \int \sqrt[4]{g\left(-\frac{x'}{2}\right) g\left(\frac{x'}{2}\right)} e^{-\frac{i}{\hbar}p_i x^i} e^{\frac{i}{\hbar}p_i \hat{x}^i} e^{-\frac{i}{\hbar}x^i \hat{p}_i} \left|\frac{x'}{2}\right\rangle \left\langle -\frac{x'}{2}\right| e^{-\frac{i}{\hbar}p_i \hat{x}^i} e^{\frac{i}{\hbar}x^i \hat{p}_i} dx' \tag{4.30}$$

$$= \int \sqrt[4]{g\left(x-\frac{x'}{2}\right) g\left(x+\frac{x'}{2}\right)} e^{-\frac{i}{\hbar}p_i x^i} e^{\frac{i}{\hbar}p_i x^i} e^{\frac{i}{2\hbar}p_i x'^i} \left|x+\frac{x'}{2}\right\rangle \left\langle x-\frac{x'}{2}\right| e^{\frac{i}{2\hbar}p_i x'^i} dx' \tag{4.31}$$

$$= \int \sqrt[4]{g\left(x-\frac{x'}{2}\right) g\left(x+\frac{x'}{2}\right)} e^{\frac{i}{\hbar}p_i x'^i} \left|x+\frac{x'}{2}\right\rangle \left\langle x-\frac{x'}{2}\right| dx' \tag{4.32}$$

$$= \int \sqrt[4]{g\left(x-\frac{x'}{2}\right) g\left(x+\frac{x'}{2}\right)} e^{\frac{i}{\hbar}p_i x'^i} \left|x+\frac{x'}{2}\right\rangle \left\langle x-\frac{x'}{2}\right| dx'. \tag{4.33}$$

The Weyl transform $\tilde{A}^g$ of the operator $\hat{A}$ is therefore

$$\tilde{A}^g = \mathrm{Tr}\left(\hat{A}\hat{\Delta}(x,p)\right) \tag{4.34}$$

$$= \int \sqrt[4]{g\left(x-\frac{x'}{2}\right) g\left(x+\frac{x'}{2}\right)} e^{\frac{i}{\hbar}p_i x'^i} \left\langle x-\frac{x'}{2}\right| \hat{A} \left|x+\frac{x'}{2}\right\rangle dx' \tag{4.35}$$

$$= \int \sqrt[4]{g\left(x+\frac{y}{2}\right) g\left(x-\frac{y}{2}\right)} e^{-\frac{i}{\hbar}p_i y^i} \left\langle x+\frac{y}{2}\right| \hat{A} \left|x-\frac{y}{2}\right\rangle dy, \tag{4.36}$$

and the Wigner function in curved space is

$$W^g(x,p) = \frac{1}{h^n} \int \sqrt[4]{g\left(x+\frac{y}{2}\right) g\left(x-\frac{y}{2}\right)} e^{-\frac{i}{\hbar}p_i y^i} \left\langle x+\frac{y}{2}\right| \hat{\rho} \left|x-\frac{y}{2}\right\rangle dy. \tag{4.37}$$

□

Compared to a flat configuration space, the Wigner distribution and the Weyl transform are modified by adding a factor dependant of the curvature of the space. It is possible to show that the Wigner distribution is however unchanged in the p representation,

$$W^g(x,p) = \frac{1}{h^n} \int e^{\frac{i}{\hbar}u_i x^i} \left\langle p+\frac{u}{2}\right| \hat{\rho} \left|p-\frac{u}{2}\right\rangle du. \tag{4.38}$$

The distribution still verifies the required properties of the Wigner quasi-probability distribution,

$$\int W^g(x,p)dp = \sqrt{g(x)} \langle x| \hat{\rho} |x\rangle, \tag{4.39}$$

$$\int W^g(x,p)dx = \langle p| \hat{\rho} |p\rangle, \tag{4.40}$$

$$\iint W^g(x,p)dxdp = \int \langle p| \hat{\rho} |p\rangle dp = 1, \tag{4.41}$$

except for the x distribution, weighted by the amount $\sqrt{g(x)}$. The product formula

$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \frac{1}{h^n} \iint \tilde{A}^g(x,p)\tilde{B}^g(x,p)dxdp \tag{4.42}$$

still holds, so the expectation values can be evaluated in the exact same way as before,

$$\langle A\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right) = \iint \tilde{A}^g(x,p)W^g(x,p)dxdp. \tag{4.43}$$

The Weyl transform of the operator $\hat{\rho}_{\psi\phi} = |\psi\rangle\langle\phi|$ gives the cross-Wigner distribution generalised to a curved space,

$$W^g_{\psi,\phi}(x,p) = \frac{1}{h^n}\int \sqrt[4]{g\left(x+\frac{y}{2}\right)g\left(x-\frac{y}{2}\right)}e^{-\frac{i}{\hbar}p_i y^i}\psi\left(x+\frac{y}{2}\right)\phi^*\left(x-\frac{y}{2}\right)dy. \tag{4.44}$$

Again, a factor depending on the curvature is present in the definition.

4.4.1 Example: Motion on a 2D curved space

Consider a particle moving on a 2D surface inside a 3D Euclidian space. The objective is to find the metric determinant to use in the Wigner distribution [39]. The z coordinate is assumed to be given by a function of x and y,

$$z = f(x,y), \tag{4.45}$$

so the configuration space is curved by this function. The infinitesimal dz is

$$dz = f_x dx + f_y dy, \tag{4.46}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. This element squared is

$$dz^2 = f_x^2 dx^2 + f_y^2 dy^2 + 2f_x f_y dxdy. \tag{4.47}$$

Therefore, an infinitesimal length element ds^2 is

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + f_x^2 dx^2 + f_y^2 dy^2 + 2f_x f_y dxdy \tag{4.48}$$

$$= \left(1+f_x^2\right)dx^2 + \left(1+f_y^2\right)dy^2 + 2f_x f_y dxdy = g_{ij}dx^i dx^j. \tag{4.49}$$

Identifying the elements of the metric gives

$$g_{xx}(x,y) = 1 + f_x^2, \tag{4.50}$$

$$g_{yy}(x,y) = 1 + f_y^2, \tag{4.51}$$

$$g_{xy}(x,y) = g_{yx}(x,y) = f_x f_y. \tag{4.52}$$

The metric determinant is then

$$g(x,y) = \det g_{ij}(x,y) = \left(1+f_x^2\right)\left(1+f_y^2\right) - (f_x f_y)^2 = 1 + f_x^2 + f_y^2. \tag{4.53}$$

By Proposition 4.3, the Wigner function is

$$W(x,y,p_x,p_y) = \frac{1}{h^2}\iint \sqrt[4]{(1+f_x^2(x_+,y_+)+f_y^2(x_+,y_+))(1+f_x^2(x_-,y_-)+f_y^2(x_-,y_-))}$$
$$e^{-\frac{i}{\hbar}(p_x x' + p_y y')}\langle x_+, y_+|\,\hat{\rho}\,|x_-, y_-\rangle dx'dy' \tag{4.54}$$

where $x_\pm = x \pm \frac{x'}{2}$ and $y_\pm = y \pm \frac{y'}{2}$.

This can be more specifically applied for a paraboloid curvature. It is defined as

$$f(x, y) = \left(\frac{x}{a}\right)^2 \pm \left(\frac{y}{b}\right)^2 \tag{4.55}$$

where the $+$ denotes an elliptic paraboloid and the $-$ is for a hyperbolic paraboloid. In both cases, the metric determinant is

$$g(x, y) = 1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}. \tag{4.56}$$

Therefore, the Wigner distribution in this curved phase space is

$$W(x, y, p_x, p_y) = \frac{1}{h^2} \iint \sqrt[4]{\left(1 + \frac{4x_+^2}{a^4} + \frac{4y_+^2}{b^4}\right)\left(1 + \frac{4x_-^2}{a^4} + \frac{4y_-^2}{b^4}\right)} \\ e^{-\frac{i}{\hbar}(p_x x' + p_y y')} \langle x_+, y_+ | \hat{\rho} | x_-, y_- \rangle \, dx' dy'. \tag{4.57}$$

4.4.2 Example: Sphere

Consider a particle moving on a sphere of radius R. The metric determinant has to be found. It is done in a different way than the previous example, by looking at the kinetic energy part of a Hamiltonian defined on the sphere [39]. The motion is described by the polar angles θ and φ and the Hamiltonian is

$$H = \frac{1}{2MR^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) = g^{ij}(\theta, \varphi)\frac{p_i p_j}{2M}. \tag{4.58}$$

By identification, we get

$$g^{\theta\theta}(\theta, \varphi) = \frac{1}{R^2}, \qquad g^{\varphi\varphi}(\theta, \varphi) = \frac{1}{R^2 \sin^2\theta}, \qquad g^{\theta\varphi}(\theta, \varphi) = g^{\varphi\theta}(\theta, \varphi) = 0. \tag{4.59}$$

The metric is diagonal, so $g_{ij}(\theta, \varphi) = (g^{ij}(\theta, \varphi))^{-1}$. The metric determinant is therefore

$$g(\theta, \varphi) = \det g_{ij}(\theta, \varphi) = \det\big(g^{ij}(\theta, \varphi)\big)^{-1} = R^4 \sin^2\theta. \tag{4.60}$$

The considered configuration space is compact, meaning that the momentum distribution is discrete. The phase space is thus $(\theta, \varphi, m_\theta, m_\varphi)$ where $m_\theta, m_\varphi \in \mathbb{Z}$. The development leading to the Wigner distribution can be remade for a compact configuration space [39]. The Wigner distribution for a particle on a sphere is

$$W(\theta, \varphi, m_\theta, m_\varphi) = \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} R^2 \sqrt{\sin\theta_+ \sin\theta_-} e^{2im_\theta\theta'} e^{im_\varphi\varphi'} \langle \theta_-, \varphi_- | \hat{\rho} | \theta_+, \varphi_+ \rangle \, d\varphi' d\theta', \tag{4.61}$$

with $\theta_\pm = \left(\theta \pm \frac{\theta'}{2}\right) \bmod \pi$ and $\varphi_\pm = \left(\varphi \pm \frac{\varphi'}{2}\right) \bmod 2\pi$.

5 Weak Measurements in Phase Space

5.1 Motivation

Weak measurements, as explained in Section 2, are particularly useful in some experiments. However, their meaning is still subject to a lot of debate. The seminal paper of Aharonov and his colleagues introducing weak values is entitled *"How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100"* [26]. This provocative title highlights the conceptual difficulty: should the weak value of the spin of the particle be interpreted as related to the authentic spin, or is it just the manifestation of an interference effect that doesn't give concrete information about the spin? Interpreting the measured weak value, exceeding the range of eigenvalues and that might even be complex, is a difficult task.

It is therefore important to link weak values and weak measurements to other concepts, to get a different point of view on their meaning. A geometrical description of weak values of projectors has been developed in [43] and extended to general observables of N-level quantum systems in [44]. Another geometric tool used is the Hopf fibration, for 4-level systems [45].

Besides the usual way to mathematically formulate quantum physics, we have seen that a phase space description exists. This formulation allows us to get an understanding of states closer to the classical intuition. Applying the phase space description of states to describe weak values and the process of weak measurement therefore makes sense to get a different light on these concepts.

Moreover, the use of the phase space description is also motivated by common features of weak values and phase space distributions. Indeed, quasi-probability distributions are used to describe the phase space and it turns out that weak values of projectors might as well be understood as quasi-probabilities.

Proposition 5.1: Weak value of projectors [46]

Consider an operator $\hat{A} = \sum_j a_j \hat{\Pi}_j = \sum_j a_j |a_j\rangle \langle a_j|$ of the system, with pre-selection $|\psi_i\rangle$ and post-selection $|\psi_f\rangle$. Then, the set of weak values of projectors,

$$\left\{ \Pi_{jw} = \frac{\langle\psi_f| \hat{\Pi}_j |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \right\}_j ,$$

form a quasi-probability distribution.

Proof. The weak value of the operator $\hat{A}$ is

$$A_w = \frac{\langle\psi_f| \hat{A} |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{\langle\psi_f| \sum_j a_j \hat{\Pi}_j |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \sum_j a_j \frac{\langle\psi_f| \hat{\Pi}_j |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \sum_j a_j \Pi_{jw}. \tag{5.1}$$

The quantities Π_{jw} are understood as a pseudo probability since performing the average weighted with Π_{jw} gives the weak value. It respects the second Kolmogorov axiom of probabilities,

$$\sum_j \Pi_{jw} = \sum_j \frac{\langle\psi_f| \hat{\Pi}_j |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{\langle\psi_f| \sum_j \hat{\Pi}_j |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{\langle\psi_f| \hat{\mathbb{1}} |\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = 1. \tag{5.2}$$

However, it may be complex or negative so the first Kolmogorov axiom is not verified and it is a quasi-probability. □

Some links between weak values and the phase space formalism already exist in the literature and a non-exhaustive state of the art is first presented. Some of these results are then extended and generalised. The von Neumann procedure and weak measurements are also reproduced in phase space.

5.2 State of the art

In this section, the literature review about the description of weak values in phase space using the Wigner formalism is presented. However, there is no description of a weak measurement in phase space and we will study it in the following sections.

5.2.1 Weak value and the cross-Wigner distribution

In phase space, any weak value can be directly related to an average over the cross-Wigner distribution W_{ψ_i,ψ_f} (Definition 3.4).

