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Analysis and control of nonlinear infinite-dimensional systems

Application to chemical and wave processes

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Analysis and control of nonlinear infinite-dimensional systems: Application to chemical and wave processes

A thesis submitted by
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in Science

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Analysis and control of nonlinear infinite-dimensional systems: Application to chemical and wave processes

by Anthony HASTIR

Abstract: Nonlinear infinite-dimensional systems are nonlinear dynamical systems whose state components lie in an infinite-dimensional space, typically a function space. Such systems, which are also called distributed parameter systems, are ubiquitous in real-life since they are able to model many physical processes, going from conservative mechanical systems to dissipative phenomena. A lot of questions may arise when dealing with such classes of systems. For instance, the well-posedness in terms of existence and uniqueness of solutions as well as the study of the equilibria, their stability and their control are paramount steps when studying these dynamical systems. On the basis of the existing literature, we pay a particular attention to the existence, the uniqueness and the stability of equilibria and the control of nonlinear distributed parameter systems. In particular, as main contributions, we extend the classical approach that allows to deduce the stability of equilibria for a nonlinear system based on the stability of a corresponding linearized version of it. Using a new concept of differentiability for nonlinear operators which takes another space as the state space into account, we show how to guarantee local exponential stability or instability of the equilibria for the original nonlinear system. This is applied to the determination of the stability of the equilibria of a nonlinear plug-flow tubular reactor model with axial dispersion for which the temperature and the reactant concentration are considered as state variables. From a control point of view, the previous results are extended to the stabilization of equilibria of nonlinear infinite-dimensional systems. Thanks to this extension, we are able to identify a class of optimally controlled systems for which the required assumptions hold. As another contribution, we study the field of tracking control, and especially funnel control, which constitutes an appropriate tool for the output of a system to track a class of reference signals. As a main contribution on this topic, we extend the available results that allow to consider linear infinite-dimensional systems as internal dynamics to the nonlinear setting. We prove that a general class of nonlinear infinite-dimensional systems that satisfy some standard assumptions admits a differential relation between the input and the output that is conducive for funnel control. A large number of theoretical results in this thesis are illustrated by means of examples and numerical simulations, especially in the field of process control. The considered applications are related to chemical reactor models, damped wave equations and damped Sine-Gordon equations.

Analyse et commande de systèmes non linéaires en dimension infinie : Application aux procédés chimiques et ondulatoires

par Anthony HASTIR

Résumé : Les systèmes non linéaires en dimension infinie sont des systèmes dynamiques non linéaires dont les composantes d'état se trouvent dans un espace de dimension infinie, typiquement un espace de fonctions. De tels systèmes, aussi appelés systèmes à paramètres distribués, sont omniprésents dans la vie de tous les jours car ils sont capables de modéliser de nombreux procédés physiques, allant de systèmes mécaniques conservatifs à des phénomènes dissipatifs. Bon nombre de questions peuvent apparaître en considérant ces classes de systèmes. Par exemple, le caractère bien posé en termes d'existence et d'unicité de solutions aussi bien que l'étude des équilibres, leur stabilité et leur contrôle sont des étapes primordiales en étudiant ces systèmes dynamiques. Sur base de la littérature existante, nous portons une attention particulière à l'étude de l'existence, de l'unicité et de la stabilité des équilibres et le contrôle de systèmes à paramètres distribués. En particulier, comme contributions principales, nous étendons l'approche classique qui permet de déduire la stabilité des équilibres pour un système non linéaire sur base de la stabilité d'un modèle linéarisé correspondant. En utilisant un nouveau concept de différentiabilité pour les opérateurs non linéaires qui prend en compte un autre espace que l'espace d'état, nous montrons comment garantir la stabilité exponentielle locale ou l'instabilité des équilibres du système non linéaire original. Cette théorie est appliquée pour déterminer la stabilité des équilibres d'un modèle de réacteur non linéaire à effet piston avec dispersion axiale de la masse pour lequel les variables d'états sont la température et la concentration en réactifs. En adoptant un point de vue "contrôle", les résultats précédents sont étendus à la stabilisation d'équilibres de systèmes non linéaires en dimension infinie. Grâce à cette extension, nous pouvons identifier une classe de systèmes contrôlés de manière optimale pour laquelle les hypothèses sont satisfaites. Comme autre contribution, nous nous intéressons au problème de poursuite de trajectoires, en particulier au contrôle "funnel", qui constitue un outil approprié pour faire suivre à la sortie d'un système une classe de signaux de référence. Comme contribution principale dans ce sujet, nous étendons les résultats disponibles qui permettent de considérer des systèmes linéaires de dimension infinie comme dynamique interne au cadre non linéaire. Nous démontrons qu'une classe générale de systèmes non linéaires en dimension infinie qui satisfait quelques hypothèses standards admet une relation différentielle entre les entrées et les sorties qui est propice au contrôle "funnel". Un grand nombre de résultats théoriques dans cette thèse sont illustrés à l'aide d'exemples et de simulations numériques, particulièrement dans le domaine du contrôle des procédés. Les applications considérées sont liées à des modèles de réacteurs chimiques, des équations d'onde amorties et des équations de Sine-Gordon amorties.

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Anthony

Contents

List of notations and abbreviations

Symbols

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of nonnegative real numbers
$\mathbb{C}$	The set of complex numbers
$\mathfrak{Re}(\lambda)$	The real part of the complex number λ
$\mathfrak{Im}(\lambda)$	The imaginary part of the complex number λ
$(a,b), a,b \in \mathbb{R}, a < b$	The open-interval $]a,b[$ of the real axis $\mathbb{R}$
N^T	The transpose of the matrix N
∂_t	The operator of first-order time derivative, i.e. $\frac{\partial}{\partial t}$
∂_z	The operator of first-order space derivative, i.e. $\frac{\partial}{\partial z}$
∂_{tt}^2	The operator of second-order time derivative, i.e. $\frac{\partial^2}{\partial t^2}$
∂_{zz}^2	The operator of second-order space derivative, i.e. $\frac{\partial^2}{\partial z^2}$
$L^p(\Omega;\mathbb{K}), 1 \leq p < \infty$	The space of $p-$integrable $\mathbb{K}-$valued functions defined on Ω
$L^\infty(\Omega;\mathbb{K})$	The space of essentially bounded $\mathbb{K}-$valued functions defined on Ω
$L^p_{loc}(\Omega;\mathbb{K}), 1 \leq p < \infty$	The space of locally $p-$integrable $\mathbb{K}-$valued functions defined on Ω
$L^\infty_{loc}(\Omega;\mathbb{K})$	The space of locally essentially bounded $\mathbb{K}-$valued functions defined on Ω

$C(\Omega;\mathbb{K})$	The space of continuous $\mathbb{K}-$valued functions defined on Ω
$C^n(\Omega;\mathbb{K}), 1 < n \leq \infty$	The space of $n-$times continuously differentiable $\mathbb{K}-$valued functions defined on Ω
$W^{k,n}(\Omega;\mathbb{K}), 1 \leq k,n \leq \infty$	The Sobolev space of $\mathbb{K}-$valued functions defined on Ω which are in $L^n(\Omega;\mathbb{K})$ and whose generalized derivatives up to order k are in $L^n(\Omega;\mathbb{K})$
$H^n(\Omega;\mathbb{K}), 1 \leq n \leq \infty$	The Sobolev space of square integrable $\mathbb{K}-$valued functions defined on Ω whose generalized derivatives up to order n are square integrable (NB: $H^n(\Omega;\mathbb{K}) = W^{2,n}(\Omega;\mathbb{K})$)
$H_0^1([0,1];\mathbb{R})$	The space of $H^1([0,1];\mathbb{R})$-functions that vanishe at 0 and 1
$l^1(\mathbb{N})$	The space of absolutely summable sequences
$l^2(\mathbb{N})$	The space of square summable sequences
$\mathscr{L}(H)$	The set of linear and bounded operators on the Hilbert space H
$\langle\cdot,\cdot\rangle_H$	The inner product on the Hilbert space H
$\|\cdot\|_H$	The norm induced by the inner product $\langle\cdot,\cdot\rangle_H$ on the Hilbert space H
I	The identity operator
$D(A)$	The domain of the operator A
A^{-1}	The right inverse of the linear operator A, whenever it exists
$A^{\frac{1}{2}}$	The square-root of the linear operator A, whenever it exists
A^*	The adjoint of the linear operator A
$\rho(A)$	The resolvent set of the linear operator A
$\sigma(A)$	The spectrum of the linear operator A
$R(\lambda,A)$	The resolvent operator of A for $\lambda \in \rho(A)$
$ran(A)$	The image of the operator A
$\|\cdot\|_A$	Graph norm induced by the linear operator A on the Hilbert space H ($\|\cdot\|_A^2 = \|A\cdot\|_H^2 + \|\cdot\|_H^2$)
$O^\perp$	The orthogonal complement of the subset O of the Hilbert space H
$\mathrm{span}\mathscr{P}$	The linear span of $\mathscr{P}$
$\overline{M}$	The closure of the set M
$M \times N$	The cartesian product of the subspaces M and N
$M \oplus N$	The direct sum of the subspaces M and N
$\sup_{n\geq 1}\{\lambda_n\}$ (vs. inf)	The supremum (vs. infimum) of the set of real numbers $\lambda_n, n \geq 1$ over the natural numbers
$\min M$ (vs. $\max M$)	The minimum (vs. maximum) value of the set M, whenever it exists

$1_{[a,b]}(z), a, b \in \mathbb{R}, a \leq b$	The indicator function defined by 1 for $z \in [a,b]$ and 0 elsewhere, i.e. the characteristic function of the interval $[a,b]$
$\mathscr{O}$	Big $\mathscr{O}$ notation (Landau notation)
o	Small o notation (Landau notation)
$df(x^e)$	The Gâteaux derivative of the nonlinear operator f at x^e
$Df(x^e)$	The Fréchet derivative of the nonlinear operator f at x^e
$D(f)$	The domain of the nonlinear operator f
$f_{\|g=0}$	The function f for which the function g is set to 0
$f_{\|[t,t+\tau]}$	Restriction of the function f defined on $\mathbb{R}^+$ to the interval $[t,t+\tau], t, \tau \geq 0$
$\Sigma_{y \to \eta}$	Dynamical system whose input is y and whose output is η
$d(x,V)$	The distance between the point x and the subset V of the Hilbert space H ($d(x,V) = \inf_{v \in V}\{\\|x - v\\|_H\}$)

Abbreviations

a.c.	absolutely continuous
a.e.	almost everywhere
BC	Boundary Control
BIBO	Bounded-Input Bounded-Output
BISBO	Bounded-Input-State Bounded-Output
CDR	Convection-Diffusion-Reaction
C_0-semigroup	Strongly-continuous semigroup
cl	closed-loop
CSTR	Continuous Stirred-Tank Reactor
DPS	Distributed parameter system
ESC	Extremum Seeking Control
LQ	Linear Quadratic
LTI	Linear Time Invariant
ODE	Ordinary Differential Equation
PD	Proportional derivative
PDE	Partial Differential Equation
PFTR	Plug-Flow Tubular Reactor
PI	Proportional Integral
PID	Proportional Integral Derivative
TRAD	Tubular Reactor with Axial Dispersion
w.r.t.	with respect to

Chemical symbols

L	Reactor length
v	Fluid superficial velocity
ΔH	Heat of the reaction
ρ	Fluid density
C_p	Specific heat
k_0	Kinetic constant
E	Activation energy
R	Gas constant
h	Wall heat transfer coefficient
d	Reactor diameter
T_w	Coolant temperature
T_{in}	Inlet temperature
C_{in}	Inlet reactant concentration
λ_{ea}	Axial energy dispersion coefficient
D_{ma}	Axial mass dispersion coefficient
D	Notation used for the diffusion coefficient in the case of equal Peclet numbers ($\lambda_{ea} =: D := D_{ma}$)
Pe_h	Thermal Peclet number
Pe_m	Mass Peclet number
Pe	Notation used for the Peclet number in the case of equal Peclet numbers ($Pe_h =: Pe := Pe_m$)
d_T	Disturbance acting on the temperature
d_C	Disturbance acting on the concentration
ε_T	Width of the window on which d_T acts
ε_C	Width of the window on which d_C acts

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Introduction

Nonlinear infinite–dimensional, that is distributed parameter systems, are nonlinear dynamical systems which admit an abstract description on an infinite–dimensional Banach or Hilbert state space. These systems model a lot of applications in real-life, e.g. either conservative mechanical systems or dissipative chemical phenomena. The variables in the equations governing such systems depend most of the time on more than one parameter, giving rise to partial differential equations together with boundary conditions. A lot of attention is paid to this kind of systems, from theoretical or practical points of view. They constitute a large field of research since a priori simple questions as existence and uniqueness of solutions (well-posedness) may be extremely challenging to deal with when dealing with such systems, whose analysis and design require notably the use of nonlinear functional analysis.

Different aspects related to this class of dynamical systems are studied in this thesis, going from the well-posedness of nonlinear distributed parameter systems to their control. The notion of strongly continuous semigroup, extensively studied for linear infinite-dimensional systems e.g. in Curtain and Zwart (1995), Jacob and Zwart (2012) and Curtain and Zwart (2020), is of great importance here and is often used to express the solution (the state trajectories) of an abstract Cauchy problem. Extensions to systems driven by nonlinear partial differential equations are also considered in (Curtain and Zwart, 2020, Chapter 11) wherein questions like existence and unicity of solutions are deeply studied. Objects like equilibria of such systems constitute an important field to explore. Depending on their stability, they are able to give information on the place where the state trajectories of the system are located for large times. Due to the distributed parameters and nonlinear natures of the systems we consider, this question concerning the equilibria may be challenging to look at. Their explicit computation is often impossible, since nonlinear ordinary differential equations with boundary conditions should be solved. Therefore, approximated solutions can be a good trade-off in order to get quite accurate and useful information on them. For this aspect our contribution concentrates on a particular distributed parameter system consisting of the dynamics of a plug-flow tubular reactor model with axial dispersion for which the reaction kinetics are given by the Arrhenius law. The latter is used to model the variation of the velocity of the reaction as a function of the temperature. Two important parameters when dealing with such kind of chemical reactors are the mass and the thermal Peclet numbers. They express two different ratios between the model parameters. As it is highlighted in this thesis, considering them equal may reduce the difficulty of the computations a lot. However, this makes not so much sense from a practical point of view. Going along the lines of Drame et al. (2008), we characterize the existence and the multiplicity of the equilibria for this system by using some perturbation based method, by considering either equal or different Peclet numbers. Approximated solutions to the equilibrium equations are also given thanks to the perturbation theory. This is also inspired by Dochain (2018) where perturbation theory has been used to tackle the equilibrium analysis for equal Peclet numbers.