Proposition 5.2: Weak value in phase space [31, 32]

Consider the pre-selected state $|\psi_i\rangle$, the post-selected state $|\psi_f\rangle$ and the operator $\hat{A}$. Then, the description of the corresponding weak value in phase space is

$$A_w = \frac{\iint W_{\psi_i,\psi_f}(x,p)\tilde{A}(x,p)dxdp}{\iint W_{\psi_i,\psi_f}(x,p)dxdp}.$$

Proof. The weak value is

$$A_w = \frac{\langle\psi_f|\,\hat{A}\,|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{5.3}$$

From Proposition 3.5, the denominator is

$$\langle\psi_f|\psi_i\rangle = \iint W_{\psi_i,\psi_f}(x,p)dxdp. \tag{5.4}$$

The numerator is evaluated using the trace property (Lemma 3.1),

$$\langle\psi_f|\,\hat{A}\,|\psi_i\rangle = \mathrm{Tr}\Big(\hat{\rho}_{\psi_i,\psi_f}\hat{A}\Big) = \frac{1}{h}\iint \tilde{A}(x,p)\tilde{\rho}_{\psi_i,\psi_f}(x,p)dxdp = \iint \tilde{A}(x,p)W_{\psi_i,\psi_f}(x,p)dxdp. \tag{5.5}$$

Consequently, the weak value is expressed in terms of the cross-Wigner distribution,

$$A_w = \frac{\iint W_{\psi_i,\psi_f}(x,p)\tilde{A}(x,p)dxdp}{\iint W_{\psi_i,\psi_f}(x,p)dxdp}. \tag{5.6}$$

□

From Proposition 3.4, the cross-Wigner distribution is interpreted as an interference occurring between the two states $|\psi_i\rangle$ and $|\psi_f\rangle$. Therefore, the operator $\hat{A}$ is averaged over the cross-Wigner distribution coming from the interference and this makes the weak value appear. One possible physical interpretation, in a time-symmetric formulation of the measurement process, is that the wave ψ_i propagates forward in time and the wave ψ_f, from the post-selection, propagates backward in time. They then interfere to give the weak value.

5.2.2 Example: Weakly Coupled Harmonic Oscillators

Consider the example of Section 2.4.3. The two weak values, y_w and p_{yw}, can also be calculated in the Wigner formalism, using Proposition 5.2. The Weyl transforms of $\hat{y}$ and $\hat{p}_y$ are simply $\tilde{y} = y$ and $\tilde{p}_y = p_y$. The weak values are therefore given by

$$y_w = \frac{\iint y W_{\psi_i,\psi_f}(y,p_y) dy dp_y}{\iint W_{\psi_s,\psi_f}(y,p_y) dy dp_y}, \qquad p_{yw} = \frac{\iint p_y W_{\psi_i,\psi_f}(y,p_y) dy dp_y}{\iint W_{\psi_s,\psi_f}(y,p_y) dy dp_y}. \tag{5.7}$$

First, we need the cross-Wigner distribution between the pre- and post-selected states,

$$\begin{aligned} W_{\psi_i,\psi_f}(y,p_y) &= \frac{1}{h}\int e^{-\frac{i}{\hbar}p_y y'}\psi_i\left(y+\frac{y'}{2}\right)\psi_f\left(y-\frac{y'}{2}\right)dy' &(5.8)\\ &= \frac{\alpha}{h}\int e^{-\frac{i}{\hbar}p_y y'}\psi_0\left(y+\frac{y'}{2}\right)\psi_0\left(y-\frac{y'}{2}\right)dy' &(5.9)\\ &\quad+\frac{1-\alpha}{h}\int e^{-\frac{i}{\hbar}p_y y'}\psi_0\left(y+\frac{y'}{2}\right)\psi_1\left(y-\frac{y'}{2}\right)dy', &(5.10)\end{aligned}$$

where $\psi_i, \psi_f, \psi_0, \psi_1$ and α are defined in the example of Section 2.4.3. The state ψ_f is not normalised, so the cross-Wigner distribution is not normalised either. However, the normalisation is an unimportant factor since it is cancelling in the ratio of the definition of the weak values. The first integral is the Wigner distribution of the ground state and is calculated in equation (3.21). The second integral is proportional to the cross-Wigner distribution between the ground state and the first excited state, $W_{\psi_0,\psi_1}(y,p_y)$. It is equal to

$$\begin{aligned} W_{\psi_0,\psi_1}(y,p_y) &= \frac{1}{h}\int e^{-\frac{i}{\hbar}p_y y'}\psi_0\left(y+\frac{y'}{2}\right)\psi_1\left(y-\frac{y'}{2}\right)dy' &(5.11)\\ &= \frac{\sqrt{2}}{h\sqrt{\pi\sigma_s^2}}\int e^{-\frac{i}{\hbar}p_y y'}e^{-\frac{\left(y+\frac{y'}{2}\right)^2}{2\sigma_s^2}}\left(y-\frac{y'}{2}\right)e^{-\frac{\left(y-\frac{y'}{2}\right)^2}{2\sigma_s^2}}dy' &(5.12)\\ &= \frac{\sqrt{2}}{h\sqrt{\pi\sigma_s^2}}\int \left(y-\frac{y'}{2}\right)e^{-\frac{i}{\hbar}p_y y'}e^{-\frac{\left(y^2+\frac{y'^2}{4}\right)}{\sigma_s^2}}dy' &(5.13)\\ &= \frac{\sqrt{2}}{h\sqrt{\pi\sigma_s^2}}e^{-\frac{y^2}{\sigma_s^2}}\left[y\int e^{-\frac{i}{\hbar}p_y y'}e^{-\frac{y'^2}{4\sigma_s^2}}dy' - \frac{1}{2}\int y' e^{-\frac{i}{\hbar}p_y y'}e^{-\frac{y'^2}{4\sigma_s^2}}dy'\right] &(5.14)\\ &= \frac{\sqrt{2}}{h\sqrt{\pi\sigma_s^2}}e^{-\frac{y^2}{\sigma_s^2}}\left[2\sigma_s y\sqrt{\pi}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}} - \frac{1}{2}\frac{-ip_y 4\sigma_s^3}{\hbar}\sqrt{\pi}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}\right] &(5.15)\\ &= \frac{2\sqrt{2}}{h\sigma_s}\left(y+\frac{ip_y\sigma_s^2}{\hbar}\right)e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}. &(5.16)\end{aligned}$$

Therefore, the cross-Wigner distribution of the pre- and post-selection, up to a normalisation factor, is

$$\begin{aligned} W_{\psi_i,\psi_f}(y,p_y) &= \frac{2\alpha}{h}e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}} + (1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\left[y+\frac{ip_y\sigma_s^2}{\hbar}\right]e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}} &(5.17)\\ &= \left[\frac{2\alpha}{h}+(1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\left(y+\frac{ip_y\sigma_s^2}{\hbar}\right)\right]e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}. &(5.18)\end{aligned}$$

With the knowledge of the cross-Wigner distribution between the pre- and post-selected states, we can now evaluate the numerator and denominator of the weak values of position and momentum. The denominator of the two weak values is the integral over y and p_y of the cross-Wigner distribution,

$$\iint W_{\psi_i,\psi_f}(y,p_y)dydp_y = \frac{2\alpha}{h}\iint e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.19}$$

$$+(1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\iint ye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.20}$$

$$+(1-\alpha)\frac{\sqrt{2}i\sigma_s}{\pi\hbar^2}\iint p_ye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y. \tag{5.21}$$

The second line is zero because of the odd argument with respect to y. The third line is zero too for the same argument over p_y. Therefore,

$$\iint W_{\psi_i,\psi_f}(y,p_y)dydp_y = \frac{2\alpha}{h}\iint e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y = \frac{2\alpha}{h}\sigma_s\sqrt{\pi}\int e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dp_y \tag{5.22}$$

$$= \frac{2\alpha}{h}\sigma_s\sqrt{\pi}\frac{\hbar}{\sigma_s}\sqrt{\pi} = \alpha. \tag{5.23}$$

The numerator of the weak value y_w is

$$\iint yW_{\psi_i,\psi_f}(y,p_y)dydp_y = \frac{2\alpha}{h}\iint ye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.24}$$

$$+(1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\iint y^2e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.25}$$

$$+(1-\alpha)\frac{\sqrt{2}i\sigma_s}{\pi\hbar^2}\iint yp_ye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y. \tag{5.26}$$

The integral of the first line has an odd argument, so it is zero. It is the same for the integral of the third line. Therefore, we have

$$\iint yW_{\psi_i,\psi_f}(y,p_y)dydp_y = (1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\iint y^2e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y = (1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\sqrt{\pi}\frac{\hbar}{\sigma_s}\int y^2e^{-\frac{y^2}{\sigma_s^2}}dy \tag{5.27}$$

$$= (1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\sqrt{\pi}\frac{\hbar}{\sigma_s}\sigma_s^3\frac{\sqrt{\pi}}{2} = (1-\alpha)\frac{\sigma_s}{\sqrt{2}}. \tag{5.28}$$

The numerator of the weak value p_{yw} is

$$\iint p_yW_{\psi_i,\psi_f}(y,p_y)dydp_y = \frac{2\alpha}{h}\iint p_ye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.29}$$

$$+(1-\alpha)\frac{2\sqrt{2}}{h\sigma_s}\iint p_yye^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y \tag{5.30}$$

$$+(1-\alpha)\frac{\sqrt{2}i\sigma_s}{\pi\hbar^2}\iint p_y^2e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydp_y. \tag{5.31}$$

The integral of the first line has an odd argument, so it is zero. The two integrals of the second line also have odd arguments. Therefore, we have

$$\iint p_yW_{\psi_i,\psi_f}(y,p_y)dydp_y = (1-\alpha)\frac{\sqrt{2}i\sigma_s}{\pi\hbar^2}\iint p_y^2e^{-\frac{y^2}{\sigma_s^2}}e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dydpy = (1-\alpha)\frac{\sqrt{2}i\sigma_s^2}{\sqrt{\pi}\hbar^2}\int p_y^2e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dpy \tag{5.32}$$

$$= (1-\alpha)\frac{\sqrt{2}i\sigma_s^2}{\sqrt{\pi}\hbar^2}\frac{\hbar^3}{\sigma_s^3}\frac{\sqrt{\pi}}{2} = (1-\alpha)\frac{i\hbar}{\sqrt{2}\sigma_s}. \tag{5.33}$$

The different parts can be put together to find the two weak values,

$$y_w = \frac{(1-\alpha)\sigma_s}{\alpha\sqrt{2}}, \qquad p_{yw} = \frac{i(1-\alpha)\hbar}{\alpha\sqrt{2}\sigma_s} \tag{5.34}$$

The objective was to illustrate that the cross-Wigner distribution can be used to calculate weak values. In the case of the bi-dimensional harmonic oscillator, the weak values obtained are the same as those demonstrated previously in equation (2.45).

5.2.3 The momentum weak value

The weak value of the momentum operator $\hat{p}$, with post-selection on the state $|x\rangle$, can be directly related to the Wigner distribution (instead of the cross-Wigner distribution).

Proposition 5.3: Real part of the weak value of momentum [47]

Consider the pre-selected state $|\psi\rangle$ and the post-selected state $|x\rangle$. The real part of the weak value of the operator $\hat{p}$ is described in phase space as

$$\operatorname{Re} \frac{\langle x|\,\hat{p}\,|\psi\rangle}{\langle x|\psi\rangle} = \frac{\int pW_\psi(x,p)dp}{\int W_\psi(x,p)dp}.$$

Proof. The Wigner distribution of the pre-selected state is

$$W_\psi(x,p) = \frac{1}{h}\int e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle dy. \tag{5.35}$$

For a given x, the partial average of p in the initial system state is

$$\begin{aligned}
\int pW_\psi(x,p)dp &= \frac{1}{h}\iint pe^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle dydp && (5.36)\\
&= \frac{1}{h}\iint i\hbar\frac{\partial}{\partial y}e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle dydp && (5.37)\\
&= \left[\frac{i\hbar}{h}\int e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle dp\right]_{y=-\infty}^{y=\infty} && (5.38)\\
&\quad -\frac{i\hbar}{h}\iint e^{-\frac{i}{\hbar}py}\frac{\partial}{\partial y}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right) dydp && (5.39)
\end{aligned}$$

where an integration by parts is done. In the first term, the wavefunction is evaluated at the infinity. However, a wavefunction is finite and must vanish at the infinity, so this first term is equal to zero. The second double integral is evaluated by distributing the derivative and by noticing that the integral over p is proportional to $\delta(y)$,

$$\int pW_\psi(x,p)dp = \int(-i\hbar)\left[\frac{\partial}{\partial y}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\right)\left\langle\psi\middle|x-\frac{y}{2}\right\rangle + \left\langle x+\frac{y}{2}\middle|\psi\right\rangle\frac{\partial}{\partial y}\left(\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right)\right]\delta(y)dy. \tag{5.40}$$

Observe that

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\right) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\right), && (5.41)\\
\frac{\partial}{\partial y}\left(\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right) &= \frac{-1}{2}\frac{\partial}{\partial x}\left(\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right), && (5.42)
\end{aligned}$$

so the partial average becomes

$$\int pW_\psi(x,p)dp = \int \frac{-i\hbar}{2}\left[\frac{\partial}{\partial x}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\right)\left\langle\psi\middle|x-\frac{y}{2}\right\rangle - \left\langle x+\frac{y}{2}\middle|\psi\right\rangle\frac{\partial}{\partial x}\left(\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right)\right]\delta(y)dy \tag{5.43}$$

$$= \frac{-i\hbar}{2}\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle - \frac{-i\hbar}{2}\langle x|\psi\rangle\frac{\partial}{\partial x}(\langle\psi|x\rangle) \tag{5.44}$$

$$= \frac{-i\hbar}{2}\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle + \overline{\left(\frac{-i\hbar}{2}\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle\right)} \tag{5.45}$$

$$= \mathrm{Re}\left[-i\hbar\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle\right] = \mathrm{Re}\left[\langle x|\hat{p}|\psi\rangle\langle\psi|x\rangle\right]. \tag{5.46}$$

This relation is related to the Terletsky-Margenau-Hill quasi-probability distribution[6]. Noting that

$$\int W_\psi(x,p)dp = \langle x|\psi\rangle\langle\psi|x\rangle\,, \tag{5.47}$$

the real part of the weak value of $\hat{p}$ is

$$\mathrm{Re}\,\frac{\langle x|\hat{p}|\psi\rangle}{\langle x|\psi\rangle} = \frac{\int pW_\psi(x,p)dp}{\int W_\psi(x,p)dp}. \tag{5.48}$$

□

This proposition describes the real part of the weak value of the operator $\hat{p}$, for the pre-selected state $|\psi\rangle$ and post-selected state $\langle x|$, using the Wigner distribution of the initial state. As was explained in Section 2.5.2, this weak value is used to determine the average trajectories of the photons in the two-slit interferometer.

5.3 Generalised weak value

Until now, the main results were coming from the literature (with a personal contribution in the different examples and illustrations). Starting from here and until the end of the thesis, only original results are presented. The Proposition 5.2 defines a relation between the cross-Wigner distribution and the weak value of the operator $\hat{A}$. In subsection 4.3, the notion of phase space quasi-probability distribution has been generalised to the Stratonovich-Weyl image $F_A^{(s)}$ (Definition 4.4). Using this, we extend the Proposition 5.2 to express weak values in the generalised phase space.

Proposition 5.4: Weak value in generalised phase space

Consider the pre-selected state $|\psi_i\rangle$, the post-selected state $|\psi_f\rangle$ and the operator $\hat{A}$. Then, the description of the corresponding weak value in the generalised phase space X is

$$A_w = \frac{\int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)F_A^{(-s)}(\Omega)d\mu(\Omega)}{\int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)d\mu(\Omega)}.$$

The exponent s decides the distribution considered, as explained in Section 4.3.

[6] $P_{p,x|\psi} = \mathrm{Re}\,\langle x|p\rangle\langle p|\psi\rangle\langle\psi|x\rangle$. It is the real part of the Kirkwood quasi-probability distribution (Table 1).

Proof. Two important conditions on the Stratonovich-Weyl image are, for any two operators $\hat{A}$ and $\hat{B}$ (Definition 4.4),

$$\mathrm{Tr}\left(\hat{B}\right) = \int_X F_B^{(s)}(\Omega)d\mu(\Omega), \qquad \mathrm{Tr}\left(\hat{A}\hat{B}\right) = \int_X F_B^{(s)}(\Omega)F_A^{(-s)}(\Omega)d\mu(\Omega). \tag{5.49}$$

These two equations are applied using $\hat{B} = \hat{\rho}_{\psi_i,\psi_f} = |\psi_i\rangle\langle\psi_f|$,

$$\mathrm{Tr}\left(\hat{\rho}_{\psi_i,\psi_f}\right) = \langle\psi_f|\psi_i\rangle = \int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)d\mu(\Omega) \tag{5.50}$$

$$\mathrm{Tr}\left(\hat{A}\hat{\rho}_{\psi_i,\psi_f}\right) = \langle\psi_f|\,\hat{A}\,|\psi_i\rangle = \int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)F_A^{(-s)}(\Omega)d\mu(\Omega). \tag{5.51}$$

Therefore, the weak value is

$$A_w = \frac{\int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)F_A^{(-s)}(\Omega)d\mu(\Omega)}{\int_X F^{(s)}_{\rho_{\psi_i,\psi_f}}(\Omega)d\mu(\Omega)}. \tag{5.52}$$

□

This proposition describes the weak value in the generalised phase space X. For $s = 0$ in the phase space $\mathbb{R}^2$, the Stratonovich-Weyl image reduce to the Weyl transform and $d\mu = dxdp$, giving the Proposition 5.2 back.

5.4 Curved space

The curved phase space situation is a particular case of the generalised phase space. The weak value can therefore be easily defined on the curved space.

Proposition 5.5: Weak value in curved configuration space

Consider the pre-selected state $|\psi_i\rangle$, the post-selected state $|\psi_f\rangle$ and the operator $\hat{A}$. Then, the description of the corresponding weak value in curved space is

$$A_w = \frac{\iint W^g_{\psi_i,\psi_f}(x,p)\tilde{A}^g(x,p)dxdp}{\iint W^g_{\psi_i,\psi_f}(x,p)dxdp}.$$

Proof. Consider the trace formula (4.42) in curved space. For $\hat{B} = \hat{\rho}_{\psi_i,\psi_f}$, we have

$$\mathrm{Tr}\left(\hat{A}\hat{\rho}_{\psi_i,\psi_f}\right) = \langle\psi_f|\,\hat{A}\,|\psi_i\rangle = \iint W^g_{\psi_i,\psi_f}(x,p)\tilde{A}^g(x,p)dxdp. \tag{5.53}$$

The denominator of the weak value can also be obtained,

$$\mathrm{Tr}\left(\hat{\rho}_{\psi_i,\psi_f}\right) = \langle\psi_i|\psi_f\rangle = \iint W^g_{\psi_i,\psi_f}(x,p)dxdp. \tag{5.54}$$

Therefore, the weak value in curved space is

$$A_w = \frac{\iint W^g_{\psi_i,\psi_f}(x,p)\tilde{A}^g(x,p)dxdp}{\iint W^g_{\psi_i,\psi_f}(x,p)dxdp}. \tag{5.55}$$

□

This can be useful to describe a weak value in a constrained configuration space.

5.5 The momentum weak value

In Proposition 5.3, the real part of the weak value of the momentum operator $\hat{p}$ post-selected on the position $|x\rangle$ is described in phase space using the Wigner distribution of the pre-selected state. The imaginary part can also be obtained, so that the complete weak value is described in phase space.

Proposition 5.6: Weak value of momentum

Consider the pre-selected state $|\psi\rangle$ and the post-selected state $|x\rangle$. The weak value of the operator $\hat{p}$ is described in phase space as

$$\frac{\langle x|\,\hat{p}\,|\psi\rangle}{\langle x|\psi\rangle} = \frac{\int pW_\psi(x,p)dp}{\int W_\psi(x,p)dp} - \frac{i\hbar}{2}\frac{\frac{\partial}{\partial x}\int W_\psi(x,p)dp}{\int W_\psi(x,p)dp}.$$

Proof. The real part of the weak value is already obtained in Proposition 5.3. The imaginary part is deduced from the following integral,

$$\int -\frac{\hbar}{2}\frac{\partial}{\partial x}W_\psi(x,p)dp = \frac{1}{h}\iint \frac{-\hbar}{2}\frac{\partial}{\partial x}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right)e^{-\frac{i}{\hbar}py}dydp \tag{5.56}$$

$$= \int \frac{-\hbar}{2}\left[\frac{\partial}{\partial x}\left(\left\langle x+\frac{y}{2}\middle|\psi\right\rangle\right)\left\langle\psi\middle|x-\frac{y}{2}\right\rangle + \left\langle x+\frac{y}{2}\middle|\psi\right\rangle\frac{\partial}{\partial x}\left(\left\langle\psi\middle|x-\frac{y}{2}\right\rangle\right)\right]\delta(y)dy \tag{5.57}$$

$$= -\frac{\hbar}{2}\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle - \frac{\hbar}{2}\langle x|\psi\rangle\frac{\partial}{\partial x}(\langle\psi|x\rangle) \tag{5.58}$$

$$= -\frac{i\hbar}{2i}\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle - \frac{i\hbar}{2i}\langle x|\psi\rangle\frac{\partial}{\partial x}(\langle\psi|x\rangle) \tag{5.59}$$

$$= \mathrm{Im}\left[-i\hbar\frac{\partial}{\partial x}(\langle x|\psi\rangle)\langle\psi|x\rangle\right] = \mathrm{Im}\left[\langle x|\,\hat{p}\,|\psi\rangle\langle\psi|x\rangle\right]. \tag{5.60}$$

Noting that

$$\int W_\psi(x,p)dp = \langle x|\psi\rangle\langle\psi|x\rangle\,, \tag{5.61}$$

the weak value is

$$\frac{\langle x|\,\hat{p}\,|\psi\rangle}{\langle x|\psi\rangle} = \frac{\int pW_\psi(x,p)dp}{\int W_\psi(x,p)dp} + i\frac{\int -\frac{\hbar}{2}\frac{\partial}{\partial x}W_\psi(x,p)dp}{\int W_\psi(x,p)dp} = \frac{\int pW_\psi(x,p)dp}{\int W_\psi(x,p)dp} - \frac{i\hbar}{2}\frac{\frac{\partial}{\partial x}\int W_\psi(x,p)dp}{\int W_\psi(x,p)dp}. \tag{5.62}$$

□

The real part of the weak value is, for a given x, the average of the momentum operator in the pre-selected state. The integral of the numerator, from equation 5.44, is

$$\int pW_\psi(x,p)dp = \frac{i\hbar}{2}\left(\psi(x)\frac{\partial}{\partial x}\overline{\psi(x)} - \overline{\psi(x)}\frac{\partial}{\partial x}\psi(x)\right) = m\vec{J}, \tag{5.63}$$

with $\vec{J}$ the probability current density [21, 22]. It is a quantum mechanical quantity. It respects a continuity equation of the probability density,

$$\boldsymbol{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}|\psi(x)|^2. \tag{5.64}$$

The imaginary part is the derivative of the probability distribution in x. From the time evolution of the Wigner distribution, equation (3.3), for a free particle, the derivative of the Wigner distribution over x is related to the time derivative of the distribution. Moreover,

$$\frac{\frac{\partial}{\partial x}\int W_\psi(x,p)dp}{\int W_\psi(x,p)dp} = \frac{\frac{\partial}{\partial x}\left(|\psi(x)|^2\right)}{|\psi(x)|^2}, \tag{5.65}$$

and this corresponds to a quantity called the osmotic velocity. This concept appears in the stochastic interpretation of quantum mechanics [21] and is also linked to the de Broglie-Bohm interpretation [22].

In Section 2.5.2, the determination of the average trajectories of photons in a two-slit interferometer is presented. The weak value considered in the experiment is the transverse momentum pre-selected on the wavefunction and post-selected on the position. It is exactly the weak value computed here, described in phase space. This experiment shows trajectories that are compatible with the de Broglie-Bohm theory.

5.6 von Neumann model of measurement in phase space

5.6.1 Motivation

The von Neumann scheme allows to define weak measurements by taking into account the measuring device in the description. Describing the process in phase space is really interesting because it provides a phase space representation of the meter and the system states, at each step. It helps to understand how the Wigner distribution evolves and how the two parts interact with each other. Describing the process in phase space also allows to check that the shift of the meter state while performing the measurement are also present in the phase space description.

The von Neumann process is first reproduced for a general measurement without post-selection. It is then developed for a weak coupling, with the additional post-selection. It is shown that the shift is effectively proportional to the weak value, as expected.

5.6.2 von Neumann measurement

The usual von Neumann measurement (Proposition 2.1) is described in phase space. It is first realised for a measurement without post-selection. In the following proposition, the operator $\hat{A}$ has a discrete spectrum but the phase space is still continuous. For example, the energy operator has a discrete spectrum in the harmonic oscillator but it is still defined in a continuous phase space.

Proposition 5.7: von Neumann measurement

Consider the joint phase space (x_s, p_s, x_m, p_m). The initial system distribution is $W_{\psi_i}(x_s, p_s)$ and the initial meter distribution is $W_\phi(x_m, p_m)$. The interaction couples the system operator $\hat{A}$ (of eigenvalues a_j and eigenvectors $|a_j\rangle$) with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total coupling strength is $\gamma = \int_0^T g(t)dt$. The final joint distribution is

$$W_\Psi(x_s, p_s, x_m, p_m) = \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar} p_m (a_j - a_k)} W_{a_j, a_k}(x_s, p_s) W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right).$$

The information available to the experimenter is the reduced state of the meter,

$$\iint W_\Psi(x_s, p_s, x_m, p_m) dx_s dp_s = \sum_j |\alpha_j|^2 W_\phi \left(x_m - \gamma a_j, p_m \right).$$

Proof. The initial joint distribution is a separable state described by the product of the Wigner distribution of the meter and the system initial states,

$$\begin{aligned} W(x_s, p_s, x_m, p_m) &= W_{\psi_i}(x_s, p_s) W_\phi(x_m, p_m) && (5.66) \\ &= \frac{1}{h^2} \int e^{-\frac{i}{\hbar} p_s y_s} \left\langle x_s + \frac{y_s}{2} \middle| \psi_i \right\rangle \left\langle \psi_i \middle| x_s - \frac{y_s}{2} \right\rangle dy_s \\ &\quad \int e^{-\frac{i}{\hbar} p_m y_m} \left\langle x_m + \frac{y_m}{2} \middle| \phi \right\rangle \left\langle \phi \middle| x_m - \frac{y_m}{2} \right\rangle dy_m && (5.67) \\ &= \frac{1}{h^2} \iint e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \middle| \psi_i, \phi \right\rangle \\ &\quad \left\langle \psi_i, \phi \middle| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m. && (5.68) \end{aligned}$$

The interaction operates through the evolution operator,

$$\hat{U} = e^{-\frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}_m}, \tag{5.69}$$

with the same hypothesis on the Hamiltonian as in Proposition 2.1. After the interaction, the Wigner distribution $W_\Psi(x_s, p_s, x_m, p_m)$ is

$$\begin{aligned} W_\Psi(x_s, p_s, x_m, p_m) = \frac{1}{h^2} \iint e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} &\left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \middle| e^{-\frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}_m} \middle| \psi_i, \phi \right\rangle \\ &\left\langle \psi_i, \phi \middle| e^{\frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}_m} \middle| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m. && (5.70) \end{aligned}$$