When moving to the study of the asymptotic behavior of a nonlinear infinite-dimensional system, what is commonly done is a linearization of the system around a given equilibrium and a study of the stability of the linearized system, like it could be appropriate when working in finite-dimensions. Then under quite common regularity assumptions, the stability properties of the linearization holds locally for the nonlinear system. However, this is not so simple when analyzing an infinite-dimensional system. In particular, the linearization is a difficult process that must have a well-defined meaning. Inspired by the works of Al Jamal and Morris (2018), Al Jamal (2013), we attack that question here by considering some useful adapted differentiability conditions to make the link between the stability of a linearized model and its corresponding nonlinear system. One important feature is that the considered stability is exponential. As it is highlighted in a counter example in Al Jamal et al. (2014), this approach does not work when considering asymptotic stability for instance.

After performing these analysis steps, the question of control pops up quite naturally. This consists in acting on a system with possibly different objectives. Interesting questions are the stabilization of an equilibrium, or the improvement of its stability margin, just to cite a few of them. The linearization process is also used a lot when considering such control problems. What is done here goes along the lines of the approach used for deducing stability. A perturbation based approach is used to compute a stabilizing control law for the nonlinear dynamical system, locally around the equilibrium of interest. Different notions related to the field of control are introduced and recalled in this part.

Looking for global control methods instead of the previous local ones, the expanding field of adaptive control has been studied. This kind of control is considered especially when parameters or parts of the dynamics are unknown and need to be estimated, see Krstic et al. (1995a) and Bastin and Dochain (1990) for instance. It is in general well suited for chemical processes as it will be highlighted in this thesis. Starting from the classical proportional integral control developed for linear infinite-dimensional systems in Pohjolainen (1982), we extend this theory by introducing some additional term that aims at dominating the nonlinear aspects of the dynamics to regulate the scalar average temperature in a plug-flow tubular reactor for which we assume that no diffusion occurs. In addition to this control problem, the notion of funnel control has been considered in e.g. Ilchmann et al. (2002), Ilchmann and Trenn (2004) and Ilchmann et al. (2005). This powerful control method, which is used for the tracking of a general class of output signals, is model-free and was shown to be suitable for an increasing number of classes of systems, as it can be seen in Berger et al. (2018) and Berger et al. (2021c) for finite-dimensional systems. A few years ago, this control method has captured the attention for its applicability to dynamical systems whose internal dynamics are infinite-dimensional as it is studied in Ilchmann et al. (2016), Berger et al. (2020), Berger et al. (2021a), Puche et al. (2021), Berger (2021) and Berger et al. (2022). Keeping our objective of studying and controlling nonlinear infinite-dimensional systems, funnel control for such a class of systems is considered in this thesis, enlarging significantly the applicability of this control approach.

Structure of the thesis

This thesis is organized as follows. The conducting application that is used to illustrate a large number of concepts and theoretical results, namely a model of nonlinear plug-flow tubular reactor with axial dispersion, is introduced in Chapter 1, where the motivations of considering this application together with the underlying mathematical model are presented.

The aim of the second chapter is to make the reader more familiar with infinite-dimensional system theory, both from linear and nonlinear points of view. Most of the reported results are borrowed from the recent book by Curtain and Zwart (2020). We take a particular attention at the consideration of useful, well-chosen and pertinent illustrations in this chapter that aims at easing the understanding of what follows in the manuscript.

A thorough study of the equilibria of the main application of this work is performed in Chapter 3. One finds for instance results on the existence, the multiplicity and the linear stability or instability of these equilibria. Numerical simulations are used to reinforce the theoretical results.

The link of these stability results with the local stability of the equilibria for the nonlinear original system is constructed in Chapter 4. Therein, a new framework for studying the stability of equilibria for nonlinear distributed parameter systems is built from a theoretical point of view with the ad-hoc assumptions. This new theory is applied to the determination of the stability of equilibria for the plug-flow tubular reactor with axial dispersion whose reaction kinetics are modeled by the Arrhenius law.

The extension of the previous results to the stabilization problem is constructed in Chapter 5, where inputs expressed as state feedbacks are considered. Local aspects of the method are discussed.

Next, with a goal of global stabilization and output regulation, we move in Chapter 6, in which we study the concept of adaptive control. Starting from the "classical" proportional integral control action, a new adaptive integral controller is developed in such a way that a scalar output corresponding to a nonlinear plug-flow tubular reactor aims at tracking constant reference profiles. Numerous numerical simulations involving notably set-point changes are used to highlight the efficiency of this approach.

Seeking for a model-free control approach for the tracking of a quite general class of signals for the output of a nonlinear infinite-dimensional system, the concept of funnel control is extensively studied in Chapter 7, where the developments are meant to be hopefully self-contained. Funnel control is shown to be appropriate for a new class of nonlinear distributed parameter systems, after a change of variables related to the Byrnes-Isidori form. Two infinite-dimensional models linked with chemical and wave processes are used to illustrate the theoretical notions and results.

Contributions

As mentioned above, when dealing with infinite-dimensional systems, and especially when they are nonlinear, it is often common to work on a linearized model to overcome technical difficulties, even when studying the analysis in terms of existence and uniqueness of solutions to the nonlinear partial differential equations. In this thesis, we adopt another point of view. In constrast to the approach that consists in changing (linearizing) the applications in order to apply existing tools from linear infinite-dimensional systems theory, we keep the dynamics nonlinear as much as possible. In that way, we develop new tools related to nonlinear functional analysis and nonlinear infinite-dimensional systems theory to perform analysis and control for nonlinear distributed parameter systems.

In particular, one of the main contributions of this thesis consists in deducing the stability of equilibria of nonlinear infinite-dimensional systems on the basis of the stability properties of a corresponding linearized system. Therefore, we extended the work of Al Jamal and Morris (2018) which uses the notion of Fréchet derivative to make the link between linear and nonlinear stability of equilibria for nonlinear infinite-dimensional systems. In particular, a new concept of differentiability for nonlinear operators is introduced here and it is shown how to get satisfactory (local) stability results thanks to this new definition. This new result can be viewed as an admissibility result since it takes into account another space as the state space, which makes the approach powerful since that new space adds a degree of freedom and may be chosen depending on the application.

The second major contribution consists in the extension of the results for deducing stability to the case of stabilization of equilibria of nonlinear distributed parameter systems. To this end, we proved how the adapted Fréchet differentiability of the nonlinear operator dynamics can be used to conclude the same fact for the closed-loop nonlinear semigroup. We noticed that boundedness (in an appropriate sense) of the feedback operator as well as boundedness of the control operator imply continuous dependence of the closed-loop system trajectories on the initial condition at 0, which is a major technical condition for getting Fréchet differentiability. Moreover, we identified a class of linear-quadratic optimally controlled systems for which the assumptions of the new framework hold. On the basis of a perturbation result for Riesz-spectral operators, we proved that the convergence of a particular series involving the eigenvalues and the eigenfunctions of the closed-loop linearized operator dynamics is a sufficient condition that guarantees that the optimal state feedback stabilizes exponentially and locally the nonlinear dynamics.

The third contribution one may mention lies in Chapter 7. There, we introduced a class of nonlinear controlled and observed infinite-dimensional systems for which the nonlinear operator has to be uniformly Lipschitz continuous. Then, under quite standard assumptions on the control and the observation operators, we show that this class of systems is conducive for funnel control. This has been performed by considering a change of variables related to the Byrnes-Isidori form, which is extensively studied in Ilchmann et al. (2016) for linear infinite-dimensional systems. This completes the contributions of Berger et al. (2018), Berger et al. (2020) and Berger et al. (2021c) since our contribution enlarges the applicability of funnel control, giving a partial answer to the following remark in Berger et al. (2020): "*While the class of functional differential equations (1) appears to be rather general and funnel control is feasible for these systems by Theorem 2.1, it is not clear exactly which kind of systems that contain PDEs are encompassed by the class (1)*", where (1) refers to a class of systems for which funnel control is considered in that paper.

Scientific communications

Talks, posters and invited seminars

- Talk at the 38th Benelux Meeting on Systems and Control, entitled "Analysis of equilibrium profiles in nonisothermal axial dispersion tubular reactors", session "Distributed Parameter Systems", March 21, 2019.

- naXys seminar entitled: "On stability and control of nonlinear infinite- dimensional systems", October 3, 2019.

- Flash-talk at the naXys research day, entitled "Stability and control of nonlinear infinite-dimensional systems: application to a nonisothermal axial dispersion tubular reactor", October 24, 2019.

- Talk at the 39th Benelux Meeting on Systems and Control, entitled "Local exponential stability of nonlinear distributed parameter systems: Application to a nonisothermal tubular reactor", session "Distributed Parameter Systems", March 10, 2020.

- Virtual Talk at the IFAC World Congress 2020, entitled "On local stability of equilibrium profiles of nonisothermal axial dispersion tubular reactors", session "On nonlinear infinite dimensional Systems", July 11–17, 2020.

- Accepted presentation at the 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS), entitled "On local exponential stability of equilibrium profiles of nonlinear distributed parameter systems", August 2020 (that talk was not given, because the symposium was cancelled for reasons related to the COVID-19 pandemic).

- Talk at the weekly seminars on Dynamical Systems and Control, UNamur, naXys, entitled "Stability and stabilization of nonlinear infinite-dimensional systems", April 2021.

- Invited presentation at the 10th workshop on Control of Distributed Parameter Systems, Warwick, UK, August 2021, entitled "Local exponential stabilization of nonlinear distributed parameter systems" (that talk was not given, because the workshop was cancelled for reasons related to the COVID-19 pandemic).

- Poster at the interactive session of the 3rd Workshop on Stability and Control of Infinite-Dimensional Systems (SCINDIS), entitled "Local exponential stabilization of nonlinear distributed parameter systems", September 27, 2021.

- Poster entitled "Local exponential stabilization of nonlinear distributed parameter systems" at the "Journée des Instituts" organized at the University of Namur (UNamur) on October 5, 2021.

- Virtual talk at the 60th Conference on Decision and Control (CDC), entitled "Local exponential stabilization of nonlinear infinite-dimensional systems", December 13-17, 2021.

Publications

This thesis is mainly based on the following papers of which I am the first author/ contributor:

- Anthony Hastir, François Lamoline, Joseph J. Winkin, Denis Dochain, Analysis of the Existence of Equilibrium Profiles in Nonisothermal Axial Dispersion Tubular Reactors, IEEE Transactions on Automatic Control, Volume 65, no. 4, pp. 1525-1536, April 2020, doi: 10. 1109/TAC. 2019. 2921675.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, Exponential stability of nonlinear infinite-dimensional systems: Application to nonisothermal axial dispersion tubular reactors, Automatica, Volume 121, November 2020, 109201, ISSN 0005-1098, doi: 10. 1016/j. automatica. 2020. 109201.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, On Local Stability of Equilibrium Profiles of Nonisothermal Axial Dispersion Tubular Reactors, IFAC-PapersOnLine, Volume 53, Issue 2, 2020, pp. 5315-5321, ISSN 2405-8963, doi: 10. 1016/j. ifacol. 2020. 12. 1217.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, On Exponential Bistability of Equilibrium Profiles of Nonisothermal Axial Dispersion Tubular Reactors, IEEE Transactions on Automatic Control, Volume 66, no. 7, pp. 3235-3242, July 2021, doi: 10. 1109/TAC. 2020. 3014457.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, On local exponential stability of equilibrium profiles of nonlinear distributed parameter systems, IFAC-PapersOnLine, Volume 54, Issue 9, 2021, pp. 390-396, ISSN 2405-8963, doi: 10. 1016/j. ifacol. 2021. 06. 097.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, Local exponential stabilization of nonlinear infinite-dimensional systems, Proceedings of the 60th Conference on Decision and Control (CDC), 2021, pp. 4038-4045, doi: 10. 1109/ CDC45484. 2021. 9683165.

- Anthony Hastir, Joseph J. Winkin, Denis Dochain, Funnel control for a general class of nonlinear infinite-dimensional systems, 2021, submitted to Systems and Control Letters, arXiv preprint, eprint: 2111. 06713, primaryClass: math.OC.

Additional contributions, which are more or less related to the topics dealt with in this thesis, have also been published or submitted:

- Anthony Hastir, Federico Califano, Hans Zwart, Well-posedness of infinite-dimensional linear systems with nonlinear feedback, Systems & Control Letters, Volume 128, June 2019, pp. 19 - 25, ISSN 0167 - 6911, doi: 10.1016/j. sysconle. 2019. 04. 002.

- Anthony Hastir, François Lamoline, Optimal equilibrium stabilization for a nonlinear infinite-dimensional plug-flow reactor model, Automatica, Volume 130, August 2021, 109722, ISSN 0005-1098, doi: 10. 1016/j.automatica. 2021. 109722.

- Bouchra Abouzaïd, Med Elarbi Achhab, Jonathan N. Dehaye, Anthony Hastir, Joseph J. Winkin, Locally positive stabilization of infinite-dimensional linear systems by state feedback, European Journal of Control, July 2021, ISSN 0947-3580, doi: 10. 1016/j. ejcon. 2021. 07. 006.

- François Lamoline, Anthony Hastir, On Dirac structure of infinite-dimensional stochastic port-Hamiltonian systems, 2021, submitted to Automatica.

Chapter 1

A conducting application: The regulation of the temperature in tubular reactors

Contents

This part of the thesis is dedicated to the presentation of the application that is taken into account for the illustration of the different theoretical concepts that are presented along the manuscript. This application is of particular interest in chemical engineering due to its ability in improving the production of products in some chemical reactions.

1.1 Motivation

Some general concepts on chemical engineering and more specifically on chemical reactors are described in this section.

A chemical reactor is basically the place wherein one chemical reaction (or more) occurs. The objective of such a device is to try to transform a reactant into a product. The basic objective is to transform raw materials into more valuable ones. To this end, a step called the separation process has to be performed on the materials that enter into the reactor in such a way that the reaction behaves well and after the reaction to get the best products as possible, see Figure 1.1 for the representation of the different steps of the process.

In order to act on these steps and to be able to have an impact on the synthe-

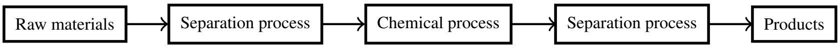

Figure 1.1 – Different steps from the transformation of the reactant into the products.

sized products thanks to mathematical tools, a mathematical model of the chemical reaction(s) will be derived.

For this, we need first to specify the type of reactions that will be considered. We shall focus on irreversible reactions here, meaning that once the products are obtained, no more transformation to the reactant is possible. A notion that will appear directly in the mathematical model is the reaction rate. This quantity describes the dependence of the rate of transformation on the variables of the systems, see Schmidt (1998) for instance. It generally depends on the temperature of the reaction like $k(T) = k_0 e^{-\frac{E}{RT}}$, where k_0 is a kinetic constant, E is the activation energy, R denotes the perfect gas constant and T is the temperature, see e.g. Schmidt (1998); Aksikas (2005). This dependency on the temperature is due to Svante Arrhenius and is often called the Arrhenius' law.

The rates may also depend on the concentration in different ways. The basic rule in pure chemistry is that they are proportional to the concentration of reactant raised to the power equal to its stochiometric coefficient, see Schmidt (1998). For instance in the reaction $2A \to B$ where A denotes the reactant and B the product, the reaction rate as a function of the reactant concentration and of the temperature will be described by the function $r(C_A, T) = k_0 e^{-\frac{E}{RT}} C_A^2$ whereas for the reaction $F \to G$, the reaction rate is given by $r(C_F, T) = k_0 e^{-\frac{E}{RT}} C_F$. This is known as the mass action law(1).

Secondly, let us distinguish different types of chemical reactors. They can either operate in batch or in flow modes. A batch reactor is sometimes called a closed reactor since no mass can be added in after time $t = 0$. However flow reactors can be loaded during the reaction.

The following distinction concerns only flow reactors. They operate between completely unmixed contents and completely mixed ones, involving different phases: solid, liquid and gas. Reactors for which the medium is homogeneous are called stirred tank reactors (e.g. the Continuous Stirred Tank Reactor (CSTR)) and if the medium can be nonhomogeneous, they are said to be tubular reactors (e.g. the Plug Flow Tubular Reactor (PFTR)).

The reactors that will be of interest in what follows are tubular reactors. Two differences may still be made. On one hand one finds the plug flow tubular reactor wherein the flow is supposed to be laminar, i.e. no turbulences are allowed inside the reactor. On the other hand, one may consider axial mixing (also called molecular dif-

(1)Note that in many instances, kinetic models differ from the mass action law and their structure is derived from identification from experimental data. We refer to (Schmidt, 1998, Chapter 2) for an overview on kinetic models and how they can be identified with data.

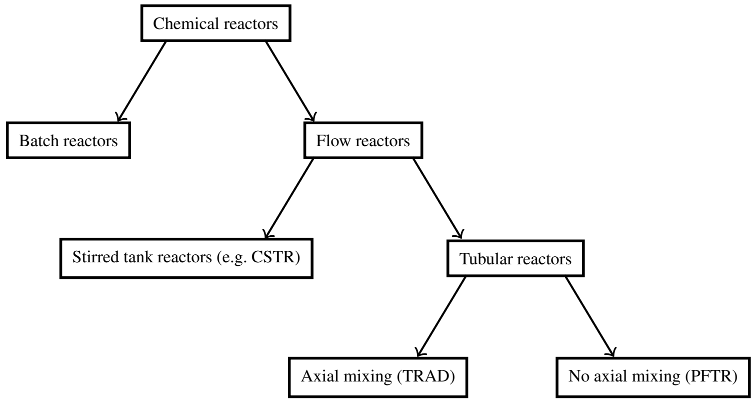

Figure 1.2 – Different types of chemical reactors.

fusion) which takes into account the fact that the contents of the reactor can move from right to left and vice versa during the reaction process, see e.g. plug flow reactors with axial mixing, sometimes called a Tubular Reactor with Axial Dispersion (TRAD). The second case is probably the most often used in practice since back-mixing of the fluid inside the reactor occurs in a large number of configurations. The latter has been modeled thanks to Fick's and Fourier's laws, see Varma and Aris (1977) or (Levenspiel, 1999, Chapter 13) for instance. The classification of the different reactors discussed above is depicted in Figure 1.2.

Tubular reactors are sometimes called diffusion-convection-reaction reactors since these three phenomena are considered when writing the mass and energy balances. The diffusion phenomenon is symbolized by the displacement of atoms or molecules from regions with high concentration to regions with low ones. The diffusion occurs in tubular reactors notably because of the axial mixing. Moreover, the convection models the heat transfer through the reactor and is due to the plug flow/laminar effect in tubular reactors. The reaction part pops up when the rate of transformation is considered, as explained above.

Moreover, note that chemical reactors are most of the time nonisothermal since reactions generate or absorb large amounts of heat (exothermic or endothermic reactions), which produces variations in the temperature in the reactor. Consequently two conflicting aspects have to be considered. On one hand, the temperature inside the reactors has to be sufficiently high to activate the reaction but on the other hand it cannot blow up or be too high. In this case, the equilibrium limitation can limit the conversion and slow the activation of the reaction down, but the most dangerous consequences of temperature runaway are the release of undesirable chemicals or even the explosion in the case of under pressure batch reactors. For this reason, control of the temperature inside chemical reactors is of great importance and will be studied in the following parts of this thesis. The device that plays the role of the control is a heat exchanger surrounding the reactor in which a fluid flows in order to regulate the temperature inside the reactor. This device is of particular interest in the case of exothermic reactions. Moreover, control of the temperature could also prevent the effects of large and concentrated variations in the spatial profiles of the temperature and/or the concentration, often called "hot spots", see Lefèvre et al. (2000) for instance.

1.2 Mathematical model for different types of reactors

The equations governing the dynamic of a nonisothermal tubular reactor are nonlinear partial differential equations (PDEs) derived from mass and energy balances, based notably on the laws of thermodynamics. The different terms that composed the PDEs come mainly from the elements that have been presented in Section 1.1. More particularly, the nonlinearities are located in the kinetic terms for which the Arrhenius law is considered to express the temperature dependence. Power dependencies for the reactant concentrations are also taken into account. The reaction that will be considered is of type $A \to B$, where A denotes the reactant and B the product, yielding a reaction rate that depends on the reactant concentration, C_A, and on the temperature, T, like $k_0 C_A e^{-\frac{E}{RT}}$ as described above, see e.g. Varma and Aris (1977). In addition to this nonlinearity, the linear part of the PDEs is called the diffusion-convection-reaction part since it includes these different phenomena. The diffusion (or the axial dispersion) is written mathematically with a second order spatial derivative operator while the convection is modeled by a first order spatial derivative operator.

As a first model, we shall consider a plug-flow tubular reactor with axial dispersion, see e.g. Dochain (2018); Varma and Aris (1977); Aksikas (2005) among others. The quantities that are considered are the temperature inside the reactor, $T[K]$, the reactant concentration, $C_A[\mathrm{mol}/l]$ and the product concentration, $C_B[\mathrm{mol}/l]$. The dynamics of these quantities are governed by the following set of nonlinear PDEs

$$\begin{cases} \frac{\partial T}{\partial \tau}(\zeta,\tau) = -v\frac{\partial T}{\partial \zeta}(\zeta,\tau) + \frac{\lambda_{ea}}{\rho C_p}\frac{\partial^2 T}{\partial \zeta^2}(\zeta,\tau) - \frac{\Delta H}{\rho C_p}k_0 C_A(\zeta,\tau)e^{-\frac{E}{RT(\zeta,\tau)}} \\ \qquad\qquad + \frac{4h}{\rho C_p d}(1_{[0,L]}(\zeta)T_w(\tau) - T(\zeta,\tau)), \\ \frac{\partial C_A}{\partial \tau}(\zeta,\tau) = -v\frac{\partial C_A}{\partial \zeta}(\zeta,\tau) + D_{ma}\frac{\partial^2 C_A}{\partial \zeta^2}(\zeta,\tau) - k_0 C_A(\zeta,\tau)e^{-\frac{E}{RT(\zeta,\tau)}}, \\ \frac{\partial C_B}{\partial \tau}(\zeta,\tau) = -v\frac{\partial C_B}{\partial \zeta}(\zeta,\tau) + D_{ma}\frac{\partial^2 C_B}{\partial \zeta^2}(\zeta,\tau) + k_0 C_A(\zeta,\tau)e^{-\frac{E}{RT(\zeta,\tau)}}, \end{cases} \tag{1.2.1}$$

where the spatial and the time variables ζ and τ lie in the intervals $[0,L]$ and $[0,+\infty)$, respectively, $L[m]$ being the length of the reactor. These are deduced by performing mass and energy balances on a slice of infinitesimal thickness $d\zeta$ during an infinitesimal time $d\tau$, under the assumption that there is no energy transfer from the reactor to the heat exchanger and that the mass flow through the heat exchanger is sufficiently large so that T_w does not depend on the position $\zeta \in [0,L]$. The characteristic function

Notation	Unit	Description
L	m	Reactor length
v	$\frac{m}{s}$	Fluid superficial velocity
ΔH	$\frac{kJ}{kg}$	Heat of the reaction
ρ	$\frac{kg}{m^3}$	Fluid density
C_p	$\frac{kJ}{kg\,K}$	Specific heat
k_0	$\frac{1}{s}$	Kinetic constant
E	$\frac{kJ}{kg}$	Activation energy
R	$\frac{kJ}{kg\,K}$	Gas constant
h	$\frac{kJ}{m^2\,K\,s}$	Wall heat transfer coefficient
d	m	Reactor diameter
T_w	K	Coolant temperature
T_{in}	K	Inlet temperature
C_{in}	$\frac{\mathrm{mol}}{l}$	Inlet reactant concentration
λ_{ea}	$\frac{kJ}{m\,s\,K}$	Axial energy dispersion coefficient
D_{ma}	$\frac{m^2}{s}$	Axial mass dispersion coefficient

Table 1.1 – System parameters.

$1_{[0,L]}(z)$ is defined as taking the value 1 for $\zeta \in [0,L]$ and the value 0 elsewhere. The meaning and the units of the parameters in (1.2.1) are summarized in Table 1.1. Note that the coefficient ΔH determines whether the reaction is exothermic or endothermic. A negative value of ΔH produces an exothermic reaction while a positive value of ΔH entails that the reaction is endothermic. The coefficients λ_{ea} and D_{ma} stand for the axial energy dispersion coefficient and the axial mass dispersion coefficient, respectively.

The variable T_w is the temperature in the heat exchanger surrounding the reactor and is distributed around the reactor through the characteristic function $1_{[0,L]}$. This variable often plays the role of a control action that enables to force the behavior of the system. A schematic profile view of a plug-flow tubular reactor with axial dispersion is depicted in Figure 1.3.

From a physical point of view, the variables T, C_A and C_B satisfy

$$0 \leq T(\zeta,\tau), 0 \leq C_A(\zeta,\tau) \leq C_{in}, 0 \leq C_B(\zeta,\tau). \tag{1.2.2}$$

This means that the temperature has to remain above the absolute zero temperature, and that the reactant concentration cannot be below 0 and cannot exceed the inlet concentration, see Laabissi et al. (2001). To the PDEs (1.2.1) we associate the following boundary conditions known as the Danckwert's boundary conditions, see Danckwerts

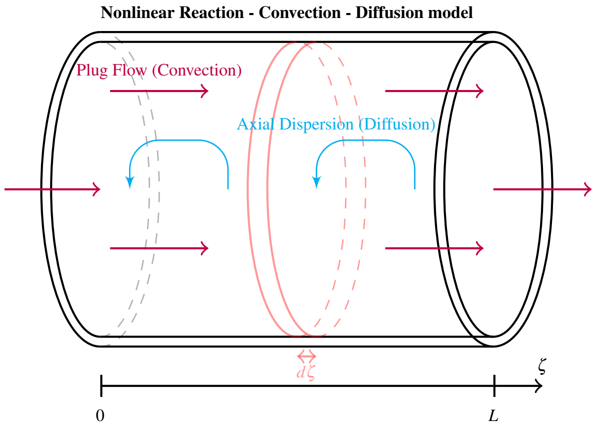

Figure 1.3 – Profile view of a plug-flow tubular reactor with axial dispersion actuated by a surrounding heat exchanger.

(1953), and expressed as

$$\begin{aligned} \frac{\lambda_{ea}}{\rho C_p}\frac{\partial T}{\partial \zeta}(0,\tau) = v(T(0,\tau)-T_{in}), D_{ma}\frac{\partial C_A}{\partial \zeta}(0,\tau) = v(C_A(0,\tau)-C_{in}),& \\ D_{ma}\frac{\partial C_B}{\partial \zeta}(0,\tau) = vC_B(0,\tau)& \\ \frac{\lambda_{ea}}{\rho C_p}\frac{\partial T}{\partial \zeta}(L,\tau) = 0, D_{ma}\frac{\partial C_A}{\partial \zeta}(L,\tau) = 0, D_{ma}\frac{\partial C_B}{\partial \zeta}(L,\tau) = 0,& \end{aligned} \tag{1.2.3}$$

where $\tau \in [0,+\infty)$. The variables T_{in} and C_{in} can play the role of boundary control actions in this context but they are going to be assumed constant in what follows. Only the distributed control variable T_w will be considered for control purposes.

Note that the case where only the plug-flow is considered leads to the convection-reaction model and is the same as (1.2.1) together with (1.2.3) where the diffusion coefficients λ_{ea} and D_{ma} are set to 0. Moreover, from (1.2.1), it is easy to see that the product concentration C_B is directly deduced from the variables T and C_A whose dynamics are independent of C_B. For this reason, the dynamics of C_B will be often omitted in what follows.