Consider the decomposition of the initial system state in the basis of the operator $\hat{A}$,

$$|\psi_i\rangle = \sum_j \alpha_j |a_j\rangle. \tag{5.71}$$

The Wigner distribution becomes

$$\begin{aligned} W_\Psi(x_s, p_s, x_m, p_m) &= \frac{1}{h^2} \iint \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \middle| e^{-\frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}_m} \middle| a_j, \phi \right\rangle \\ &\quad \left\langle a_k, \phi \middle| e^{\frac{i}{\hbar}\gamma \hat{A} \otimes \hat{p}_m} \middle| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m && (5.72) \\ &= \frac{1}{h^2} \iint \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \middle| e^{-\frac{i}{\hbar}\gamma a_j \hat{p}_m} \middle| a_j, \phi \right\rangle \\ &\quad \left\langle a_k, \phi \middle| e^{\frac{i}{\hbar}\gamma a_k \hat{p}_m} \middle| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m && (5.73) \end{aligned}$$

$$= \frac{1}{h^2} \sum_{jk} \alpha_j \overline{\alpha}_k \int e^{-\frac{i}{\hbar} p_s y_s} \left\langle x_s + \frac{y_s}{2} \middle| a_j \right\rangle \left\langle a_k \middle| x_s - \frac{y_s}{2} \right\rangle dy_s$$
$$\int e^{-\frac{i}{\hbar} p_m y_m} \left\langle x_m + \frac{y_m}{2} \middle| e^{-\frac{i}{\hbar}\gamma a_j \hat{p}_m} \middle| \phi \right\rangle \left\langle \phi \middle| e^{\frac{i}{\hbar}\gamma a_k \hat{p}_m} \middle| x_m - \frac{y_m}{2} \right\rangle dy_m \tag{5.74}$$

$$= \frac{1}{h^2} \sum_{jk} \alpha_j \overline{\alpha}_k \int e^{-\frac{i}{\hbar} p_s y_s} \left\langle x_s + \frac{y_s}{2} \middle| a_j \right\rangle \left\langle a_k \middle| x_s - \frac{y_s}{2} \right\rangle dy_s$$
$$\int e^{-\frac{i}{\hbar} p_m y_m} \left\langle x_m + \frac{y_m}{2} - \gamma a_j \middle| \phi \right\rangle \left\langle \phi \middle| x_m - \frac{y_m}{2} - \gamma a_k \right\rangle dy_m. \tag{5.75}$$

The first integral corresponds to a cross-Wigner distribution between the two eigenvectors a_j and a_k of $\hat{A}$,

$$\frac{1}{h} \int e^{-\frac{i}{\hbar} p_s y_s} \left\langle x_s + \frac{y_s}{2} \middle| a_j \right\rangle \left\langle a_k \middle| x_s - \frac{y_s}{2} \right\rangle dy_s = W_{a_j,a_k}(x_s, p_s). \tag{5.76}$$

The second integral is a shifted Wigner distribution of the meter. Indeed, by the change of variables $x'_m = x_m - \frac{\gamma}{2}(a_j + a_k)$ and $y'_m = y_m - \gamma(a_j - a_k)$, we have

$$\int e^{-\frac{i}{\hbar} p_m y_m} \left\langle x_m + \frac{y_m}{2} - \gamma a_j \middle| \phi \right\rangle \left\langle \phi \middle| x_m - \frac{y_m}{2} - \gamma a_k \right\rangle dy_m \tag{5.77}$$

$$= \int e^{-\frac{i}{\hbar}\gamma p_m (a_j - a_k)} e^{-\frac{i}{\hbar} p_m y'_m} \left\langle x'_m + \frac{\gamma}{2}(a_j + a_k) + \frac{y'_m}{2} + \frac{\gamma}{2}(a_j - a_k) - \gamma a_j \middle| \phi \right\rangle$$
$$\left\langle \phi \middle| x'_m + \frac{\gamma}{2}(a_j + a_k) - \frac{y'_m}{2} - \frac{\gamma}{2}(a_j - a_k) - \gamma a_k \right\rangle dy'_m \tag{5.78}$$

$$= e^{-\frac{i}{\hbar} p_m (a_j - a_k)} \int e^{-\frac{i}{\hbar} p_m y'_m} \left\langle x'_m + \frac{y'_m}{2} \middle| \phi \right\rangle \left\langle \phi \middle| x'_m - \frac{y'_m}{2} \right\rangle dy'_m \tag{5.79}$$

$$= e^{-\frac{i}{\hbar} p_m (a_j - a_k)} h W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right). \tag{5.80}$$

The Wigner distribution of the joint state is therefore

$$W_\Psi(x_s, p_s, x_m, p_m) = \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar} p_m (a_j - a_k)} W_{a_j,a_k}(x_s, p_s) W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right). \tag{5.81}$$

The experimenter observes only the meter and the system part is traced out. This is done, in phase space, by integrating the coordinates of the system x_s and p_s,

$$\iint W_\Psi(x_s, p_s, x_m, p_m) dx_s dp_s = \iint \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar} p_m (a_j - a_k)} W_{a_j,a_k}(x_s, p_s) W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right) dx_s dp_s \tag{5.82}$$

$$= \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar} p_m (a_j - a_k)} W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right) \iint W_{a_j,a_k}(x_s, p_s) dx_s dp_s \tag{5.83}$$

$$= \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar} p_m (a_j - a_k)} W_\phi \left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m \right) \langle a_j | a_k \rangle \tag{5.84}$$

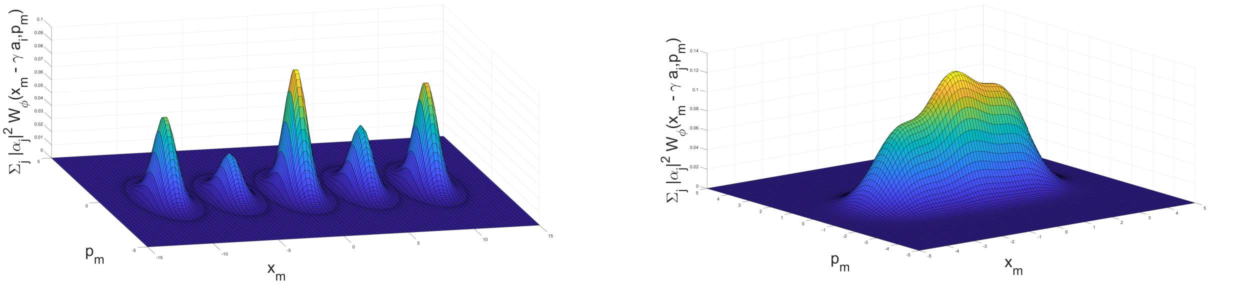
$$= \sum_j |\alpha_j|^2 W_\phi \left(x_m - \gamma a_j, p_m \right). \tag{5.85}$$

□

The joint Wigner distribution describes the entanglement between the system and the meter, depending on the eigenvalues of the operator $\hat{A}$ and the decomposition of $|\psi_i\rangle$. From the point of view of the experimenter, the resulting Wigner distribution of the meter describes a mixture of shifted meter distributions, proportional to each of the eigenvalues of the operator $\hat{A}$.

5.6.3 Example: Coupled Harmonic Oscillators

Consider the situation of the coupled harmonic oscillators, 2.3.2. Taking the ground state of the oscillator in the x direction as the meter initial state ϕ, the Wigner distribution of the meter is the equation (3.21). The representation of the meter distribution after the measurement (Proposition 5.7) is shown in Figure 11. If the interaction is strong, the shifted Wigner distributions are separated and the system is projected on an eigenvector, when performing the measurement of the meter (figure 11a). For a weak interaction, the distributions are not separated and the system is not projected on an eigenvector of the operator $\hat{A}$ (figure 11b). The experimenter observe a merging of the Gaussian distributions linked to each eigenvalue.



(a) Wigner distribution of the meter for a strong von Neumann measurement.

(b) Wigner distribution of the meter for a weak von Neumann measurement.

Figure 11: Meter distribution after a von Neumann non post-selected measurement.

5.6.4 Weak post-selected measurement

The von Neumann model can also be studied in phase space to describe a weak measurement with post-selection, in parallel to Proposition 2.2. For a weak interaction, this gives a shift of the meter Wigner distribution proportional to the weak value.

Proposition 5.8: Weak post-selected von Neumann measurement

Consider the joint phase space (x_s, p_s, x_m, p_m). The initial system distribution is $W_{\psi_i}(x_s, p_s)$ and the initial meter distribution is $W_\phi(x_m, p_m)$. The post-selected system distribution is $W_{\psi_f}(x_s, p_s)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$ and the weak value A_w is assumed to be real. The final (non normalised) post-selected joint distribution associated to the state $|\Psi_j\rangle$ (equation 2.14) is, to the first order in γ,

$$W_{\Psi_j}(x_s, p_s, x_m, p_m) = |\langle\psi_f|\psi_s\rangle|^2 W_{\psi_f}(x_s, p_s) W_\phi(x_m - \gamma A_w, p_m).$$

Proof. The initial joint distribution and the evolution operator are the same as for a strong coupling. The interaction corresponds to the application of the evolution operator and the post-selection is realised using the projection operator $\hat{\Pi}_f = |\psi_f\rangle\langle\psi_f| \otimes \hat{\mathbb{1}}_m$. The projection is not unitary, so the resulting Wigner distribution is not normalised. The final joint Wigner distribution is obtained in the same way than the equation (5.70),

$$W_{\Psi_j}(x_s, p_s, x_m, p_m) = \frac{1}{h^2} \iint e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \right| \hat{\Pi}_f e^{-\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}_m} \left| \psi_i, \phi \right\rangle \left\langle \psi_i, \phi \right| e^{\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}_m} \hat{\Pi}_f^\dagger \left| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m \tag{5.86}$$

$$= \frac{1}{h^2} \iint e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2} \middle| \psi_f \right\rangle \left\langle \psi_f, x_m + \frac{y_m}{2} \right| e^{-\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}_m} \left| \psi_i, \phi \right\rangle \left\langle \psi_i, \phi \right| e^{\frac{i}{\hbar}\gamma \hat{A}\otimes\hat{p}_m} \left| \psi_f, x_m - \frac{y_m}{2} \right\rangle \left\langle \psi_f \middle| x_s - \frac{y_s}{2} \right\rangle dy_s dy_m. \tag{5.87}$$

The bracket term $\langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}} |\psi_i\rangle$, in the system space, is

$$\langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m} |\psi_i\rangle = \mathrm{Tr}\left(|\psi_i\rangle\langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m}\right) = \iint W_{\psi_i,\psi_f}(x'_s, p'_s) \widetilde{e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m}} dx'_s dp'_s. \tag{5.88}$$

The Weyl transform of the exponential evolution operator must be evaluated. For a weak interaction, γ is small. Therefore, the exponential can be approximated to the first order as

$$e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m} \approx 1 - \frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m, \tag{5.89}$$

so that

$$\widetilde{e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m}} \approx \int e^{-\frac{i}{\hbar}p'_s y_s} \left\langle x'_s + \frac{y_s}{2} \right| \left(1 - \frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m\right) \left| x'_s - \frac{y_s}{2} \right\rangle dy_s \tag{5.90}$$

$$= 1 - \frac{i}{\hbar}\gamma \int e^{-\frac{i}{\hbar}p'_s y_s} \left\langle x'_s + \frac{y_s}{2} \right| \hat{A} \left| x'_s - \frac{y_s}{2} \right\rangle dy_s\ \hat{p}_m = 1 - \frac{i}{\hbar}\gamma\tilde{A}\hat{p}_m. \tag{5.91}$$

This gives for the bracket term, applying the approximation of the exponential in reverse,

$$\langle\psi_f| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m} |\psi_i\rangle = \iint W_{\psi_i,\psi_f}(x'_s, p'_s) dx'_s dp'_s - \frac{i}{\hbar}\gamma \iint W_{\psi_i,\psi_f}(x'_s, p'_s)\tilde{A} dx'_s dp'_s\ \hat{p}_m \tag{5.92}$$

$$= \iint W_{\psi_i,\psi_f}(x'_s, p'_s) dx'_s dp'_s \left(1 - \frac{i}{\hbar}\gamma A_w \hat{p}_m\right) = \langle\psi_f|\psi_i\rangle e^{-\frac{i}{\hbar}\gamma A_w \hat{p}_m}, \tag{5.93}$$

by setting

$$A_w = \frac{\iint W_{\psi_s,\psi_f}(x_s, p_s)\tilde{A}(x_s, p_s) dx_s dp_s}{\iint W_{\psi_s,\psi_f}(x_s, p_s) dx_s dp_s} \tag{5.94}$$

as the weak value. For simplicity, it is assumed to be real in the remaining of the calculations. The complex case is dealt with later. The other bracket term in equation (5.87) can be treated in the same way to obtain

$$\langle\psi_i| e^{\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m} |\psi_f\rangle = \langle\psi_i|\psi_f\rangle e^{\frac{i}{\hbar}\gamma\overline{A}_w\hat{p}_m}. \tag{5.95}$$