Despite the fact that this thesis is centered mainly on reactions of the form $A \to B$, other more involved reactions such as the Van der Vusse or the Williams-Otto reactions could be considered, see e.g. Hudon et al. (2008) and Hudon et al. (2005), respectively. On one hand, the Van der Vusse reaction reads as $A \to B \to C, 2A \to D$. This takes three reactions into account, the first transforming A into B, the second going from B to C and the last changing $2A$ into C. For taking this into account, one should consider three quantities in the PDE, the temperature, the concentration of A and the concentration of B. As an example, the three reaction rates are expressed in the same manner as the one introduced in (1.2.1) by $r_1 = k_1 C_A e^{-\frac{E}{RT}}, r_2 = k_2 C_B e^{-\frac{E}{RT}}$ and $r_3 = k_3 C_A^2 e^{-\frac{E}{RT}}$, where T, C_A, C_B and R have the same meanings as before. The parameters $k_i, i = 1,2,3$ are kinetic constants. On the other hand the Williams-Otto reaction is given by $A+B \to 2C, B+2C \to P+2E, 2C+P \to 3G$, yielding to consider the concentration of six chemical components, C_A, C_B, C_C, C_P, C_E and C_G. Many others reactions are also available in Schmidt (1998).

The first question that will be under investigation is whether the PDE system (1.2.1) together with the boundary conditions (1.2.3) possesses a unique solution that has some useful properties such as continuous dependence on the initial profiles of temperature and concentrations or invariance properties induced by the physical constraints (1.2.2). This will be discussed at the end of Chapter 2 in the case where no control action is given to the system, i.e. $T_w(\tau) \equiv 0$ for all $\tau \geq 0$. Then, still by considering the homogeneous case[(2)], the challenging question of existence and stability of equilibria of (1.2.1) with (1.2.3) is studied in Chapter 3. In particular, perturbation methods are used to achieve this objective. Although the question of existence of equilibria is studied for the nonlinear equation, the stability is first deduced on a linearized version of (1.2.1). The extension of this question to the nonlinear case with appropriate tools of functional analysis is available in Chapter 4. The control aspects related to (1.2.1) with particular objectives such as the production of the best output concentration are developed in Chapters 6 and 7 in the case where the diffusion coefficients are set to 0, i.e. for the case of a plug-flow reactor.

[(2)] Homogeneous means that $T_w(\tau) \equiv 0, \tau \geq 0$ here.

Chapter 2

Infinite-dimensional system theory

Contents

This chapter aims at introducing the general framework of this thesis. Concepts of functional analysis together with linear and nonlinear tools of system theory are presented. The definitions and the theorems will be always motivated regarding the application to chemical processes encountered in Chapter 1. In particular, theorems that aim at proving the existence and uniqueness of solutions for the partial differential equation modeling the dynamics of a chemical reactor will be used.

2.1 Linear system theory on Hilbert spaces

Let H be a (separable) Hilbert space, equipped with the inner product $\langle \cdot, \cdot \rangle_H$. For $x_0 \in H$ and $t \geq 0$, we consider the following abstract differential equation

$$\begin{cases} \dot{x}(t) = Ax(t), \\ x(0) = x_0, \end{cases} \tag{2.1.1}$$

where the (possibly unbounded) time independent linear operator $A : D(A) \subset H \to H$ is defined on a linear, closed and dense subspace $D(A)$ of H, which is called the domain of the operator A. The notation $(A, D(A))$ will be frequently used in what follows to speak of the operator A together with its domain. The question we ask is under which conditions on A and $D(A)$ (2.1.1) possesses a unique solution which lies in the space H, and even in $D(A)$. The answer in the case where H is finite-dimensional is quite direct since the solution of (2.1.1) is given by the matrix exponential $x(t) = e^{tA}x_0$. This solution possesses some properties that are "the minimum we may ask" for a solution of an ordinary differential equation (ODE). It gives the initial condition back when evaluated at $t = 0$, it satisfies the composition law $e^{(t+s)A} = e^{tA}e^{sA}$ for all $t, s \geq 0$ and it is continuous with respect to time at 0, which implies the continuity at any time due to the composition law. The extension to infinite-dimensions is not direct and is intricate. To answer this question we need the concept of semigroup of bounded linear operators, see (Curtain and Zwart, 2020; Pazy, 1983) among other books for an overview on that topic.

Definition 2.1.1 *Let us consider the operator-valued function $T(t)$ in the set of linear and bounded operators on the Hilbert space H, denoted $\mathscr{L}(H)$, for any $t \in \mathbb{R}^+$. It is called a strongly-continuous semigroup (C_0-semigroup) if it satisfies the following properties:*

1. $T(0) = I_H$;
2. $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$;
3. $\|T(t)x_0 - x_0\|_H \to 0$ *as* $t \to 0^+$ *for all* $x_0 \in H$.

Note that the matrix exponential defined above in the case of a finite-dimensional space H satisfies the properties of Definition 2.1.1. More generally, for any operator $A \in \mathscr{L}(H)$, one may define the operator-valued function

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \tag{2.1.2}$$

for $t \geq 0$. It is well-defined since the operator $A \in \mathscr{L}(H)$ and it can be easily verified that it satisfied the three properties of Definition 2.1.1. Therefore it defines a C_0-semigroup of bounded linear operators on H.

In order to be able to characterize the solution of (2.1.1) the following definition is useful.

Definition 2.1.2 *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on H. The infinitesimal generator A of $(T(t))_{t\geq 0}$ is defined as*

$$Ax_0 = \lim_{t\to 0^+} \frac{T(t)x_0 - x_0}{t}, \tag{2.1.3}$$

whenever the limit exists. The set of all x_0 for which the limit exists is called the domain of A and is noted $D(A)$.

This definition allows us to link the solution of (2.1.1) with Definition 2.1.1. Provided that the operator A is the infinitesimal generator of a strongly continuous semigroup

$(T(t))_{t\geq 0}$ on H, the solution of (2.1.1) is given by

$$x(t) = T(t)x_0,$$

where $x_0 \in H$ is the initial condition given to the dynamical system (2.1.1).

Different types of solution of (2.1.1) may be distinguished, relying mainly on the regularity of the initial condition.

Definition 2.1.3 *A differentiable function $x : [0,\infty) \to H$ is called a classical solution of (2.1.1) if for all $t \geq 0$ it holds that $x(t) \in D(A)$ and equation (2.1.1) is satisfied.*

Definition 2.1.4 *A continuous function $x : [0,\infty) \to H$ is called a mild solution of (2.1.1) if $\int_0^t x(s)ds \in D(A), x(0) = x_0$ and*

$$x(t) - x_0 = A\int_0^t x(s)ds,$$

for all $t \geq 0$.

The main difference between these two notions of solution of (2.1.1) is its regularity, dealing with a solution which may be differentiated in the case it is classical and which is continuous but not necessarily differentiable when it is mild.

2.1.1 Generation theorems

This part is dedicated to the theorems that give necessary and sufficient conditions in order to prove that a closed, linear and densely defined operator A is the infinitesimal generator of a C_0-semigroup on a Hilbert space H. We shall focus on a particular class of semigroups, the contraction semigroups, i.e. semigroups $(T(t))_{t\geq 0}$ that satisfy

$$\|T(t)\| \leq 1,$$

for any $t \geq 0$. Note that in what follows we shall tacitly assumed that when speaking of an operator A, the latter is closed, linear and densely defined.

To start we need the following definition.

Definition 2.1.5 *Let A be a closed, linear and densely defined operator on H. The set of $\lambda \in \mathbb{C}$ that are such that the operator $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on a dense domain of H is called the resolvent set of A and is denoted $\rho(A)$. For $\lambda \in \rho(A)$, the operator $(\lambda I - A)^{-1} =: R(\lambda, A)$ is called the resolvent operator of A.*

In the next example it is shown how to compute the resolvent operator of the diffusion operator and how to determine the corresponding resolvent set.

Example 2.1.1 *Let $H := L^2([0,1];\mathbb{R})$ and let the linear operator A be defined as $Ax = \frac{d^2x}{dz^2}$ for $x \in D(A)$ expressed as*

$$D(A) = \left\{ x \in H, x \text{ a.c. }, \frac{dx}{dz} \text{ a.c. }, \frac{d^2x}{dz^2} \in H, \frac{dx}{dz}(0) = 0 = \frac{dx}{dz}(1) \right\}, \tag{2.1.4}$$

where a.c. means absolutely continuous. Computing the function $(\lambda I - A)^{-1} f$ for $\lambda \in \rho(A)$ and $f \in H$ consists in finding $x \in D(A)$ such that the equation $(\lambda I - A)x = f$ is satisfied. Equivalently, by defining the variables $x_1(z) = x(z)$ and $x_2(z) = \frac{dx}{dz}(z), z \in [0,1]$, it can be rewritten as

$$\frac{d}{dz}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ f \end{pmatrix}$$

whose solution is given by

$$\begin{aligned} \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix} &= \begin{pmatrix} \cosh(\sqrt{\lambda}z) & \frac{\sinh(\sqrt{\lambda}z)}{\sqrt{\lambda}} \\ \sqrt{\lambda}\sinh(\sqrt{\lambda}z) & \cosh(\sqrt{\lambda}z) \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\ &+ \int_0^z \begin{pmatrix} \cosh(\sqrt{\lambda}(z-\zeta)) & \frac{\sinh(\sqrt{\lambda}(z-\zeta))}{\sqrt{\lambda}} \\ \sqrt{\lambda}\sinh(\sqrt{\lambda}(z-\zeta)) & \cosh(\sqrt{\lambda}(z-\zeta)) \end{pmatrix}\begin{pmatrix} 0 \\ f(\zeta) \end{pmatrix} d\zeta. \end{aligned} \tag{2.1.5}$$

By taking the boundary conditions into account, one gets that $x_2(0) = 0$ and $x_2(1) = 0$. This implies that the constant $x_1(0)$ is given by

$$x_1(0) = \frac{\int_0^1 \cosh(\sqrt{\lambda}(1-\zeta)) f(\zeta) d\zeta}{\sqrt{\lambda}\sinh(\sqrt{\lambda})}. \tag{2.1.6}$$

Combining (2.1.5), (2.1.6) and the relation $x_1 = x$ one gets that

$$\begin{aligned} ((\lambda I - A)^{-1} f)(z) = &\cosh(\sqrt{\lambda}z)\left[\frac{\int_0^1 \cosh(\sqrt{\lambda}(1-\zeta)) f(\zeta) d\zeta}{\sqrt{\lambda}\sinh(\sqrt{\lambda})}\right] \\ &- \int_0^z \frac{\sinh(\sqrt{\lambda}(z-\zeta))}{\sqrt{\lambda}} f(\zeta) d\zeta. \end{aligned}$$

The values of λ which are such that the previous expression makes sense are those that satisfy $\sqrt{\lambda}\sinh(\sqrt{\lambda}) \neq 0$. This is equivalent to impose that $\lambda \neq 0$ and $-j\sin(j\sqrt{\lambda}) = \sinh(\sqrt{\lambda}) \neq 0$, where j is such that $j^2 = -1$. The last condition is satisfied provided that $j\sqrt{\lambda} \neq n\pi, n \in \mathbb{Z}$. Equivalently, $\lambda \neq -n^2\pi^2, n \in \mathbb{N}$. Consequently, the resolvent set of A is given by

$$\rho(A) = \mathbb{C} \setminus \left\{ -n^2\pi^2, n \in \mathbb{N} \right\}. \tag{2.1.7}$$

In order to be able to characterize the contraction property, the next definition will play an important role.

Definition 2.1.6 *A linear operator $A : D(A) \subset H \to H$ is called dissipative, if*

$$\mathfrak{Re}\langle Ax, x\rangle_H \leq 0, x \in D(A). \tag{2.1.8}$$

This definition generalizes the concept of negative definite matrices which is expressed as $p^T \mathbb{A} p \leq 0$ for any vector $p \in \mathbb{R}^n$ and a matrix $\mathbb{A} \in \mathbb{R}^{n\times n}, n \in \mathbb{N}$. When the matrix $\mathbb{A}$ is associated to the dynamical system $\dot{p}(t) = \mathbb{A}p(t), p(0) = p_0 \in \mathbb{R}^n, t \geq 0$, the fact that $\mathbb{A}$ is negative definite translates in the fact that the energy $p^2(t)$ is dissipating with time, or in other words that the system is dissipative. This gives intuition about the terminology "dissipative".

Example 2.1.2 *Let $H := L^2([0,1];\mathbb{R})$ be equipped with the standard inner product*

$$\langle f, g\rangle_H := \int_0^1 f(z)g(z)dz, \tag{2.1.9}$$

for $f, g \in H$ and let the linear operator A be defined as in Example 2.1.1. By taking $x \in D(A)$, it holds that

$$\begin{aligned}\mathfrak{Re}\langle Ax, x\rangle_H = \int_0^1 \frac{d^2x}{dz^2}(z)x(z)dz &= \left[x(z)\frac{dx}{dz}(z)\right]_0^1 - \int_0^1 \left(\frac{dx}{dz}(z)\right)^2 dz \\ &= -\int_0^1 \left(\frac{dx}{dz}(z)\right)^2 dz \leq 0.\end{aligned}$$

This shows that the one-dimensional diffusion operator with Neumann boundary conditions is dissipative on the real Hilbert space of square integrable functions.

The next theorem (special case of the Hille-Yosida theorem) gives necessary and sufficient conditions in order to show that a closed, linear and densely defined operator is the infinitesimal generator of a C_0- semigroup on a separable Hilbert space, see e.g. (Jacob and Zwart, 2012, Theorem 6.1.3). The conditions that have to be satisfied are related to the resolvent set and the resolvent operator.

Theorem 2.1.1 *A necessary and sufficient condition for a closed, densely defined, linear operator A on a Hilbert space H to be the infinitesimal generator of a contraction semigroup is that $(0,\infty) \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \textit{ for all } \lambda > 0. \tag{2.1.10}$$

We shall now give an illustration of this Theorem for a perturbed convection operator.

Example 2.1.3 *Consider the closed, densely defined and linear operator $Ax :=$*

$-\frac{dx}{dz} - \alpha x, \alpha > 0$ *for*

$$x \in D(A) := \left\{ x \in H^1([0,1];\mathbb{R}), x(0) = 0 \right\},$$

where $H^1([0,1];\mathbb{R})$ is the Sobolev space of square integrable functions whose generalized first order derivative is square integrable. The state space associated to this operator is $H = L^2([0,1];\mathbb{R})$ equipped with the inner product (2.1.9). Let us consider $\lambda > 0$ and $f \in H$. The resolvent operator $R(\lambda, A)$ applied to f at some point $z \in [0,1]$ is given by

$$(R(\lambda,A)f)(z) = \int_0^z e^{(-\alpha-\lambda)(z-\zeta)} f(\zeta) d\zeta, \tag{2.1.11}$$

which means that any positive λ yields a linear and bounded resolvent operator, leading to $(0,\infty) \subset \rho(A)$. Moreover, observe that, for any $x \in D(A)$, the relation

$$\langle Ax, x \rangle = -\frac{1}{2} x^2(1) - \alpha \int_0^1 x^2(\zeta) d\zeta \leq 0$$

holds. This implies that the operator A is dissipative. According to the Cauchy-Schwarz inequality, one may write that

$$\|(\lambda I - A)x\| \|x\| \geq \langle (\lambda I - A)x, x \rangle = \lambda \|x\|^2 - \langle Ax, x \rangle \geq \lambda \|x\|^2, \tag{2.1.12}$$

where the relation holds for any $x \in D(A)$ and any $\lambda > 0$. For an arbitrary $f \in H$, let us define $x := (\lambda I - A)^{-1} f$, see (2.1.11). Plugging this x in (2.1.12) entails that

$$\|R(\lambda, A)f\| \leq \frac{1}{\lambda} \|f\|.$$

The Hille-Yosida Theorem leads to the conclusion that A is the infinitesimal generator of a contraction semigroup on H.

Let us present another theorem (Lumer-Philipps theorem) that provides alternative necessary and sufficient conditions for the generation of a contration semigroup. The latter is based on the dissipativity of the linear operator and a range condition, see (Jacob and Zwart, 2012, Theorem 6.1.7) among others.

Theorem 2.1.2 *Let A be a closed, linear and densely defined operator on a Hilbert space H. Then the operator A is the infinitesimal generator of a contraction C_0-semigroup on H if and only if A is dissipative and $ran(I - A) = H$.*

In order to test that theorem, let us consider the operator of Example 2.1.1 and let us show that is the generator of a C_0-semigroup on $H = L^2([0,1];\mathbb{R})$.

Example 2.1.4 *The dissipativity of the operator A defined in Example 2.1.1 is proved in Example 2.1.2. In order to show that A is the infinitesimal generator of a C_0-semigroup, it remains to show the range condition $ran(I - A) = H$. The* *inclusion* $ran(I-A) \subseteq H$ *is easy to see. Let us now take* $x \in H$ *and let us prove that there exists* $y \in D(A)$ *such that* $(I-A)y = x$*. According to Example 2.1.1 such a* y *is given by*

$$((I-A)^{-1}x)(z) = \frac{\cosh(z)}{\sinh(1)} \int_0^1 \cosh((1-\zeta)) f(\zeta) d\zeta - \int_0^z \sinh((z-\zeta)) f(\zeta) d\zeta.$$

It is well-defined since $1 \in \rho(A)$*, see (2.1.7), and it lies in the domain of the operator* A*. This implies that the one-dimensional diffusion operator with Neumann boundary conditions generates a contraction* C_0- *semigroup on* H*.*

A last variant we shall go through is a corollary of the Lumer-Philipps Theorem, reported in (Jacob and Zwart, 2012, Theorem 6.1.8) for instance.

Theorem 2.1.3 *Let* A *be a linear, densely defined and closed operator on a Hilbert space* H*. Then* A *is the infinitesimal generator of a contraction* C_0- *semigroup if and only if* A *and* A^* *are dissipative, where* A^* *is the adjoint operator of* A*.*

The example on which the corollary of the Lumer-Phillips Theorem is applied corresponds to the linear part of the PDE introduced in Chapter 1.

Example 2.1.5 *Let us consider the Hilbert state space of square integrable functions* $H = L^2([0,1];\mathbb{R})$ *equipped with the inner product (2.1.9). We define the operator* A *by* $Ax = \frac{1}{Pe}\frac{d^2x}{dz^2} - \frac{dx}{dz} - k_0 x$ *for* $x \in D(A) := \{x \in H^2([0,1];\mathbb{R}), \frac{dx}{dz}(0) = Pex(0), \frac{dx}{dz}(1) = 0\}$*, where the parameters* $Pe > 0$ *and* $k_0 > 0$*. A calculation reveals that the adjoint operator of* A*, denoted by* A^**, is given by* $A^*y = \frac{1}{Pe}\frac{d^2y}{dz^2} + \frac{dy}{dz} - k_0 y$ *for* $y \in D(A^*) = \{y \in H^2([0,1];\mathbb{R}), \frac{dy}{dz}(0) = 0, \frac{dy}{dz}(1) = -Pey(1)\}$*. By taking* $x \in D(A)$ *we have that*

$$\begin{aligned}\langle Ax, x\rangle &= \frac{1}{Pe}\int_0^1 \frac{d^2x}{dz}(z)x(z)dz - \int_0^1 \frac{dx}{dz}(z)x(z)dz - k_0\int_0^1 x^2(z)dz \\ &= -\frac{1}{2}x^2(0) - \frac{1}{2}x^2(1) - \int_0^1 \left(\frac{dx}{dz}\right)^2(z)dz - k_0\int_0^1 x^2(z)dz \leq 0,\end{aligned}$$

which means that the operator A *is dissipative. Same types of arguments may be used to show that, for* $y \in D(A^*)$

$$\langle A^*y, y\rangle \leq -\frac{1}{2}y^2(1) - \frac{1}{2}y^2(0) - \frac{1}{Pe}\int_0^1 \left(\frac{dy}{dz}\right)^2(z)dz - k_0\int_0^1 y^2(z)dz \leq 0,$$

i.e. A^* *is a dissipative operator. By using Theorem 2.1.3 one may conclude that the operator* A *is the infinitesimal generator of a contraction semigroup on the Hilbert space* H*.*

2.1.2 Riesz-spectral operators

In this section, we introduce a class of operators which satisfy a spectral decomposition as matrices in finite dimensions and for which the spectrum is composed of eigenvalues only. First let us consider the following definition of a Riesz basis, see (Curtain and Zwart, 2020, Definition 3.2.1).

Definition 2.1.7 *A sequence of vectors $\{\phi_n\}_{n\geq 1}$ in a Hilbert space H forms a Riesz basis for H if the following conditions hold:*

1. $\overline{\operatorname{span}_{n\geq 1}\{\phi_n\}} = H$;
2. *There exist $m > 0$ and $M > 0$ such that for arbitrary $N \in \mathbb{N}$ and arbitrary scalars $\{\alpha_n\}_{n\geq 1}$,*

$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \| \sum_{n=1}^{N} \alpha_n \phi_n \|^2 \leq M \sum_{n=1}^{N} |\alpha_n|^2. \tag{2.1.13}$$

This definition entails that any orthonormal basis of a Hilbert space H is a Riesz basis for H. The notion of Riesz basis can be viewed as an extension of orthonormal basis according to the following lemma, see (Curtain and Zwart, 2020, Lemma 3.2.4).

Lemma 2.1.1 *Let $\{\phi_n\}_{n\geq 1}$ be a Riesz basis of a Hilbert space H. Then it holds that:*

1. *There exists a unique biorthogonal sequence $\{\psi_n\}_{n\geq 1}$, i.e. $\langle \phi_n, \psi_m \rangle = \delta_{nm}$;*
2. *Every $z \in H$ can be represented uniquely by*

$$z = \sum_{n=1}^{\infty} \langle z, \psi_n \rangle \phi_n,$$

and

$$m \sum_{n=1}^{\infty} |\langle z, \psi_n \rangle|^2 \leq \|z\|^2 \leq M \sum_{n=1}^{\infty} |\langle z, \psi_n \rangle|^2,$$

where the constants m and M come from (2.1.13).

Moreover, we want to stress in the following lemma that any Riesz basis can be obtained from an orthonormal basis thanks to an invertible linear and bounded operator, see (Curtain and Zwart, 2020, Lemma 3.2.2).

Lemma 2.1.2 *Suppose that $\{e_n\}_{n\geq 1}$ is an orthonormal basis for the Hilbert space H. Then $\{\phi_n\}_{n\geq 1}$ forms a Riesz basis for H if and only if there exists an operator $T \in \mathscr{L}(H)$ such that T is boundedly invertible and $Te_n = \phi_n$. For a given Riesz basis this operator T can be chosen such that $\|T\|^2 \leq M$ and*

$\|T^{-1}\|^2 \leq m^{-1}$. Moreover, if the Riesz basis is constructed via $Te_n = \phi_n$, then $M = \|T\|^2$ and $m^{-1} = \|T^{-1}\|^2$.

Before defining a Riesz spectral operator, let us consider the following characterization coming from (Curtain and Zwart, 2020, Lemma 3.2.5).

Lemma 2.1.3 *Suppose that a closed, linear operator A on a Hilbert space H has simple eigenvalues $\{\lambda_n\}_{n\geq 1}$ and that its corresponding eigenvectors $\{\phi_n\}_{n\geq 1}$ form a Riesz basis in H. If $\{\psi_n\}_{n\geq 1}$ are the eigenvectors of the adjoint of A corresponding to the eigenvalues $\{\overline{\lambda_n}\}_{n\geq 1}$, then the $\{\psi_n\}_{n\geq 1}$ can be suitably scaled so that $\{\phi_n\}_{n\geq 1}$ and $\{\psi_n\}_{n\geq 1}$ are biorthogonal.*

The previous considerations allow to introduce the definition of a Riesz-spectral operator, see (Curtain and Zwart, 2020, Definition 3.2.6).

Definition 2.1.8 *Suppose that A is a linear, closed operator on a Hilbert space H. Assume that A has simple eigenvalues $\{\lambda_n\}_{n\geq 1}$ with corresponding eigenvectors $\{\phi_n\}_{n\geq 1}$. If*

1. *$\{\phi_n\}_{n\geq 1}$ form a Riesz basis of H;*
2. *The set of eigenvalues $\{\lambda_n\}_{n\geq 1}$ has at most finitely many accumulation points,*

then A is called a Riesz-spectral operator.

In order to illustrate the concept of Riesz operator, we shall use the operator of Example 2.1.1 again. It has been shown in (2.1.7) that the resolvent set of A is given by $\rho(A) = \mathbb{C} \setminus \{-n^2\pi^2, n \in \mathbb{N}\}$, which means that the spectrum of the operator A is composed of only eigenvalues which are given by $\{-n^2\pi^2\}_{n\geq 0}$. Moreover, it can be easily seen that the eigenfunctions of the operator A are given by $1_{[0,1]}(z) \cup \{\sqrt{2}\cos(n\pi z)\}_{n\geq 1}$. This set of eigenfunctions is an orthonormal basis of $H := L^2([0,1];\mathbb{R})$ and hence a Riesz basis. According to Definition 2.1.8, the diffusion operator with Neumann boundary conditions is a Riesz spectral operator.

The class of Riesz spectral operators possesses nice properties in terms of generation of C_0-semigroups, resolvent operators, ... This is summarized in the following Theorem, see (Curtain and Zwart, 2020, Theorem 3.2.8) for instance.

Theorem 2.1.4 *Suppose that A is a Riesz-spectral operator with simple eigenvalues $\{\lambda_n\}_{n\geq 1}$ and corresponding eigenvectors $\{\phi_n\}_{n\geq 1}$. Let $\{\psi_n\}_{n\geq 1}$ be the eigenvectors of A^* such that $\langle \phi_n, \psi_m \rangle = \delta_{nm}$. Then A satisfies the following properties:*

- *The resolvent set $\rho(A)$ is given by $\{\lambda \in \mathbb{C}, \inf_{n\geq 1} |\lambda - \lambda_n| > 0\}$, the spectrum* *of A has the expression $\sigma(A) = \overline{\{\lambda_n\}_{n\geq 1}}$ and for $\lambda \in \rho(A)$, it holds that*

$$(\lambda I - A)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle \cdot, \psi_n \rangle \phi_n;$$

- *The operator A has the representation*

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n \tag{2.1.14}$$

for $x \in D(A)$ given by $D(A) = \{x \in H, \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 < \infty\}$;

- *A is the infinitesimal generator of a C_0−semigroup $(T(t))_{t\geq 0}$ if and only if $\sup_{n\geq 1} \mathfrak{Re}(\lambda_n) < \infty$ and $T(t)$ is given by*

$$T(t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle \cdot, \psi_n \rangle \phi_n; \tag{2.1.15}$$

- *The growth bound of $(T(t))_{t\geq 0}$ is given by*

$$\omega_0 = \inf_{t>0} \left(\frac{1}{t} \log \|T(t)\| \right) = \sup_{n\geq 1} \mathfrak{Re}(\lambda_n). \tag{2.1.16}$$

Note that the relation (2.1.16) means that in the case of a Riesz-spectral operator, the growth bound of the semigroup of which it is the infinitesimal generator may be viewed as a generalization of what is called the spectral abscissa for matrices.

According to the latter theorem, the spectral representation of the linear diffusion operator with Neumann boundary conditions is given by

$$Ax = -\sum_{n=1}^{\infty} 2n^2\pi^2 \langle x, \cos(n\pi z) \rangle \cos(n\pi z),$$

for $x \in D(A)$ given by (2.1.4). Moreover, the corresponding generated semigroup $(T(t))_{t\geq 0}$ is expressed as

$$T(t)x = \langle x, 1_{[0,1]} \rangle 1_{[0,1]}(z) + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle x, \cos(n\pi z) \rangle \cos(n\pi z),$$

for $t \geq 0$ and $x \in H$.

2.1.2.1 Sturm-Liouville operators

Finding the spectrum and the eigenfunctions of an operator may be difficult in many cases. Therefore, we shall consider a particular class of systems that is of prime importance in the context of Riesz-spectral operators and which is called the class of Sturm-Liouville systems.

Let us consider the state space of square integrable functions $H = L^2([a,b];\mathbb{K})$ where $a,b \in \mathbb{R}, a < b$ and $\mathbb{K}$ is an arbitrary vector field. Let us consider the class of operators A defined on the domain $D(A)$ expressed as

$$D(A) = \left\{ x \in H^2([a,b];\mathbb{K}), \alpha_a \frac{dx}{dz}(a) + \beta_a x(a) = 0, \alpha_b \frac{dx}{dz}(b) + \beta_b x(b) = 0 \right\}, \tag{2.1.17}$$

where $(\alpha_a \quad \beta_a) \neq (0 \quad 0), (\alpha_b \quad \beta_b) \neq (0 \quad 0)$. The operator A is said to be a Sturm-Liouville operator if it admits the representation

$$(Ax)(z) = \frac{1}{r(z)} \left(\frac{d}{dz} \left(-p(z) \frac{dx}{dz}(z) \right) + q(z)x(z) \right), \tag{2.1.18}$$

where $p, \frac{dp}{dz}, q$ and r are real-valued and continuous functions such that $r > 0, p > 0$, see e.g. Delattre et al. (2003). The following lemma makes the link between Sturm-Liouville and Riesz-spectral operators, see (Delattre et al., 2003, Lemma 1).