Finally, the joint Wigner distribution after post-selection becomes

$$
\begin{aligned}
W_{\Psi_j}(x_s,p_s,x_m,p_m) = \frac{1}{h^2}\iint & e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}\middle|\psi_f\right\rangle \langle\psi_f|\psi_i\rangle \left\langle x_m + \frac{y_m}{2}\middle| e^{-\frac{i}{\hbar}\gamma A_w \hat{p}_m} \middle| \phi\right\rangle \\
& \left\langle \phi\middle| e^{\frac{i}{\hbar}\gamma \overline{A}_w \hat{p}_m} \middle| x_m - \frac{y_m}{2}\right\rangle \langle\psi_i|\psi_f\rangle \left\langle\psi_f\middle|x_s - \frac{y_s}{2}\right\rangle dy_s dy_m \qquad (5.96)\\
= \frac{1}{h^2}\iint & e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \left\langle x_s + \frac{y_s}{2}\middle|\psi_f\right\rangle \langle\psi_f|\psi_i\rangle \left\langle x_m + \frac{y_m}{2} - \gamma A_w\middle|\phi\right\rangle \\
& \left\langle\phi\middle|x_m - \frac{y_m}{2} - \gamma A_w\right\rangle \langle\psi_i|\psi_f\rangle \left\langle\psi_f\middle|x_s - \frac{y_s}{2}\right\rangle dy_s dy_m \qquad (5.97)\\
= \frac{1}{h^2}|\langle\psi_f|\psi_i\rangle|^2 & \int e^{-\frac{i}{\hbar}p_s y_s} \left\langle x_s + \frac{y_s}{2}\middle|\psi_f\right\rangle \left\langle\psi_f\middle|x_s - \frac{y_s}{2}\right\rangle dy_s \\
& \int e^{-\frac{i}{\hbar}p_m y_m} \left\langle x_m + \frac{y_m}{2} - \gamma A_w\middle|\phi\right\rangle \left\langle\phi\middle|x_m - \frac{y_m}{2} - \gamma A_w\right\rangle dy_m \qquad (5.98)\\
= |\langle\psi_f|\psi_i\rangle|^2 & W_{\psi_f}(x_s,p_s) W_\phi(x_m - \gamma A_w, p_m). \qquad (5.99)
\end{aligned}
$$

□

The shift of the Wigner distribution of the meter state is coherent with the shift obtained in the usual von Neumann scheme (Proposition 2.2). The system is also rightfully in the post-selected state. Moreover, it can be readily seen that the amplitude of the (not normalised) Wigner distribution is given by $|\langle\psi_f|\psi_i\rangle|^2$. When a weak value is chosen to have an amplification effect, the pre- and post-selection are taken nearly orthogonal. The scalar product is therefore close to zero, so the amplitude of the Wigner distribution will be small in that case.

In this case, contrary to Proposition 5.7, there is no need to perform a partial trace. Indeed, the information accessible to the experimenter is directly readable in the result. This comes from the post-selection, that has disentangled the system and the meter and no superposition is present anymore. The state is therefore directly separable.

To find the shift in the x_m and p_m coordinates for a complex weak value, the Lemma 2.1 is shown to hold in the phase space.

Lemma 5.1: Average value of the meter

Let $W_{\psi_i}(x_s,p_s)$ and $W_{\psi_f}(x_s,p_s)$ be the pre- and post-selected distributions of the weak measurement, respectively. The initial meter distribution is $W_\phi(x_m,p_m)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A}\otimes\hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0,T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. The post-selected joint state is $|\Psi_j\rangle$ with Wigner distribution $W_{\Psi_j}(x_s,p_s,x_m,p_m)$. For any observable $\hat{M}$ on the meter space, the average value in the post-selected joint state is

$$\frac{\langle\Psi_j|\,\hat{\mathbb{1}}_s\otimes\hat{M}\,|\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle}=\frac{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)\tilde{M}(x_m,p_m)dx_mdp_m}{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)dx_mdp_m}$$
$$=\langle\phi|\,\hat{M}\,|\phi\rangle+\frac{i}{\hbar}\gamma\operatorname{Re}(A_w)\,\langle\phi|\,[\hat{p}_m,\hat{M}]\,|\phi\rangle+\frac{1}{\hbar}\gamma\operatorname{Im}(A_w)\,\langle\phi|\,\{\hat{p}_m,\hat{M}\}\,|\phi\rangle$$
$$-\frac{2}{\hbar}\gamma\operatorname{Im}(A_w)\,\langle\phi|\,\hat{M}\,|\phi\rangle\,\langle\phi|\,\hat{p}_m\,|\phi\rangle\,.$$

Proof. Since we work with the operator $\hat{p}$, it is more convenient to write the Wigner distribution in the p basis instead of the x one. The total joint state, up to the first order, is calculated in the same way than to obtain (5.96). By approximating the exponential, we find

$$W_{\Psi_j}(x_s,p_s,x_m,p_m)=\frac{1}{h^2}\iint e^{\frac{i}{\hbar}(x_su_s+x_mu_m)}\left\langle p_s+\frac{u_s}{2}\middle|\psi_f\right\rangle\langle\psi_f|\psi_i\rangle\left\langle p_m+\frac{u_s}{2}\middle|\left(1-\frac{i}{\hbar}\gamma A_w\hat{p}_m\right)\middle|\phi\right\rangle$$
$$\left\langle\phi\middle|\left(1+\frac{i}{\hbar}\gamma\overline{A}_w\hat{p}_m\right)\middle|p_m-\frac{u_s}{2}\right\rangle\langle\psi_i|\psi_f\rangle\left\langle\psi_f\middle|p_s-\frac{u_s}{2}\right\rangle du_sdu_m \tag{5.100}$$

$$=\frac{1}{h}W_{\psi_f}(x_s,p_s)|\langle\psi_f|\psi_i\rangle|^2\int e^{\frac{i}{\hbar}x_mu_m}\left\langle p_m+\frac{u_s}{2}\middle|\left(1-\frac{i}{\hbar}\gamma A_w\hat{p}_m\right)\middle|\phi\right\rangle$$
$$\left\langle\phi\middle|\left(1+\frac{i}{\hbar}\gamma\overline{A}_w\hat{p}_m\right)\middle|p_m-\frac{u_s}{2}\right\rangle du_m \tag{5.101}$$

$$=\frac{1}{h}W_{\psi_f}(x_s,p_s)|\langle\psi_f|\psi_i\rangle|^2$$
$$\left[\int e^{\frac{i}{\hbar}x_mu_m}\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle du_m\right. \tag{5.102}$$
$$-\frac{i}{\hbar}\gamma A_w\int e^{ix_mu_m/\hbar}\left\langle p_m+\frac{u_s}{2}\middle|\hat{p}_m\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle du_m \tag{5.103}$$
$$\left.+\frac{i}{\hbar}\gamma\overline{A}_w\int e^{\frac{i}{\hbar}x_mu_m}\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|\hat{p}_m\middle|p_m-\frac{u_s}{2}\right\rangle du_m\right]. \tag{5.104}$$

The two parentheses have been distributed and the term proportional to γ^2 is neglected. The first integral (5.102) is the Wigner distribution of the initial meter state, $W_\phi(x_m,p_m)$. We want to evaluate the normalised average value of the operator $\hat{M}$ in the final state, with respect to the meter coordinates. By the properties of the Wigner distribution (Proposition 3.1),

$$\frac{\langle\Psi_j|\,\hat{\mathbb{1}}_s\otimes\hat{M}\,|\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle}=\frac{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)\tilde{M}(x_m,p_m)dx_mdp_m}{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)dx_mdp_m}. \tag{5.105}$$

The denominator is first calculated. The double integration acts on each of the three integrals, (5.102),(5.103),(5.104). The first term (5.102), integrated, is proportional to

$$\iint W_\phi(x_m,p_m)dx_mdp_m=\langle\phi|\phi\rangle=1. \tag{5.106}$$

The application of the double integrals on the second term (5.103) is proportional to

$$\frac{1}{h}\left(\frac{-i}{\hbar}\gamma A_w\right)\iiint e^{\frac{i}{\hbar}x_m u_m}\left\langle p_m+\frac{u_s}{2}\right|\hat{p}_m\left|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle du_m dx_m dp_m \tag{5.107}$$

$$=\frac{1}{h}\left(\frac{-i}{\hbar}\gamma A_w\right)\iiint e^{\frac{i}{\hbar}x_m u_m}\left(p_m+\frac{u_m}{2}\right)\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle du_m dx_m dp_m \tag{5.108}$$

$$=\left(\frac{-i}{\hbar}\gamma A_w\right)\iint \delta(u_m)\left(p_m+\frac{u_m}{2}\right)\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle du_m dp_m \tag{5.109}$$

$$=\left(\frac{-i}{\hbar}\gamma A_w\right)\int p_m\left\langle p_m\middle|\phi\right\rangle\left\langle\phi\middle|p_m\right\rangle dp_m=\left(\frac{-i}{\hbar}\gamma A_w\right)\int\left\langle\phi\middle|p_m\right\rangle p_m\left\langle p_m\middle|\phi\right\rangle dp_m=\left(\frac{-i}{\hbar}\gamma A_w\right)\left\langle\phi\right|\hat{p}_m\left|\phi\right\rangle. \tag{5.110}$$

A similar development can be made for the third part, (5.104). Finally, the denominator of equation (5.105) is

$$\iint W_\alpha(x_s,p_s,x_m,p_m)dx_m dp_m=W_{\psi_f}(x_s,p_s)|\left\langle\psi_f\middle|\psi_i\right\rangle|^2\left(\left\langle\phi\middle|\phi\right\rangle-\frac{i}{\hbar}\gamma A_w\left\langle\phi\right|\hat{p}_m\left|\phi\right\rangle+\frac{i}{\hbar}\gamma\overline{A}_w\left\langle\phi\right|\hat{p}_m\left|\phi\right\rangle\right). \tag{5.111}$$

The numerator of equation (5.105) is now studied. We first need the Weyl transform of the operator $\hat{M}$ (in the p basis),

$$\tilde{M}(x_m,p_m)=\int e^{\frac{i}{\hbar}x_m v_m}\left\langle p_m+\frac{v_m}{2}\right|\hat{M}\left|p_m-\frac{v_m}{2}\right\rangle dv_m. \tag{5.112}$$

Again, three different parts of the numerator are present because of the three integrals (5.102),(5.103),(5.104) in the definition of the Wigner distribution $W_\alpha(x_s,p_s,x_m,p_m)$. The first one (5.102), averaged on the operator $\hat{M}$, is proportional to

$$\iint W_\phi(x_m,p_m)\tilde{M}(x_m,p_m)dx_m dp_m=\mathrm{Tr}\left(\hat{\rho}_\phi\hat{M}\right)=\left\langle\phi\right|\hat{M}\left|\phi\right\rangle \tag{5.113}$$

by the properties of the Wigner distribution. It is the average value of the operator $\hat{M}$, in the initial meter state. The second term (5.103), once averaged, is proportional to

$$\left(-\frac{i}{\hbar}A_w\right)\frac{1}{h}\iiiint e^{\frac{i}{\hbar}x_m(u_m+v_m)}\left\langle p_m+\frac{u_s}{2}\right|\hat{p}_m\left|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle$$
$$\left\langle p_m+\frac{v_m}{2}\right|\hat{M}\left|p_m-\frac{v_m}{2}\right\rangle du_m dv_m dx_m dp_m \tag{5.114}$$

$$=\left(-\frac{i}{\hbar}A_w\right)\iiint\delta(u_m+v_m)\left(p_m+\frac{u_m}{2}\right)\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle$$
$$\left\langle p_m+\frac{v_m}{2}\right|\hat{M}\left|p_m-\frac{v_m}{2}\right\rangle du_m dv_m dp_m \tag{5.115}$$

$$=\left(-\frac{i}{\hbar}A_w\right)\iint\left(p_m+\frac{u_m}{2}\right)\left\langle p_m+\frac{u_s}{2}\middle|\phi\right\rangle\left\langle\phi\middle|p_m-\frac{u_s}{2}\right\rangle$$
$$\left\langle p_m-\frac{u_s}{2}\right|\hat{M}\left|p_m+\frac{u_s}{2}\right\rangle du_m dp_m \tag{5.116}$$

$$=\left(-\frac{i}{\hbar}A_w\right)\iint p\left\langle p\middle|\phi\right\rangle\left\langle\phi\middle|w\right\rangle\left\langle w\right|\hat{M}\left|p\right\rangle dp dw \qquad w=p_m-\frac{u_m}{2},p=p_m+\frac{u_m}{2} \tag{5.117}$$

$$=\left(-\frac{i}{\hbar}A_w\right)\int\left\langle\phi\right|\hat{M}\left|p\right\rangle p\left\langle p\middle|\phi\right\rangle dp=\left(-\frac{i}{\hbar}A_w\right)\left\langle\phi\right|\hat{M}\hat{p}_m\left|\phi\right\rangle. \tag{5.118}$$

A similar development can be made for the third part, equation (5.104). The numerator of equation (5.105) is therefore

$$\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)\tilde{M}(x_m,p_m)dx_m dp_m = W_{\psi_f}(x_s,p_s)|\langle\psi_f|\psi_i\rangle|^2 \left(\langle\phi|\hat{M}|\phi\rangle - \frac{i}{\hbar}\gamma A_w\langle\phi|\hat{M}\hat{p}_m|\phi\rangle + \frac{i}{\hbar}\gamma\overline{A}_w\langle\phi|\hat{p}_m\hat{M}|\phi\rangle\right). \tag{5.119}$$

Putting the two results in equation (5.105) gives the average value of the operator $\hat{M}$,

$$\frac{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)\tilde{M}(x_m,p_m)dx_m dp_m}{\iint W_{\Psi_j}(x_s,p_s,x_m,p_m)dx_m dp_m} = \frac{\langle\phi|\hat{M}|\phi\rangle - \frac{i}{\hbar}\gamma A_w\langle\phi|\hat{M}\hat{p}_m|\phi\rangle + \frac{i}{\hbar}\gamma\overline{A}_w\langle\phi|\hat{p}_m\hat{M}|\phi\rangle}{\langle\phi|\phi\rangle - \frac{i}{\hbar}\gamma A_w\langle\phi|\hat{p}_m|\phi\rangle + \frac{i}{\hbar}\gamma\overline{A}_w\langle\phi|\hat{p}_m|\phi\rangle}. \tag{5.120}$$

This is the same result as in the proof of Proposition 2.1, in equation (2.26). The remaining of the calculations are identical, to find the final result. □

The lemma just proven gives the same results as Lemma 2.1. Therefore, it can be applied to get the shift in x_m and p_m, as has been done in Proposition 2.3. The shift of the Wigner distribution after a weak post-selected measurement can finally be obtained, for a complex weak value.