Lemma 2.1.4 *Let A be the negative of a Sturm-Liouville operator (2.1.18) defined on its domain (2.1.17). Then*

1. *A is a Riesz-spectral operator,*
2. *A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on H.*

As an illustration, let us consider the operator A of Example 2.1.5. It can be seen that its opposite admits the representation (2.1.18) where the functions r, p and q are given by

$$r(z) = e^{-Pe\,z}, p(z) = \frac{1}{Pe} r(z), q(z) = k_0 r(z).$$

Moreover, the domain of A may be written as (2.1.17) with

$$(\alpha_a \quad \beta_a) = (1 \quad -Pe), (\alpha_b \quad \beta_b) = (1 \quad 0).$$

Consequently, the operator A described in Example 2.1.5 is a Riesz-spectral operator.

2.1.2.2 Perturbation of Riesz-spectral operators

A natural question one may asked is under which types of perturbation a Riesz-spectral operator preserves that property. Let us consider the state Hilbert space H equipped with the inner product $\langle \cdot, \cdot \rangle_H$. Let $A : D(A) \subset H \to H$ be a Riesz-spectral operator on H whose spectrum is composed only of the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. On the spectrum of A, we make the following assumption:

$$\sup_{m\in\mathbb{N}} \sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{1}{|\lambda_n - \lambda_m|^2} = \kappa, \tag{2.1.19}$$

for some positive and finite constant κ. This entails that the following condition holds:

$$\inf_{n,m\in\mathbb{N},n\neq m} |\lambda_n - \lambda_m| = \mu, \mu > 0. \tag{2.1.20}$$

Indeed, assume by contradiction that this does not hold. Let us take any $k \in \mathbb{N}$ such that $\kappa \leq k^2$. Then there exists $\lambda_{n_k}, \lambda_{m_k}$ such that $|\lambda_{n_k} - \lambda_{m_k}| \leq 1/k$, which implies that the sum of the series in (2.1.19) exceeds k^2, leading to a contradiction.

The condition (2.1.20) is imposed in order to be able to characterize the spectrum of a particular perturbation of the operator A, see later in (2.1.21). It could be viewed as a bit restrictive but it is useful to preserve what is called the *spectrum determined growth assumption* (SDGA). This says that the growth bound of a C_0-semigroup is given by the supremum of the real part of the eigenvalues of its infinitesimal generator, which is a priori not guaranteed in infinite dimensions. For an overview on the root locii and the preservation of the SDGA under bounded perturbation, we refer to Jacob and Morris (2016).

Let us consider $b, c \in H$ and define the operator A_p as the following perturbation of the operator A:

$$A_p = A + b\langle c, \cdot\rangle_H. \tag{2.1.21}$$

Thanks to condition (2.1.19), the operator A_p defined in (2.1.21) is still a Riesz-spectral operator whose spectrum is composed of eigenvalues only. This result may be found in (Shun-Hua, 1981, Theorem 2.1) in the case where A is self-adjoint. An extension to non necessarily self-adjoint operators has been considered in (Curtain, 1985, Appendix B). This result may be useful for instance in the case where optimal control of the following class of systems is considered:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \in H \\ y(t) = Cx(t), \end{cases}$$

where the (unbounded) linear operator $A : D(A) \subset H \to H$ is a Riesz-spectral operator whose eigenvalues satisfy (2.1.19), the control operator $B : \mathbb{R} \to H$ is defined as $Bu = bu$ for some function $b \in H$ and the observation operator $C : H \to \mathbb{R}$ is given as $Cx = \langle c, x\rangle_H$, with $c \in H$. An optimal control law $u(t)$ which minimizes the functional cost

$$J(u, x_0) = \int_0^\infty \left(\|Cx(t)\|^2 + \|u(t)\|^2\right) dt$$

is known to be given[(1)] by the linear state feedback $u(t) = K_o x(t) = \langle k_o, x(t)\rangle_H$ for some function $k_o \in H$, see (Winkin et al., 2004, Section 4.2) for instance. This entails that the closed-loop operator $A + b\langle k_o, \cdot\rangle_H$ is a Riesz-spectral operator.

As an illustration let us consider the operator A of Example 2.1.1 again. Let us fix an arbitrary small $\sigma > 0$ and let us define the operator A_p as

$$A_p x = Ax - \frac{1}{\ell} 1_{[0,\ell]}(z)\langle x, 1_{[0,1]}\rangle, \tag{2.1.22}$$

[(1)] We make here the assumptions that the pairs (A, B) and (C, A) are exponentially stabilisable and detectable, respectively, see Definition 2.1.10 in the next section.

where $x \in D(A_p) = D(A)$ given in (2.1.4). The operator A_p arises for instance when trying to positively stabilize the heat equation with Neumann boundary conditions with the integral state feedback $u(t) = -\int_0^1 x(z,t)dz$ distributed on a window of width ℓ right of $z = 0$, see (Abouzaïd et al., 2021, Section 5.2). As already noticed, the operator A is of Riesz-spectral type with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} = \{-n^2\pi^2\}_{n\in\mathbb{N}}$. Therefore it can be checked that the condition (2.1.19) holds. In particular there holds

$$\sup_{m\in\mathbb{N}} \sum_{\substack{n=0\\ n\neq m}}^{\infty} \frac{1}{|\lambda_n - \lambda_m|^2} \leq \frac{1}{6\pi^2}.$$

Consequently, the operator A_p is a Riesz spectral operator. Computations reveal that the eigenvalues of A_p are given by $\{\lambda_n^p\}_{n\in\mathbb{N}} = \{-1\} \cup \{-n^2\pi^2\}_{n\geq 1}$ and the corresponding eigenfunctions are expressed as

$$\begin{aligned}
\phi_0^p(z) = K_{\phi_0^p} \Bigg[& \cos(z) \left(\frac{\sin(\frac{\ell}{2})}{\frac{\ell}{2}} \frac{\cos(\frac{2-\ell}{2})}{\sin(1)} \right) \\
& + 1_{[\ell,1]}(z) \frac{\sin(\frac{\ell}{2})}{\frac{\ell}{2}} \sin(z - \frac{\ell}{2}) + \frac{1}{\ell} 1_{[0,\ell]}(z)(1 - \cos(z)) \Bigg], \\
\phi_n^p(z) = \sqrt{2}\cos(n\pi z), & n \in \mathbb{N}_0.
\end{aligned}$$

The eigenfunctions of the adjoint operator[(2)] of A_p are given by

$$\begin{aligned}
\psi_0^p(z) &= K_{\psi_0} 1_{[0,1]}(z), \\
\psi_n^p(z) &= K_{\psi_n} \left[\frac{n\pi\ell}{\sin(n\pi\ell)} \left(1 - \frac{1}{n^2\pi^2}\right) \cos(n\pi z) + \frac{1}{n^2\pi^2} \right],
\end{aligned}$$

where $n \in \mathbb{N}_0$. The constants $K_{\phi_0^p}, K_{\psi_n}, n \geq 0$ are such that the sequences $\{\phi_n^p\}_{n\in\mathbb{N}}$ and $\{\psi_n^p\}_{n\in\mathbb{N}}$ are biorthogonal, i.e. $\langle \phi_n^p, \psi_m^p \rangle_H = \delta_{nm}$. Thanks to Theorem 2.1.4 the operator A_p (2.1.22) may be decomposed as

$$A_p x = \sum_{n=0}^{\infty} \lambda_n^p \langle x, \psi_n^p \rangle_H \phi_n^p,$$

for any $x \in D(A_p)$. Moreover, this operator is known to generate a C_0-semigroup of bounded linear operators whose expression is given by

$$T_p(t)x = \sum_{n=0}^{\infty} e^{\lambda_n^p t} \langle x, \psi_n^p \rangle_H \phi_n^p,$$

where x is allowed to be chosen arbitrarily in H.

(2) It is expressed as $A_p^* = A^* + \frac{1}{\ell} 1_{[0,1]}(z) \langle \cdot, 1_{[0,\ell]} \rangle_H = A + \frac{1}{\ell} 1_{[0,1]}(z) \langle \cdot, 1_{[0,\ell]} \rangle_H$.

2.1.3 Stability analysis

Here we aim at describing the asymptotic behavior of solutions of the abstract differential equation (2.1.1). In particular, we shall consider the exponential decay of the trajectories of (2.1.1), i.e. of the C_0-semigroup $(T(t))_{t\geq 0}$, also called exponential stability, see Curtain and Zwart (2020).

Definition 2.1.9 *A strongly continuous semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on a Hilbert space H is exponentially stable if there exist positive constants M and α such that*

$$\|T(t)\| \leq Me^{-\alpha t},\ t \geq 0. \tag{2.1.23}$$

If the C_0-semigroup $(T(t))_{t\geq 0}$ whose infinitesimal generator is the operator A from (2.1.1) is exponentially stable, then the solution of (2.1.1) with initial condition $x_0 \in H$, i.e. $x(t) = T(t)x_0$, converges exponentially fast to 0 when t tends to ∞.

Some characterizations are available for determining whether a C_0-semigroup is exponentially stable or not. Let us first consider Datko's lemma which gives necessary and sufficient conditions for the exponential stability.

Lemma 2.1.5 *The C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H is exponentially stable if and only if for every $x \in H$ there exists a positive constant $\gamma_x < \infty$ (that may depend on x) such that*

$$\int_0^\infty \|T(t)x\|_H^2 \leq \gamma_x. \tag{2.1.24}$$

As an example of Datko's lemma, let us consider the operator A defined by

$$Ax = \frac{d^2x}{dz^2} + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}+\sqrt{6}}{2}1_{[0,1]}(z)\int_0^1 x(z)dz, \tag{2.1.25}$$

where $x \in D(A) = \{x \in H^2([0,1];\mathbb{R}), \frac{dx}{dz}(0) = 0 = \frac{dx}{dz}(1)\}$ and for which the state space is $H = L^2([0,1];\mathbb{R})$. This operator can be viewed as a perturbation of the diffusion operator defined in Example 2.1.1. It may pop up as the result of the resolution of the following optimal control[(3)] problem

$$\begin{cases} \min_{u(\cdot)\in L^2([0,\infty);\mathbb{R})} \int_0^\infty \left(\left(\int_0^1 x(z,t)dz\right)^2 + u^2(t)\right)dt \\ \text{subject to } \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \frac{\sqrt{2}}{2}x + 1_{[0,1]}(z)u(t), x(0,t) = x_0(z). \end{cases} \tag{2.1.26}$$

It can be shown that the latter is a Riesz-spectral operator whose eigenvalues are simple and given by the set $\{\lambda_n\}_{n\in\mathbb{N}} = \{-\frac{\sqrt{6}}{2}\} \cup \{-n^2\pi^2 + \frac{\sqrt{2}}{2}\}_{n\in\mathbb{N}_0}$. The corresponding eigenfunctions are the same as the ones of the non-perturbed diffusion operator, i.e.

(3) The resulting optimal control $u^*(t) = -\frac{\sqrt{2}+\sqrt{6}}{2}\int_0^1 x(z,t)dz$ and has been obtained by solving an appropriate Riccati equation.

$\{\phi_n\}_{n\in\mathbb{N}} = \{1_{[0,1]}(z)\} \cup \{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}_0}$. An easy computation reveals also that the operator A defined in (2.1.25) is self-adjoint with respect to the classical inner product whose H is equipped with. Consequently, A admits the spectral representation

$$Ax = \sum_{n=0}^{\infty} \lambda_n \langle x, \phi_n \rangle_H \phi_n,$$

where the function $x \in D(A)$. Hence, the corresponding semigroup $(T(t))_{t\geq 0}$ is given by

$$(T(t)x)(z) = e^{\frac{-\sqrt{6}}{2}t} \langle x, 1_{[0,1]} \rangle_H 1_{[0,1]}(z) + 2\sum_{n=1}^{\infty} e^{(-n^2\pi^2 + \frac{\sqrt{2}}{2})t} \langle x, \cos(n\pi z) \rangle_H \cos(n\pi z),$$

where x may be chosen in H. A computation of $\|T(t)x\|_H^2 = \langle T(t)x, T(t)x \rangle_H$ gives rise to

$$\|T(t)x\|_H^2 = e^{-\sqrt{6}t} \langle x, 1_{[0,1]} \rangle_H^2 + 2\sum_{n=1}^{\infty} e^{(-2n^2\pi^2 + \sqrt{2})t} \langle x, \cos(n\pi z) \rangle_H^2,$$

which has the consequence that

$$\int_0^\infty \|T(t)x\|_H^2 dt = \frac{1}{\sqrt{6}} \langle x, 1_{[0,1]} \rangle_H^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - \sqrt{2}} \langle x, \cos(n\pi z) \rangle_H^2.$$

According to the Cauchy-Schwarz inequality, it can be shown that

$$\int_0^\infty \|T(t)x\|_H^2 dt \leq \|x\|_H^2 \left(\frac{1}{\sqrt{6}} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - \sqrt{2}} \right) =: \gamma_x < \infty,$$

which is a statement of exponential stability according to Lemma 2.1.5.

Remark 2.1.1 *Note that a generalization of Datko's Lemma consists in showing that the integral*

$$\int_0^\infty \|T(t)x\|_H^p dt$$

is convergent for every $x \in H$, for some $p \in [1, \infty)$. This result can be found in Buse et al. (2006) and (Pazy, 1983, Theorem 4.1) for instance and is valid also when H is not necessarily a Hilbert space.

An information that can help to conclude on exponential stability of a C_0- semigroup is its Growth bound. The latter has already been defined in Theorem 2.1.4. In that way, the following result makes the link between the sign of the growth bound of a C_0-semigroup $(T(t))_{t\geq 0}$ and the exponential stability of $(T(t))_{t\geq 0}$.

Proposition 2.1.5 *Let us consider the strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H. Then, $(T(t))_{t\geq 0}$ is exponentially stable if and only if its growth bound is negative.*

Proof. Let us start with the sufficiency. Let $\omega_0 = \inf_{t>0} \frac{1}{t} \log \|T(t)\|$ be the growth bound of $(T(t))_{t\geq 0}$. According to (Curtain and Zwart, 2020, Theorem 2.1.7 e.), for any $\omega > \omega_0$, there exists M_ω such that the relation $\|T(t)\| \leq M_\omega e^{\omega t}$ holds for all $t \geq 0$. Hence, the negativity of ω_0 implies exponential stability of $(T(t))_{t\geq 0}$.

Assume now that $(T(t))_{t\geq 0}$ is exponentially stable. Then, for any $t \geq 0$, there holds $\|T(t)\| \leq Me^{-\alpha t}$, for positive constants M and α. Without loss of generality let us assume that $M > 1$ here (this is always possible since $Me^{-\alpha t}$ is a positive and increasing function of M). Define the time instant $t^* := \frac{\log(2M)}{\alpha} > 0$. Then for t^*, we have that $\|T(t^*)\| \leq \frac{1}{2}$, which means that $\omega_0 = \inf_{t>0} \frac{1}{t} \log \|T(t)\| \leq \frac{1}{t^*} \log \|T(t^*)\| < 0$. □

This proposition has the consequence that for a Riesz-spectral operator A with simple eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, the corresponding C_0-semigroup is exponentially stable if and only if $\sup_{n\in\mathbb{N}} \mathfrak{Re}(\lambda_n) < 0$, see (2.1.16).

To highlight this result, let us take the operator A of Example 2.1.5 again. It has been shown that the latter is a Riesz-spectral operator. The calculation of its eigenvalues reveals that

$$\{\lambda_n\}_{n\in\mathbb{N}} = \left\{ -\frac{Pe}{4} s_n^2 - \frac{Pe}{4} - k_0 \right\}_{n\in\mathbb{N}}, \tag{2.1.27}$$

where the positive numbers $\{s_n\}_{n\in\mathbb{N}}$ are the solutions of the resolvent equation

$$\tan\left(\frac{Pe}{2} s_n\right) = \frac{2s_n}{s_n^2 - 1}.$$

Note also as an information that the eigenfunctions of the operator A are expressed as

$$\{\phi_n(z)\}_{n\in\mathbb{N}} = K_n \left(e^{\frac{Pe}{2}z} \left[\sin\left(\frac{Pe}{2} z\right) + s_n \cos\left(\frac{Pe}{2} z\right) \right] \right), z \in [0,1],$$

where K_n is a constant that normalizes ϕ_n. Given the expression of $\{\lambda_n\}_{n\in\mathbb{N}}$ (2.1.27), it is easy to see that

$$\sup_{n\in\mathbb{N}} \mathfrak{Re}(\lambda_n) < -\frac{Pe}{4} - k_0 < 0,$$

which means that the C_0-semigroup generated by the operator A is exponentially stable.

We shall now consider another tool that gives necessary and sufficient conditions for proving exponential stability of a C_0-semigroup.

Theorem 2.1.6 *Assume that the operator A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H. Then $(T(t))_{t\geq 0}$ is exponentially stable if and only if there exists a positive operator $P \in \mathscr{L}(H)$ such that*

$$\langle Ax, Px\rangle + \langle Pz, Az\rangle = -\langle x, x\rangle, \tag{2.1.28}$$

for all $x \in D(A)$.

This theorem is known as the Lyapunov Theorem and (2.1.28) is called a Lyapunov equation. Note that another version of that theorem exists, where the relation (2.1.28) is replaced by

$$\langle Ax, Px\rangle + \langle Pz, Az\rangle \leq -\langle x, x\rangle, x \in D(A). \tag{2.1.29}$$

As an illustration, let us consider the application of (Curtain and Zwart, 2020, Example 2.3.5) which deals with the following abstract differential equation

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + A_0 x = 0, x(0) = x_0, \frac{dx}{dt}(0) = x_1, \tag{2.1.30}$$

where it is assumed that the operator $A_0 : D(A_0) \subset H \to H$ is a self-adjoint and coercive operator[(4)] on the Hilbert space H (equipped with the real inner product $\langle\cdot,\cdot\rangle_H$) and $\alpha > 0$. Let us introduce the Hilbert space $Z = D(A_0^{\frac{1}{2}}) \times H$ with the inner product

$$\langle w, \tilde{w}\rangle_Z = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} \tilde{w}_1\rangle_H + \langle w_2, \tilde{w}_2\rangle_H, \tag{2.1.31}$$

for $w = (w_1 \quad w_2)^T \in D(A_0^{\frac{1}{2}}) \times H$ and $\tilde{w} = (\tilde{w}_1 \quad \tilde{w}_2)^T \in D(A_0^{\frac{1}{2}}) \times H$. Note that (2.1.30) may also be written as

$$\dot{X}(t) = AX(t), X(0) = X_0, \tag{2.1.32}$$

where the state vector $X(t) = (x(t) \quad \frac{dx}{dt}(t))^T$ and the operator $A : D(A) = D(A_0) \times D(A_0^{\frac{1}{2}}) \subset Z \to Z$ is defined by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix}. \tag{2.1.33}$$

According to (Curtain and Zwart, 2020, Example 2.3.5) the operator A is the infinitesimal generator of a C_0-semigroup on $Z = D(A_0^{\frac{1}{2}}) \times H$. We shall show in the next proposition that the latter is exponentially stable.

Proposition 2.1.7 *The C_0-semigroup $(T(t))_{t\geq 0}$ whose operator A defined in (2.1.33) is the infinitesimal generator, is exponentially stable on $Z = D(A_0^{\frac{1}{2}}) \times H$.*

[(4)]This means that there exists a positive constant k such that $\langle A_0 x, x\rangle_H \geq k\|x\|_H^2$ for every $x \in H$. As a consequence, the operator A_0 is boundedly invertible, i.e. $A_0^{-1} \in \mathscr{L}(H)$, see (Curtain and Zwart, 2020, Lemma A.3.85)

Proof. Let us define the operator $P : Z \to Z$ by

$$P = \begin{pmatrix} \frac{1}{\alpha}I + \frac{\alpha}{2}A_0^{-1} & \frac{1}{2}A_0^{-1} \\ \frac{1}{2}I & \frac{1}{\alpha}I \end{pmatrix}. \tag{2.1.34}$$

Let us check that the operator A_0^{-1} is bounded from $D(A_0^{\frac{1}{2}})$ into itself. For this pick any $w_1 \in D(A_0^{\frac{1}{2}})$ and observe that

$$\|A_0^{-1}w_1\|^2_{D(A_0^{\frac{1}{2}})} = \langle A_0^{\frac{1}{2}}A_0^{-1}w_1, A_0^{\frac{1}{2}}A_0^{-1}w_1\rangle_H = \langle A_0^{-1}w_1, w_1\rangle_H \leq \|A_0^{-1}\|\|w_1\|_H^2.$$

As the operator A_0 is boundedly invertible, so is its square root $A_0^{\frac{1}{2}}$, see (Curtain and Zwart, 2020, Lemma A.3.84). Consequently, it holds that

$$\begin{aligned}\|A_0^{-1}w_1\|^2_{D(A_0^{\frac{1}{2}})} &\leq \|A_0^{-1}\|\|w_1\|_H^2 = \|A_0^{-1}\|\|A_0^{-\frac{1}{2}}A_0^{\frac{1}{2}}w_1\|_H^2 \\ &\leq \|A_0^{-1}\|\|A_0^{-\frac{1}{2}}\|^2\langle A_0^{\frac{1}{2}}w_1, A_0^{\frac{1}{2}}w_1\rangle_H = C\|w_1\|^2_{D(A_0^{\frac{1}{2}})},\end{aligned}$$

where $C := \|A_0^{-1}\|\|A_0^{-\frac{1}{2}}\|^2$. Using the boundedness of A_0^{-1} on H and the previous arguments implies that the operator P is bounded from Z into Z. Moreover, according to (Curtain and Zwart, 2020, Exercise 4.11), the operator $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ is self-adjoint for the inner product defined in (2.1.31) if and only if P_{22} is self-adjoint, $P_{11}D(A_0) \subset D(A_0)$, $P_{11}^*A_0 = A_0P_{11}$ on $D(A_0)$ and $P_{12}^*A_0 = P_{21}$ on $D(A_0)$. According to (2.1.34), it is easy to see that P_{22} is self-adjoint and that $P_{11}D(A_0) \subset D(A_0)$. The facts that $P_{11}^*A_0 = A_0P_{11}$ on $D(A_0)$ and $P_{12}^*A_0 = P_{21}$ on $D(A_0)$ come from the relation[(5)] $(A_0^{-1})^* = A_0^{-1}$. Moreover for $w = (w_1 \quad w_2)^T \in Z$, it holds that

$$\begin{aligned}\langle Pw, w\rangle_Z &= \langle \frac{1}{\alpha}A_0^{\frac{1}{2}}w_1 + \frac{\alpha}{2}A_0^{\frac{1}{2}}A_0^{-1}w_1, A_0^{\frac{1}{2}}w_1\rangle_H + \langle \frac{1}{2}A_0^{\frac{1}{2}}A_0^{-1}w_2, A_0^{\frac{1}{2}}w_1\rangle_H \\ &\quad + \frac{1}{2}\langle w_1, w_2\rangle_H + \frac{1}{\alpha}\langle w_2, w_2\rangle_H \\ &= \frac{1}{\alpha}\langle A_0^{\frac{1}{2}}w_1, A_0^{\frac{1}{2}}w_1\rangle_H + \frac{\alpha}{2}\langle w_1, w_1\rangle_H + \langle w_1, w_2\rangle_H + \frac{1}{\alpha}\langle w_2, w_2\rangle_H \\ &\geq \frac{1}{2\alpha}\langle A_0^{\frac{1}{2}}w_1, A_0^{\frac{1}{2}}w_1\rangle_H + \langle \frac{\sqrt{\alpha}}{\sqrt{2}}w_1 + \frac{1}{\sqrt{2\alpha}}w_2, \frac{\sqrt{\alpha}}{\sqrt{2}}w_1 + \frac{1}{\sqrt{2\alpha}}w_2\rangle_H \\ &\quad + \frac{1}{2\alpha}\langle w_2, w_2\rangle_H \geq \frac{1}{2\alpha}\|w\|_Z^2,\end{aligned}$$

which shows that P is coercive. Furthermore, by taking $w = (w_1 \quad w_2)^T \in D(A)$ and by looking at the quantity $\langle Aw, Pw\rangle_Z + \langle Pw, Aw\rangle_Z$, one gets that

$$\langle Aw, Pw\rangle_Z + \langle Pw, Aw\rangle_Z = 2\langle Aw, Pw\rangle_Z$$

[(5)] This holds since A_0 is self-adjoint and has a bounded inverse.

$$
\begin{aligned}
&= 2\left\langle \begin{pmatrix} w_2 \\ -A_0 w_1 - \alpha w_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{\alpha} w_1 + \frac{\alpha}{2} A_0^{-1} w_1 + \frac{1}{2} A_0^{-1} w_2 \\ \frac{1}{2} w_1 + \frac{1}{\alpha} w_2 \end{pmatrix} \right\rangle_Z \\
&= \frac{2}{\alpha} \langle A_0^{\frac{1}{2}} w_2, A_0^{\frac{1}{2}} w_1 \rangle_H + \alpha \langle A_0^{\frac{1}{2}} w_2, A_0^{\frac{1}{2}} A_0^{-1} w_1 \rangle_H + \langle A_0^{\frac{1}{2}} w_2, A_0^{\frac{1}{2}} A_0^{-1} w_2 \rangle_H \\
&- \langle A_0 w_1, w_1 \rangle_H - \frac{2}{\alpha} \langle A_0 w_1, w_2 \rangle_H - \alpha \langle w_2, w_1 \rangle_H - 2 \langle w_2, w_2 \rangle_H .
\end{aligned} \tag{2.1.35}
$$

As the elements $A_0^{-1} w_1$ and $A_0^{-1} w_2$ are in $D(A_0)$ by construction and as the relation $\langle A_0 w_1, w_1 \rangle_H = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_1 \rangle_H$ holds for any $w_1 \in D(A_0)$, the relation (2.1.35) can also be written as

$$
\begin{aligned}
&\langle Aw, Pw \rangle_Z + \langle Pw, Aw \rangle_Z = \frac{2}{\alpha} \langle w_2, A_0 w_1 \rangle_H + \alpha \langle w_2, w_1 \rangle_H + \langle w_2, w_2 \rangle_H \\
&- \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_1 \rangle_H - \frac{2}{\alpha} \langle A_0 w_1, w_2 \rangle_H - \alpha \langle w_2, w_1 \rangle_H - 2 \langle w_2, w_2 \rangle_H \\
&= -\langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_1 \rangle_H - \langle w_2, w_2 \rangle_H = - \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_Z^2 .
\end{aligned}
$$

The conclusion follows thanks to Theorem 2.1.6. □

This example pops up for instance when looking at the damped wave equation with Dirichlet boundary condition at one hand and Neumann boundary condition at the other hand. The displacement of the wave at time $t \in [0, \infty)$ and position $z \in [0, 1]$, denoted by $x(t, z)$, is subject to the PDE

$$
\begin{cases} \frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2} - \alpha \frac{\partial x}{\partial t} \\ x(t, 0) = 0, \frac{\partial x}{\partial z}(t, 1) = 0, \end{cases} \tag{2.1.36}
$$

where $\alpha > 0$ is a positive parameter coming from the damping. The PDE (2.1.36) admits the representation (2.1.32) with A being defined as (2.1.33), where $A_0 : D(A_0) \subset H \to H, H = L^2([0, 1]; \mathbb{R})$ is given by $A_0 = -\frac{d^2}{dz^2}$ on $D(A_0) = \{x \in H^2([0, 1]; \mathbb{R}), x(0) = 0 = \frac{dx}{dz}(1)\}$. Thanks to the Poincaré inequality, it can be shown that

$$
\langle A_0 x, x \rangle_H = \int_0^1 \left(\frac{dx}{dz} \right)^2 dz \geq \frac{\pi^2}{4} \|x\|_H^2,
$$

which implies that A_0 is coercive. As it is also self-adjoint, it admits a unique nonnegative square-root whose domain is expressed as

$$
D(A_0^{\frac{1}{2}}) = \{x \in H^1([0, 1]; \mathbb{R}), x(0) = 0\},
$$

see e.g. (Tucsnak and Weiss, 2009, Chapter 3). This allows to consider the Hilbert state space $Z = D(A_0^{\frac{1}{2}}) \times H$ equipped with the inner product

$$
\langle w, \tilde{w} \rangle_Z = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} \tilde{w}_1 \rangle_H + \langle w_2, \tilde{w}_2 \rangle_H = \langle \frac{dw_1}{dz}, \frac{d\tilde{w}_1}{dz} \rangle_H + \langle w_2, \tilde{w}_2 \rangle_H, \tag{2.1.37}
$$