Proposition 5.9: Meter shift for a complex weak value

Let $W_{\psi_i}(x_s,p_s)$ and $W_{\psi_f}(x_s,p_s)$ be the pre- and post-selected distributions, respectively. The initial meter distribution is $W_\phi(x_m,p_m)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A}\otimes\hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0,T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. Then, the post-selected joint Wigner distribution is

$$W_{\Psi_j}(x_s,p_x,x_m,p_m) = |\langle\psi_f|\psi_i\rangle|^2 W_{\psi_f}(x_s,p_s)\, W_\phi\left(x_m - \gamma\,\mathrm{Re}\,A_w - \frac{m}{\hbar}\gamma\,\mathrm{Im}\,A_w\frac{d\,\mathrm{Var}_i(x_m)}{dt},\; p_m - \frac{2}{\hbar}\gamma\,\mathrm{Im}\,A_w\,\mathrm{Var}_i(p_m)\right).$$

Proof. This is an application of Lemma 5.1 for $\hat{M} = \hat{x}_m$ and $\hat{M} = \hat{p}_m$ and the translation property of the Wigner distribution. □

5.6.5 Example: Weakly Coupled Harmonic Oscillators

Consider the coupled harmonic oscillators of Section 2.3.2. The Proposition 5.9 gives the shift of the Wigner distribution of the meter. The weak values are already known from equation (5.34). The Wigner distribution of the meter is also known, equation (3.21). It is represented in Figure 12, for different values of the orthogonality parameter α. The variance σ_m is assumed to be constant so the Wigner distribution is

$$W_\phi\left(x-\gamma\frac{(1-\alpha)\sigma_s}{\alpha\sqrt{2}}+\frac{2}{\hbar}\gamma\frac{(1-\alpha)\hbar}{\alpha\sqrt{2}\sigma_s}\sigma_m, p_x\right). \tag{5.121}$$

Only the x coordinate is shifted by the measurement, so the p distribution is left unchanged. In Figure 12, four different values of α are taken and the Wigner distribution of the meter is shown for each case (a shift in the p distribution is artificially added for clarity of the representation). The amplitude parameter is taken into account, to show the low measurement probability for increasing amplification. The higher distribution, on the left, corresponds to α close to one. The shift of the distribution is therefore small but the amplitude is high. The last distribution, very small on the right, is obtained with α close to zero. The shift of the meter distribution is more important but the amplitude is low.

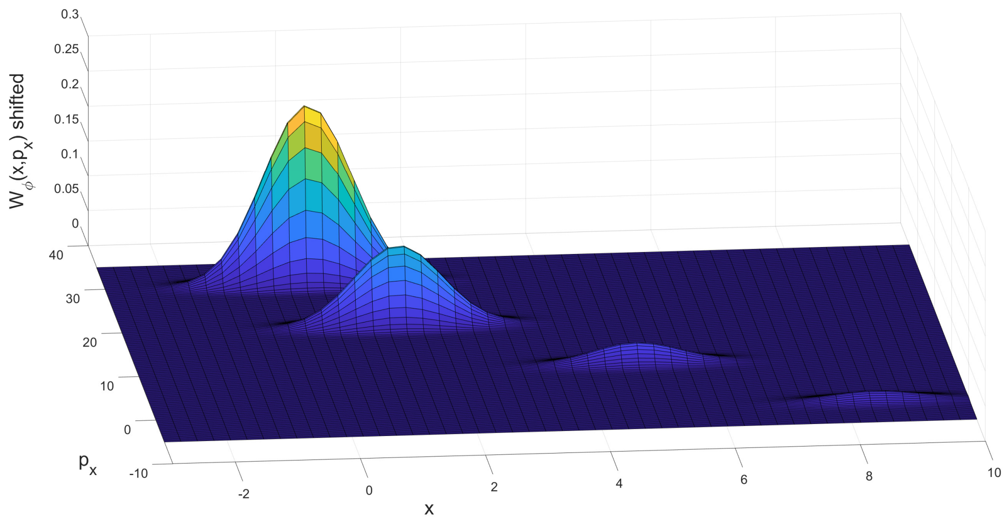


Figure 12: Shift of the meter Wigner distribution for different values of the orthogonality parameter α. Around $x = 9$ is the smaller value of α, hence showing a large amplification but a small height.

5.7 von Neumann measurement in curved configuration space

The complete von Neumann section can be generalised to a curved space. Indeed, the Wigner distribution and Weyl transforms are modified by a factor depending on the metric determinant (Proposition 4.3). By paying attention to the application of the displacement operator (that also displaces the metric determinant, equation (4.24)), the von Neumann process is generalised.

This development is interesting because, in the usual von Neumann process, a curvature of the space cannot be easily considered. However, in the phase space formalism, it is much more straightforward. This allows us to consider a measurement involving curved meter and/or system spaces. The curvature might be intrinsic, like the curvature of space-time, or it may come from constraints applied on a system. Some examples are the motion of a particle constrained on a sphere, or an optic measurement involving metamaterials.

The meter and the system are defined on independent Hilbert spaces. The phase spaces are therefore also different, and we assume that the degrees of freedom of the two spaces are completely independent. Therefore, the curvature is considered as decoupled between the two spaces. The metric determinant of the system space is $g_s(x_s, p_s)$ and the metric determinant of the meter space is $g_m(x_m, p_m)$. It makes sense to consider a curvature of the meter space because the system and meter are often two properties of a same particle, with a common global curvature.

First, the von Neumann measurement for a measurement without post-selection is reproduced in curved phase space.

Proposition 5.10: von Neumann measurement in curved space

Consider the curved joint phase space (x_s, p_s, x_m, p_m). The initial system distribution is $W^{g_s}_{\psi_i}(x_s, p_s)$ and the initial meter distribution is $W^{g_m}_{\phi}(x_m, p_m)$. The interaction couples the system operator $\hat{A}$ (of eigenvalues a_j and eigenvectors $|a_j\rangle$) with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total coupling strength is $\gamma = \int_0^T g(t)dt$. The final joint distribution is

$$W^{g_s g_m}_{\Psi}(x_s, p_s, x_m, p_m) = \sum_{jk} \alpha_j \overline{\alpha}_k e^{-\frac{i}{\hbar}p_m(a_j - a_k)} W^{g_s}_{a_j, a_k}(x_s, p_s) W^{g_m}_{\phi}\left(x_m - \frac{\gamma}{2}(a_j + a_k), p_m\right).$$

The information available to the experimenter is the reduced state of the meter,

$$\iint W^{g_s g_m}_{\Psi}(x_s, p_s, x_m, p_m) dx_s dp_s = \sum_j |\alpha_j|^2 W^{g_m}_{\phi}\left(x_m - \gamma a_j, p_m\right).$$

Proof. The proof is very similar to the proof of Proposition 5.7, with the metric determinant rightfully positioned. The initial joint distribution is

$$W^{g_s g_m}(x_s, p_s, x_m, p_m) = W^{g_s}_{\psi_i}(x_s, p_s) W^{g_m}_{\phi}(x_m, p_m) \tag{5.122}$$

$$= \frac{1}{h^2} \int \sqrt[4]{g_s\left(x_s + \frac{y_s}{2}\right) g_s\left(x_s - \frac{y_s}{2}\right)} e^{-\frac{i}{\hbar}p_s y_s} \left\langle x_s + \frac{y_s}{2} \middle| \psi_i \right\rangle \left\langle \psi_i \middle| x_s - \frac{y_s}{2} \right\rangle dy_s$$
$$\int \sqrt[4]{g_m\left(x_m + \frac{y_m}{2}\right) g_m\left(x_m - \frac{y_m}{2}\right)} e^{-\frac{i}{\hbar}p_m y_m} \left\langle x_m + \frac{y_m}{2} \middle| \phi \right\rangle \left\langle \phi \middle| x_m - \frac{y_m}{2} \right\rangle dy_m \tag{5.123}$$

$$= \frac{1}{h^2} \iint e^{-i(p_s y_s + p_m y_m)/\hbar} \sqrt[4]{g_s\left(x_s + \frac{y_s}{2}\right) g_s\left(x_s - \frac{y_s}{2}\right)}$$
$$\sqrt[4]{g_m\left(x_m + \frac{y_m}{2}\right) g_m\left(x_m - \frac{y_m}{2}\right)} \left\langle x_s + \frac{y_s}{2}, x_m + \frac{y_m}{2} \middle| \psi_i, \phi \right\rangle$$
$$\left\langle \psi_i, \phi \middle| x_s - \frac{y_s}{2}, x_m - \frac{y_m}{2} \right\rangle dy_s dy_m. \tag{5.124}$$

Up to the equation (5.74), the development of the proof is the same. Therefore, the Wigner distribution after the interaction is

$$
\begin{aligned}
W_\Psi^{g_s g_m}(x_s,p_s,x_m,p_m) = \frac{1}{h^2}\sum_{jk}\alpha_j\overline{\alpha}_k \int e^{-\frac{i}{\hbar}p_s y_s}\sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right)g_s\left(x_s-\frac{y_s}{2}\right)}\left\langle x_s+\frac{y_s}{2}\middle|a_j\right\rangle\left\langle a_k\middle|x_s-\frac{y_s}{2}\right\rangle dy_s \\
\int e^{-\frac{i}{\hbar}p_m y_m}\sqrt[4]{g_m\left(x_m+\frac{y_m}{2}\right)g_m\left(x_m-\frac{y_m}{2}\right)} \\
\left\langle x_m+\frac{y_m}{2}\middle| e^{-\frac{i}{\hbar}\gamma a_j\hat{p}_m}\left|\phi\right\rangle\left\langle\phi\right| e^{\frac{i}{\hbar}\gamma a_k\hat{p}_m}\middle|x_m-\frac{y_m}{2}\right\rangle dy_m.
\end{aligned}
\tag{5.125}
$$

The translation operator is then applied. From the equation (4.24), the metric determinant of the meter should also be shifted. Therefore, the resulting state is

$$
\begin{aligned}
W_\Psi^{g_s g_m}(x_s,p_s,x_m,p_m) = \frac{1}{h^2}\sum_{jk}\alpha_j\overline{\alpha}_k \int e^{-\frac{i}{\hbar}p_s y_s}\sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right)g_s\left(x_s-\frac{y_s}{2}\right)}\left\langle x_s+\frac{y_s}{2}\middle|a_j\right\rangle\left\langle a_k\middle|x_s-\frac{y_s}{2}\right\rangle dy_s \\
\int e^{-\frac{i}{\hbar}p_m y_m}\sqrt[4]{g_m\left(x_m+\frac{y_m}{2}-\gamma a_j\right)g_m\left(x_m-\frac{y_m}{2}-\gamma a_k\right)} \\
\left\langle x_m+\frac{y_m}{2}-\gamma a_j\middle|\phi\right\rangle\left\langle\phi\middle|x_m-\frac{y_m}{2}-\gamma a_k\right\rangle dy_m.
\end{aligned}
\tag{5.126}
$$

The development can then be pursued with no further difference except the metric factor, so that the joint Wigner distribution is

$$
W_\Psi^{g_s g_m}(x_s,p_s,x_m,p_m) = \sum_{jk}\alpha_j\overline{\alpha}_k e^{-\frac{i}{\hbar}p_m(a_j-a_k)} W^{g_s}_{a_j,a_k}(x_s,p_s) W^{g_m}_\phi\left(x_m-\frac{\gamma}{2}(a_j+a_k),p_m\right),
\tag{5.127}
$$

and the experimenter observes the partial traced distribution

$$
\iint W_\Psi^{g_s g_m}(x_s,p_s,x_m,p_m)dx_s dp_s = \sum_j |\alpha_j|^2 W^{g_m}_\phi\left(x_m-\gamma a_j,p_m\right).
\tag{5.128}
$$

□

The Wigner distributions, compared to a flat space, are different but the shift of the meter state is identical in curved space and in flat space.

5.7.1 Weak post-selected measurement

The same generalisation can be made for a weak post-selected von Neumann measurement in curved space.

Proposition 5.11: Weak von Neumann measurement in curved space

Consider the curved joint phase space (x_s,p_s,x_m,p_m). The initial system distribution is $W^{g_s}_{\psi_i}(x_s,p_s)$ and the initial meter distribution is $W^{g_m}_\phi(x_m,p_m)$. The post-selected system distribution is $W^{g_s}_{\psi_f}(x_s,p_s)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$
\hat{H}_{int} = g(t)\hat{A}\otimes\hat{p}_m,
$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$ and the weak value A_w is assumed to be real. The final post-selected joint distribution is

$$W_{\Psi_j}^{g_s g_m}(x_s, p_s, x_m, p_m) = |\langle\psi_f|\psi_s\rangle|^2 W_{\psi_f}^{g_s}(x_s, p_s) W_\phi^{g_m}(x_m - \gamma A_w, p_m).$$

The proof is very similar to the proof of Proposition 5.8, and is given in Appendix F. For a weak measurement, the shift in curved space is the same as in flat space. As in Proposition 5.8, it is not necessary to perform a partial trace because the post-selection destroyed the superposition.

The lemma 5.1 is also generalised to describe the average value of an observable of the meter, after weak measurement and post-selection, in curved space.

Lemma 5.2: Average value of the meter in curved space

Let $W_{\psi_i}^{g_s}(x_s, p_s)$ and $W_{\psi_f}^{g_s}(x_s, p_s)$ be the pre- and post-selected distributions of the weak measurement, respectively. The initial meter distribution is $W_\phi^{g_m}(x_m, p_m)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. The post-selected joint state is $|\Psi_j\rangle$ with Wigner distribution $W_{\Psi_j}^{g_s g_m}(x_s, p_s, x_m, p_m)$. For any observable $\hat{M}$ on the meter space, the average value in the post-selected joint state is

$$\begin{aligned}
\frac{\langle\Psi_j|\hat{M}|\Psi_j\rangle}{\langle\Psi_j|\Psi_j\rangle} &= \frac{\iint W_{\Psi_j}^{g_s g_m}(x_s, p_s, x_m, p_m)\tilde{M}^{g_m}(x_m, p_m)dx_m dp_m}{\iint W_{\Psi_j}^{g_s g_m}(x_s, p_s, x_m, p_m)dx_m dp_m} \\
&= \langle\phi|\hat{M}|\phi\rangle + \frac{i}{\hbar}\gamma\,\mathrm{Re}(A_w)\langle\phi|[\hat{p}_m, \hat{M}]|\phi\rangle + \frac{1}{\hbar}\gamma\,\mathrm{Im}(A_w)\langle\phi|\{\hat{p}_m, \hat{M}\}|\phi\rangle \\
&\quad - \frac{2}{\hbar}\gamma\,\mathrm{Im}(A_w)\langle\phi|\hat{M}|\phi\rangle\langle\phi|\hat{p}_m|\phi\rangle.
\end{aligned}$$

Proof. The proof is identical to the proof of Lemma 5.1. Indeed, the development is realised in the p basis of the Weyl transform and the Wigner distribution and the equation (4.38) shows that the Wigner distribution is unchanged in the p distribution. □

For a complex weak value, applying the lemma gives the shift of the Wigner distribution in curved phase space.

Proposition 5.12: Meter shift for a complex weak value on curved space

Let $W_{\psi_i}^{g_s}(x_s, p_s)$ and $W_{\psi_f}^{g_s}(x_s, p_s)$ be the pre- and post-selected distributions, respectively. The initial meter distribution is $W_\phi^{g_m}(x_m, p_m)$. The interaction couples the system operator $\hat{A}$ with the momentum meter operator $\hat{p}_m$ through

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{p}_m,$$

with a coupling $g(t)$ depending on time and equal to zero out of the range $[0, T]$. The total weak coupling strength is $\gamma = \int_0^T g(t)dt$. Then, the post-selected joint Wigner distribution is

$$W_{\alpha}^{g_s g_m}(x_s, p_x, x_m, p_m) = |\langle\psi_f|\psi_i\rangle|^2 W_{\psi_f}^{g_s}(x_s, p_s) \\ W_{\phi}^{g_m}\left(x_m - \gamma \operatorname{Re} A_w - \frac{m}{\hbar}\gamma \operatorname{Im} A_w \frac{d\operatorname{Var}_i(x_m)}{dt},\right. \\ \left. p_m - \frac{2}{\hbar}\gamma \operatorname{Im} A_w \operatorname{Var}_i(p_m)\right)$$

Proof. This is a direct application of Proposition 2.3 and the translation property of the Wigner distribution. □

6 Conclusion and Outlooks

The physical interpretation of weak measurements and weak values is a complex task. That motivates the search for alternative ways to describe weak values and weak measurements, in a manner that is easier to understand physically. Since weak measurements are also involved in the description of many quantum paradoxes [11, 12, 25], such a task is also important to understand the meaning of these paradoxes.

An interesting tool to use in this context is the quantum phase space [16, 17, 18, 19, 30]. The classical physical reasoning takes place in phase space, so a description of the quantum theory in phase space is important to bring more insights into its meaning. This is why the Wigner distribution was introduced, by reasoning in terms of statistical physics at first [19]. This gives a representation of a state (in an abstract Hilbert space) into the phase space (x, p).

The concept of weak value has therefore been studied in phase space. It has been shown in the literature that it is obtained from the cross-Wigner distribution between the pre-selection and post-selection, interpreted as an interference occurring between the two states [31, 32]. The weak value is the average of the cross-Wigner distribution. This was extended to generalised phase spaces and to curved configuration spaces. The weak value can therefore be described in any phase space as an interference between the pre-selection and the post-selection represented in the given phase space.

Another interesting result is the determination, in phase space, of the weak value of the momentum, post-selected on the position. Its real part is proportional to the probability current while its imaginary part is linked to the osmotic velocity. These concepts are widely used in the stochastic interpretation of quantum mechanics [21], as well as in the de Broglie-Bohm theory [22]. Therefore, this gives a clear meaning to the weak value in the framework of these two interpretations. Moreover, this specific weak value appears in one of the most important weak experiment, the two-slit interferometer [10]. The average trajectories of the photons are obtained from this weak value, and the results also support the de Broglie-Bohm interpretation of quantum physics.

The von Neumann model is a description of quantum measurements taking into account the system and the measuring device, with a given coupling strength between the two [3, 4, 5]. This model allows to define weak as well as strong measurements and post-selected ones. The process, in all cases, has been reproduced in phase space. This makes a shift of the Wigner distribution appear, in the x and p coordinates, depending on the eigenvalues of the operator if the measurement is not post-selected, depending on the weak value if it is. The Wigner distribution can therefore be described, for the meter and for the system, at each step of the measurement. The experimenter, in a weak post-selected measurement, will observe the weak value.

The description of the von Neumann model in phase space has then been generalised to describe the process of measurement in curved configuration spaces. Interestingly, the weak value is the same in a flat or a curved space, so the shift is identical. Only the shape of the Wigner function itself is modified by the curvature. This can be used, for example, for a weak measurement of a particle constrained on the surface of a sphere. In practice, this could be useful to study rotating or vibrating molecules [40], or in transformation optics where the light propagates in a metamaterial as it would in a curved space.

Most of these concepts and results have been illustrated on the example of a bi-dimensional harmonic oscillator, coupled through a magnetic field. This example has the advantage to be easy to treat and to give intuitive representations of the concepts in phase space. Moreover, the ground state of the oscillator is a coherent state, interpreted as a state close to a classical one. This is useful to understand the generalised phase space from the point of view of generalised coherent states.

For a future work, the Wigner distribution should be evolved directly in phase space, through the time evolution equation of the Wigner distribution. It is here described before and after the measurement, however a description of the distribution during the measurement would be very interesting. However, the post-selection step is a part that should be difficult to treat in phase space.

Another outlook is to use the formalism of generalised phase space presented in this thesis to describe discrete systems, such as the spin. The phase space can then be discrete and the Wigner distribution is described differently [33, 34, 35]. This is important because many weak measurements involve the spin or the polarisation of a particle. Furthermore, the phase space of a given system is not uniquely defined. In this thesis, we used only the phase space constructed from the Wigner distribution but other phase space distributions on (x, p) exist and could provide insight into weak measurements.

Lastly, the weak value of the momentum post-selected on the position is hugely interesting. From its interpretation with concepts of the statistical and Bohmian interpretations of quantum physics, the study of this weak value can provide insights in the foundations of the quantum theory. Evidencing the links between weak measurements and the de Broglie-Bohm theory, in phase space, can be interesting to better understand these two concepts together.

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7 Appendices

Appendix A: Integral calculation

A particular integral I appears in multiple places of the thesis. In general, its form is

$$I = \int e^{-\frac{i}{\hbar}px} e^{-\frac{x^2}{a^2}} dx. \tag{7.1}$$

It can be evaluated by noting that

$$\frac{x^2}{a^2} + \frac{ipx}{\hbar} = \frac{x^2}{a^2} + \frac{ipx}{\hbar} - \frac{p^2a^2}{4\hbar^2} + \frac{p^2a^2}{4\hbar^2} = \left(\frac{x}{a} + \frac{ipa}{2\hbar}\right)^2 + \frac{p^2a^2}{4\hbar^2}, \tag{7.2}$$

so the integral becomes

$$I = e^{-\frac{p^2a^2}{4\hbar^2}} \int e^{-\left(\frac{x}{a}+\frac{ipa}{2\hbar}\right)^2} dx \tag{7.3}$$

$$= e^{-\frac{p^2a^2}{4\hbar^2}} \int e^{-y^2} a \, dy \qquad\qquad y = \frac{x}{a} + \frac{ipa}{2\hbar} \tag{7.4}$$

$$= e^{-\frac{p^2a^2}{4\hbar^2}} a \int e^{-y^2} dy. \tag{7.5}$$

The remaining integral is called the Gauss integral,

$$\int e^{-y^2} dy = \sqrt{\pi}. \tag{7.6}$$

Finally, we get

$$I = a\sqrt{\pi} e^{-\frac{p^2a^2}{4\hbar^2}}. \tag{7.7}$$

Appendix B: Fourier tranforms of the harmonic oscillator

The Fourier transform $\psi_0(p)$ of the ground state of the harmonic oscillator $\psi_0(x)$ is

$$\begin{aligned}
\psi_0(p) &= \frac{1}{\sqrt{h}} \int e^{-\frac{i}{\hbar} p_x x} \psi_0(x) dx && (7.8)\\
&= \frac{1}{\sqrt{h}} \int e^{-\frac{i}{\hbar} p_x x} \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-\frac{x^2}{2\sigma^2}} dx && (7.9)\\
&= \frac{1}{\sqrt{h}\sqrt[4]{\pi}\sqrt{\sigma}} \int e^{-\frac{i}{\hbar} p_x x} e^{-\frac{x^2}{2\sigma^2}} dx. && (7.10)
\end{aligned}$$

This integral is solved in Appendix A. Therefore, the distribution is

$$\psi_0(p) = \frac{1}{\sqrt{h}\sqrt[4]{\pi}\sqrt{\sigma}} \sqrt{2}\sigma\sqrt{\pi} e^{-\frac{p_x^2 2\sigma^2}{4\hbar^2}} = \frac{\sqrt{2\pi\sigma}}{\sqrt{h}\sqrt[4]{\pi}} e^{-\frac{p_x^2\sigma^2}{2\hbar^2}}. \tag{7.11}$$

The Fourier transform $\psi_1(p)$ of the ground state of the harmonic oscillator $\psi_1(x)$ is

$$\begin{aligned}
\psi_1(p) &= \frac{1}{\sqrt{h}} \int e^{-\frac{i}{\hbar} px} \psi_1(x) dx && (7.12)\\
&= \frac{1}{\sqrt{h}\sqrt[4]{\pi}} \sqrt{\frac{2}{\sigma}} \frac{1}{\sigma} \int x e^{-\frac{i}{\hbar} px} e^{-\frac{x^2}{2\sigma^2}} dx && (7.13)\\
&= \frac{1}{\sqrt{h}\sqrt[4]{\pi}\sigma_s} \sqrt{\frac{2}{\sigma}} \int x e^{-\left(\frac{x}{\sqrt{2}\sigma} + \frac{ip\sigma}{\sqrt{2}\hbar}\right)^2} dx e^{-\frac{p^2\sigma^2}{2\hbar^2}} && (7.14)\\
&= \frac{1}{\sqrt{h}\sqrt[4]{\pi}\sigma} \sqrt{\frac{2}{\sigma}} e^{-\frac{p^2\sigma^2}{2\hbar^2}} \sqrt{2}\sigma \int \left(\sqrt{2}\sigma x' - \frac{ip\sigma^2}{\hbar}\right) e^{-x'^2} dx' && (7.15)\\
&= \frac{2}{\sqrt{h\sigma}\sqrt[4]{\pi}} e^{-\frac{p^2\sigma^2}{2\hbar^2}} \left(\sqrt{2}\sigma \int x' e^{-x'^2} dx' - \frac{ip\sigma^2}{\hbar} \int e^{-x'^2} dx'\right) && (7.16)\\
&= -\frac{2ip\sigma^2\sqrt[4]{\pi}}{\sqrt{2\pi\hbar\sigma}\hbar} e^{-\frac{p^2\sigma^2}{2\hbar^2}}. && (7.17)
\end{aligned}$$

Appendix C: Calculation of the weak values (2.45) of the coupled harmonic oscillators

The numerator of the weak value y_w is

$$\langle\psi_f|\,\hat{y}\,|\psi_i\rangle = \int y\,\langle\psi_f|y\rangle\,\langle y|\psi_i\rangle\,dy \tag{7.18}$$

$$= \alpha\int y\,\langle\psi_0|y\rangle\,\langle y|\psi_0\rangle\,dy + (1-\alpha)\int y\,\langle\psi_1|y\rangle\,\langle y|\psi_0\rangle\,dy \tag{7.19}$$

$$= \frac{\alpha}{\sqrt{\pi}\sigma_s}\int ye^{-\frac{y^2}{\sigma_s^2}}dy + \frac{(1-\alpha)\sqrt{2}}{\sqrt{\pi}\sigma_s^2}\int y^2e^{-\frac{y^2}{\sigma_s^2}}dy \tag{7.20}$$

$$= \frac{(1-\alpha)\sqrt{2}}{\sqrt{\pi}\sigma_s^2}\sigma_s^3\int y'^2e^{-y'^2}dy' \qquad y' = \frac{y}{\sigma_s}. \tag{7.21}$$

where the integral of the first term is equal to zero because the argument is odd. The remaining integral is

$$\int y'^2e^{-y'^2}dy' = \frac{\sqrt{\pi}}{2}. \tag{7.22}$$

The numerator of the weak value becomes

$$\langle\psi_f|\,\hat{y}\,|\psi_i\rangle = (1-\alpha)\frac{\sigma_s}{\sqrt{2}}, \tag{7.23}$$

with the denominator $\langle\psi_f|\psi_i\rangle = \alpha$, so the weak value is

$$y_w = \frac{(1-\alpha)\sigma_s}{\alpha\sqrt{2}}. \tag{7.24}$$

The numerator of the weak value p_{yw} is

$$\langle\psi_f|\,\hat{p}_y\,|\psi_i\rangle = \int p_y\,\langle\psi_f|p_y\rangle\,\langle p_y|\psi_i\rangle\,dp_y \tag{7.25}$$

$$= \alpha\int p_y\,\langle\psi_0|p_y\rangle\,\langle p_y|\psi_0\rangle\,dp_y + (1-\alpha)\int p_y\,\langle\psi_1|p_y\rangle\,\langle p_y|\psi_0\rangle\,dp_y \tag{7.26}$$

$$= \frac{\alpha\sigma_s}{\hbar\sqrt{\pi}}\int p_ye^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dp + \frac{(1-\alpha)i2\sigma_s^2}{\sqrt{2\pi}\hbar^2}\int p_y^2e^{-\frac{p_y^2\sigma_s^2}{\hbar^2}}dp_y \tag{7.27}$$

$$= \frac{(1-\alpha)i2\sigma_s^2}{\sqrt{2\pi}\hbar^2}\frac{\hbar^3\sqrt{\pi}}{2\sigma_s^3} = \frac{i(1-\alpha)\hbar}{\sqrt{2}\sigma_s}, \tag{7.28}$$

so the weak value is

$$p_{yw} = \frac{i(1-\alpha)\hbar}{\alpha\sqrt{2}\sigma_s}. \tag{7.29}$$

Appendix D: Proof of Proposition 3.2

1. The distribution is recovered by integration. The first marginal is realised using the Wigner distribution in the x basis, and the second is done in the p basis. This gives

$$\int W(x,p)dp = \frac{1}{h}\iint e^{-\frac{i}{\hbar}py}\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle dydp \tag{7.30}$$

$$= \int \delta(y)\left\langle x+\frac{y}{2}\right|\hat{\rho}\left|x-\frac{y}{2}\right\rangle dy = \langle x|\,\hat{\rho}\,|x\rangle\,. \tag{7.31}$$

$$\int W(x,p)dx = \frac{1}{h}\iint e^{\frac{i}{\hbar}xu}\left\langle p+\frac{u}{2}\right|\hat{\rho}\left|p-\frac{u}{2}\right\rangle dudx \tag{7.32}$$

$$= \int \delta(u)\left\langle p+\frac{u}{2}\right|\hat{\rho}\left|p-\frac{u}{2}\right\rangle du = \langle p|\,\hat{\rho}\,|p\rangle\,. \tag{7.33}$$

2. The Weyl transform of the identity operator is $\tilde{\mathbb{1}} = 1$. Using this and Lemma 3.1, we have

$$\iint W(x,p)dxdp = \iint W(x,p)\tilde{\mathbb{1}}dxdp = \mathrm{Tr}\big(\hat{\rho}\hat{\mathbb{1}}\big) = \mathrm{Tr}(\hat{\rho}) = 1. \tag{7.34}$$

3. To show that the distribution is real, the complex conjugate of the distribution is shown to be equal to the distribution itself,

$$W^*(x,p) = \frac{1}{h}\int e^{\frac{i}{\hbar}py}\left\langle x-\frac{y}{2}\right|\hat{\rho}\left|x+\frac{y}{2}\right\rangle dy \qquad y=-y' \tag{7.35}$$

$$= \frac{1}{h}\int e^{-\frac{i}{\hbar}py'}\left\langle x+\frac{y'}{2}\right|\hat{\rho}\left|x-\frac{y'}{2}\right\rangle dy' = W(x,p). \tag{7.36}$$

4. To show that the Wigner distribution is not always positive, a counterexample is shown. Let's consider two density operators $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{\rho}_2 = |\psi_2\rangle\langle\psi_2|$. We have

$$\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) = \int \langle u|\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2|u\rangle\, du = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_1\rangle = |\,\langle\psi_1|\psi_2\rangle\,|^2. \tag{7.37}$$

Using Lemma 3.1, we also have

$$\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) = \frac{1}{h}\iint \tilde{\rho}_1(x,p)\tilde{\rho}_2(x,p)dxdp = h\iint W_1(x,p)W_2(x,p)dxdp. \tag{7.38}$$

If we assume that the two states are orthogonal, we obtain

$$\iint W_1(x,p)W_2(x,p)dxdp = |\,\langle\psi_1|\psi_2\rangle\,|^2 = 0. \tag{7.39}$$

The two functions cannot be zero on the whole space so at least one of them must take negative values in some regions of the phase space.

Appendix E: Proof of Proposition 4.2

2. For any $\Omega \in X$ and any operator $\hat{A}$,

$$F_A^{(s)}(\Omega) = \left(F_{A^\dagger}^{(s)}(\Omega)\right)^* \tag{7.40}$$

$$\Leftrightarrow \quad \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right) = \mathrm{Tr}\left(\hat{A}^\dagger\hat{\Delta}^{(s)}(\Omega)\right)^* \tag{7.41}$$

$$\Leftrightarrow \quad \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right) = \mathrm{Tr}\left((\hat{\Delta}^{(s)}(\Omega))^\dagger\hat{A}\right) \tag{7.42}$$

$$\Leftrightarrow \quad \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right) = \mathrm{Tr}\left(\hat{A}(\hat{\Delta}^{(s)}(\Omega))^\dagger\right). \tag{7.43}$$

It is valid for any $\hat{A}$, so

$$\hat{\Delta}^{(s)}(\Omega) = (\hat{\Delta}^{(s)}(\Omega))^\dagger. \tag{7.44}$$

3. For any operator $\hat{A}$,

$$\mathrm{Tr}\left(\hat{A}\right) = \mathrm{Tr}\left(\hat{A}\hat{\mathbb{1}}\right) = \int_X F_A^{(s)}(\Omega)d\mu(\Omega) = \int_X \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right)d\mu(\Omega) = \mathrm{Tr}\left(\hat{A}\int_X \hat{\Delta}^{(s)}(\Omega)d\mu(\Omega)\right). \tag{7.45}$$

It is valid for any $\hat{A}$, so

$$\int_X \hat{\Delta}^{(s)}(\Omega)d\mu(\Omega) = \hat{\mathbb{1}}.$$

4. For any operator $\hat{A}$ and any $\Omega \in X$, using the Proposition 4.1 and the Definition 4.5,

$$F_A^{(s)}(\Omega) = \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(\Omega)\right) = \int_X F_A^{s'}(\Omega')\,\mathrm{Tr}\left(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\right)d\mu(\Omega') \tag{7.46}$$

$$= \int_X \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{s'}(\Omega')\right)\mathrm{Tr}\left(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\right)d\mu(\Omega') \tag{7.47}$$

$$= \mathrm{Tr}\left(\hat{A}\int_X \hat{\Delta}^{s'}(\Omega')\,\mathrm{Tr}\left(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\right)d\mu(\Omega')\right). \tag{7.48}$$

It is valid for any $\hat{A}$, so

$$\hat{\Delta}^{(s)}(\Omega) = \int_X \hat{\Delta}^{s'}(\Omega')\,\mathrm{Tr}\left(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\right)d\mu(\Omega'). \tag{7.49}$$

The factor $\mathrm{Tr}\left(\hat{\Delta}^{(s)}(\Omega)\hat{\Delta}^{(-s')}(\Omega')\right)$ acts like a delta function on X.

5. For any operator $\hat{A}$ and any $\Omega \in X$,

$$F_{g\cdot A}^{(s)}(\Omega) = F_A^{(s)}(g^{-1}\Omega) \tag{7.50}$$

$$\Leftrightarrow \quad \mathrm{Tr}\left(T(g)\hat{A}T^{-1}(g)\hat{\Delta}^{(s)}(\Omega)\right) = \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(g^{-1}\Omega)\right) \tag{7.51}$$

$$\Leftrightarrow \quad \mathrm{Tr}\left(\hat{A}T^{-1}(g)\hat{\Delta}^{(s)}(\Omega)T(g)\right) = \mathrm{Tr}\left(\hat{A}\hat{\Delta}^{(s)}(g^{-1}\Omega)\right). \tag{7.52}$$

It is valid for any $\hat{A}$, so

$$\hat{\Delta}^{(s)}(g^{-1}\Omega) = T^{-1}(g)\hat{\Delta}^{(s)}(\Omega)T(g). \tag{7.53}$$

Appendix F: Proof of Proposition 5.11

The proof is very similar to the proof of Proposition 5.8. The joint Wigner distribution, after interaction and post-selection, is

$$\begin{aligned}W^{g_s g_m}_{\Psi_j}(x_s,p_s,x_m,p_m) = \frac{1}{h^2}\iint & e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right) g_s\left(x_s-\frac{y_s}{2}\right)} \\ & \sqrt[4]{g_m\left(x_m+\frac{y_m}{2}\right) g_m\left(x_m-\frac{y_m}{2}\right)} \left\langle x_s+\frac{y_s}{2}\middle|\psi_f\right\rangle \left\langle\psi_f\middle|x_s-\frac{y_s}{2}\right\rangle \\ & \left\langle \psi_f, x_m+\frac{y_m}{2}\middle| e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m}\middle|\psi_i,\phi\right\rangle \left\langle \psi_i,\phi\middle| e^{\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}_m}\middle|\psi_f, x_m-\frac{y_m}{2}\right\rangle dy_s dy_m.\end{aligned} \tag{7.54}$$

The bracket term can be evaluated in the same way as in flat space, to get

$$\langle\psi_f|\, e^{-\frac{i}{\hbar}\gamma\hat{A}\otimes\hat{p}}\,|\psi_i\rangle = \langle\psi_f|\psi_i\rangle\, e^{-\frac{i}{\hbar}\gamma A_w \hat{p}_m}, \tag{7.55}$$

by setting

$$A_w = \frac{\iint W^{g_s}_{\psi_i,\psi_f}(x_s,p_s)\tilde{A}^{g_s}(x_s,p_s)dx_s dp_s}{\iint W^{g_s}_{\psi_i,\psi_f}(x_s,p_s)dx_s dp_s} \tag{7.56}$$

as the weak value. For simplicity, the weak value is assumed to be real. Finally, the joint Wigner distribution in curved space after post-selection becomes

$$\begin{aligned}W^{g_s g_m}_{\Psi_j}(x_s,p_s,x_m,p_m) = \frac{1}{h^2}\iint & e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right) g_s\left(x_s-\frac{y_s}{2}\right)} \left\langle x_s+\frac{y_s}{2}\middle|\psi_f\right\rangle \\ & \sqrt[4]{g_m\left(x_m+\frac{y_m}{2}\right) g_m\left(x_m-\frac{y_m}{2}\right)} \langle\psi_f|\psi_i\rangle \left\langle x_m+\frac{y_m}{2}\middle| e^{-\frac{i}{\hbar}\gamma A_w\hat{p}_m}\middle|\phi\right\rangle \\ & \left\langle\phi\middle| e^{\frac{i}{\hbar}\gamma\overline{A}_w\hat{p}_m}\middle|x_m-\frac{y_m}{2}\right\rangle \langle\psi_i|\psi_f\rangle \left\langle\psi_f\middle|x_s-\frac{y_s}{2}\right\rangle dy_s dy_m \end{aligned} \tag{7.57}$$

$$\begin{aligned}= \frac{1}{h^2}\iint & e^{-\frac{i}{\hbar}(p_s y_s + p_m y_m)} \sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right) g_s\left(x_s-\frac{y_s}{2}\right)} \left\langle x_s+\frac{y_s}{2}\middle|\psi_f\right\rangle \\ & \sqrt[4]{g_m\left(x_m+\frac{y_m}{2}-\gamma A_w\right) g_m\left(x_m-\frac{y_m}{2}-\gamma A_w\right)} \langle\psi_f|\psi_i\rangle\langle\psi_i|\psi_f\rangle \\ & \left\langle x_m+\frac{y_m}{2}-\gamma A_w\middle|\phi\right\rangle \left\langle\phi\middle|x_m-\frac{y_m}{2}-\gamma A_w\right\rangle \left\langle\psi_f\middle|x_s-\frac{y_s}{2}\right\rangle dy_s dy_m\end{aligned} \tag{7.58}$$

$$\begin{aligned}= \frac{1}{h^2}|\langle\psi_f|\psi_i\rangle|^2 \int & e^{-\frac{i}{\hbar}p_s y_s} \sqrt[4]{g_s\left(x_s+\frac{y_s}{2}\right) g_s\left(x_s-\frac{y_s}{2}\right)} \\ & \left\langle x_s+\frac{y_s}{2}\middle|\psi_f\right\rangle \left\langle\psi_f\middle|x_s-\frac{y_s}{2}\right\rangle dy_s \\ \int & e^{-\frac{i}{\hbar}p_m y_m} \sqrt[4]{g_m\left(x_m+\frac{y_m}{2}-\gamma A_w\right) g_m\left(x_m-\frac{y_m}{2}-\gamma A_w\right)} \\ & \left\langle x_m+\frac{y_m}{2}-\gamma A_w\middle|\phi\right\rangle \left\langle\phi\middle|x_m-\frac{y_m}{2}-\gamma A_w\right\rangle dy_m\end{aligned} \tag{7.59}$$

$$= |\langle\psi_f|\psi_i\rangle|^2 W^{g_s}_{\psi_f}(x_s,p_s) W^{g_m}_{\phi}(x_m-\gamma A_w, p_m). \tag{7.60}$$