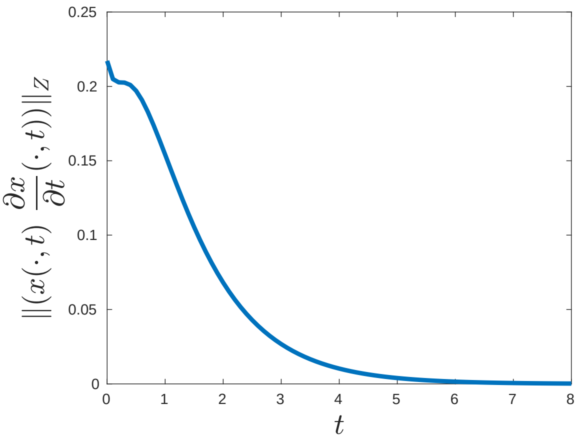

Figure 2.1 – $Z-$norm of the solution pair $(x(z,t)\ \frac{\partial x}{\partial t}(z,t))$ of (2.1.36).

for $w = (w_1, w_2) \in D(A_0^{\frac{1}{2}})$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in H$. Hence, according to Proposition 2.1.7, the $Z-$norm of the solution pair $(x(z,t)\ \ \frac{\partial x}{\partial t}(z,t))$ of (2.1.36) converges exponentially fast to 0 when t goes to ∞. This is illustrated in Figure 2.1 wherein the exponential decay can be observed. Moreover the state trajectories $x(z,t)$ and $\frac{\partial x}{\partial t}(z,t)$ are depicted in Figures 2.2 and 2.3. Note that the initial conditions have been fixed to $x(z,0) = \frac{1}{6}\sin(\frac{\pi}{2}z)$ and $\frac{\partial x}{\partial t}(z,0) = \frac{1}{5}(2z^2 - z^4)$ while the damping parameter is set to $\alpha = \pi + \frac{1}{3}$.

Remark 2.1.2 *The numerical method that has been used is based on a space discretization of the operator A (2.1.33) by means of finite differences. The spatial coordinate has been discretized into n equal pieces,* $n = 50$*. Based on this, a finite-dimensional approximation of the operator A has been obtained, let us denote it by* $A_n \in \mathbb{R}^{2n\times 2n}$*. Let us also denote by* $X_n \in \mathbb{R}^{2n}$ *the approximation of the state vector X. Its components are given by*

$$X_n^i(t) = x((i-1)h,t), X_n^{i+n}(t) = \frac{\partial x}{\partial z}((i-1)h,t), i = 1,\ldots,n,$$

where h stands for the discretization step ($h = \frac{1}{n-1}$)*. Then, the linear finite-dimensional approximation of (2.1.32),* $\dot{X}_n(t) = A_n X_n(t)$*, has been numerically integrated via the routine* `ode15s` *of Matlab©.*

We end this section by recalling the notions of exponential stabilizability and exponential detectability, which will be of interest later in this thesis, see e.g. (Curtain

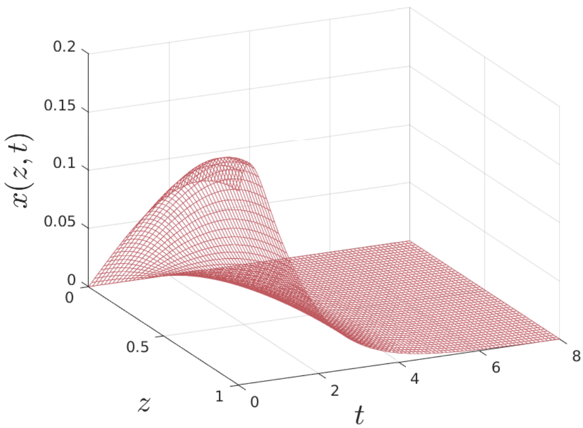

Figure 2.2 – State trajectory $x(z,t)$ of (2.1.36).

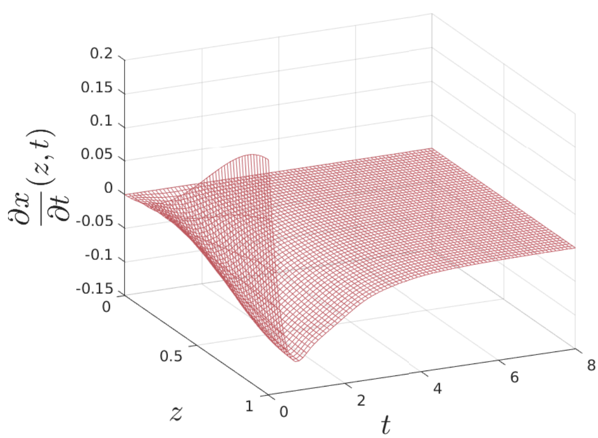

Figure 2.3 – State trajectory $\frac{\partial x}{\partial t}(z,t)$ of (2.1.36).

and Zwart, 1995, Definition 5.2.1).

Definition 2.1.10 *Let us consider an operator $A : D(A) \subset X \to X$ defined on the Hilbert space X. It is assumed that A is the infinitesimal generator of a C_0- semigroup of bounded linear operators on X. Moreover, take two operators $B \in \mathscr{L}(U,X)$ and $C \in \mathscr{L}(X,Y)$ for some Hilbert spaces U and Y. The pair (A,B) is called exponentially stabilizable if there exists an operator $K \in \mathscr{L}(X,U)$ such that the semigroup generated by the operator $A+BK$ is exponentially stable. Furthermore, the pair (C,A) is called exponentially detectable if there exists an operator $L \in \mathscr{L}(Y,X)$ such that the operator $A+LC$ generates an exponentially stable C_0-semigroup.*

2.2 Nonlinear systems on Hilbert spaces

The objective of this section is characterizing the existence and uniqueness (well-posedness) of solutions to the abstract differential equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)), \\ x(0) = x_0 \in H, \end{cases} \tag{2.2.1}$$

where $A : D(A) \subset H \to H$ is a linear operator that generates a C_0-semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on the Hilbert space H and $f : D(f) \subseteq H \to H$ is a nonlinear operator. This question of well-posedness can be quite different depending on the domain of the nonlinear operator f and of its Lipschitz continuity, which is defined as follows.

Definition 2.2.1 *A mapping $f : D(f) \subseteq H \to H$ is locally Lipschitz continuous on $D(f)$ if for every $r > 0$ there exists $L(r)$ such that for all $x_1, x_2 \in D(f)$ satisfying $\|x_1\|_H, \|x_2\|_H \leq r$ there holds*

$$\|f(x_1) - f(x_2)\|_H \leq L(r)\|x_1 - x_2\|_H. \tag{2.2.2}$$

If $L(r)$ can be chosen independently of r, then the mapping f is called uniformly Lipschitz continuous on $D(f)$.

Let us first investigate the case where f is defined on the whole space H, i.e. $D(f) = H$.

The following theorem gives sufficient conditions under which a solution of (2.2.1) exists and is unique, see (Curtain and Zwart, 2020, Theorem 11.1.5).

Theorem 2.2.1 *Let us consider the abstract differential equation (2.2.1) where the operator A is the infinitesimal generator of a C_0-semigroup on the Hilbert state space H and assume that $f : H \to H$. If f is locally Lipschitz continuous, then there exists a $t_{max} > 0$ such that (2.2.1) has a unique mild solution on $[0, t_{max})$ with the following properties:*

- *For $0 \leq t \leq t_{max}$ the solution depends continuously on the initial condition, uniformly on any bounded interval $[0,\tau] \subset [0,t_{max})$,*
- *If $x_0 \in D(A)$, then the mild solution is a classical solution on $[0,t_{max})$.*

If $t_{max} < \infty$, then $\lim_{t\uparrow t_{max}} \|x(t)\| = \infty$. Moreover, if the nonlinear operator f is uniformly Lipschitz continuous, then $t_{max} = \infty$.

Note that, by mild solution, we mean that the solution of the integral form of (2.2.1), i.e. the solution of

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds \tag{2.2.3}$$

defines a continuous function, see e.g. (Curtain and Zwart, 2020, Defintion 11.1.3). In addition, a classical solution of (2.2.1) on the time interval $[0,\tau), \tau > 0$ is a function $x(t)$ that possesses the following properties

- $x(t) \in C^1([0,\tau);H)$;
- $x(t) \in D(A)$ for all $t \in [0,\tau)$;
- $x(t)$ satisfies (2.2.1) for all $t \in [0,\tau)$,

see the definition in (Curtain and Zwart, 2020, Definition 11.1.2).

To illustrate Theorem 2.2.1, we shall consider a nonlinear PDE built from the linear damped wave equation introduced in (2.1.36) and known as the damped Sine-Gordon equation. It models many physical phenomena such as the dynamics of a Josephson junction, see e.g. Temam (1997); Cuevas-Maraver et al. (2014). From a more mechanical point of view, this PDE arises also when studying the nonlinear dynamics of mechanical transmission lines, see Cirillo et al. (1981) among others. The corresponding PDE is

$$\begin{cases} \frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2} - \alpha \frac{\partial x}{\partial t} - \beta \sin(x) \\ x(t,0) = 0, \frac{\partial x}{\partial z}(t,1) = 0, \end{cases} \tag{2.2.4}$$

where $\beta > 0$. As for the damped wave equation, let us define the state variable $X(t) = (x(t) \quad \frac{dx}{dt}(t))^T =: (X_1(t) \quad X_2(t))^T$. Then (2.2.4) admits the representation (2.2.1) with A being expressed by (2.1.33) on the domain $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$ where the considered state space $Z = D(A_0^{\frac{1}{2}}) \times H$. The nonlinear operator $f : Z \to Z$ is

$$f(X(t)) = (0 \quad -\beta \sin(X_1(t)))^T. \tag{2.2.5}$$

As explained in the previous section, the operator A (2.1.33) is the infinitesimal generator of a C_0-semigroup on Z. To prove that the PDE (2.2.4) possesses a unique mild solution, let us examine the Lipschitz continuity of the nonlinear operator f. Now take $X = (X_1 \quad X_2)^T \in Z, \tilde{X} = (\tilde{X}_1 \quad \tilde{X}_2)^T \in Z$ and let us compute the quantity $\|f(X) - f(\tilde{X})\|_Z^2$. It holds that

$$\|f(X) - f(\tilde{X})\|_Z^2 = \beta^2 \|\sin(X_1) - \sin(\tilde{X}_1)\|_H^2 = \beta^2 \int_0^1 (\sin(X_1(z)) - \sin(\tilde{X}_1(z))^2 dz.$$

Note that, by defining the scalar function $\mathfrak{f}:\mathbb{R}\to[-1,1],\mathfrak{f}(y)=\sin(y)$ the relation $\sup_{y\in\mathbb{R}}|\mathfrak{f}'(y)|=1$ holds true. This has the consequence that

$$\|f(X)-f(\tilde{X})\|_Z^2\leq\beta^2\int_0^1(X_1(z)-\tilde{X}_1(z))^2dz=\beta^2\|X_1-\tilde{X}_1\|_H^2.$$

Thanks to the invertibility of $A_0^{\frac{1}{2}}$ and the boundedness of $A_0^{-\frac{1}{2}}$ the following holds

$$\begin{aligned}\|f(X)-f(\tilde{X})\|_Z^2&\leq\beta^2\|X_1-\tilde{X}_1\|_H^2=\beta^2\|A_0^{-\frac{1}{2}}A_0^{\frac{1}{2}}(X_1-\tilde{X}_1)\|_H^2\\&\leq\beta^2\|A_0^{-\frac{1}{2}}\|^2\|A_0^{\frac{1}{2}}(X_1-\tilde{X}_1)\|_H^2\leq\beta^2\|A_0^{-\frac{1}{2}}\|^2\|X-\tilde{X}\|_Z^2,\end{aligned}$$

where (2.1.31) has been used. This proves that the nonlinear operator f is uniformly Lipschitz continuous on Z with $\beta\|A_0^{-\frac{1}{2}}\|$ as one Lipschitz constant. Hence, according to Theorem 2.2.1 the nonlinear PDE (2.2.4) possesses a unique mild solution on $[0,\infty)$. Moreover, if the initial condition pair $(x(z,0)\quad\frac{\partial x}{\partial t}(z,0))^T$ is chosen to be in the domain of the operator A, the solution is classical.

The question of existence and uniqueness of solutions of (2.2.1) becomes quite more complicated when the nonlinear operator f is not defined on the whole space H. It is assumed here that $D(f)\subsetneq H$ is a closed and convex subset[(6)] of the Hilbert state space H. Note that the domain $D(f)$ often encompasses the physical constraints required on the state variables of the dynamical system (2.2.1) as it will be seen in Section 2.3 herebelow. First let us consider the following definition, see Laabissi et al. (2001).

Definition 2.2.2 *The operator $f:D(f)\to H$ is called dissipative if*

$$\langle f(x)-f(y),x-y\rangle_H\leq 0,\tag{2.2.6}$$

for any $x,y\in D(f)$.

Note that, if the operator f is linear, this definition coincides with Definition 2.1.6. Intuitively speaking, the terminology "dissipative" can be related to some energy dissipation, as it is explained above with Definition 2.1.6 for linear operators. This means that, if a dissipative operator f is attached to a dynamical system, it does not contribute to a production of energy with time. As the property of dissipativity is important for existence and uniqueness of solutions for linear infinite-dimensional systems, see Theorem 2.1.2, it also plays a role when looking at nonlinear systems, see Theorem 2.2.2 hereafter.

The existence and uniqueness of solutions of (2.2.1) is characterized in the following theorem, see (Laabissi et al., 2001, Theorem 2.1) or (Martin, 1987, Chapter 8, Theorem 5.1) for instance.

---

(6) The closedness of the domain $D(f)$ is required for the well-posedness of the abstract equation (2.2.1), see (Martin, 1987, Chapter 8, Theorem 5.1), while its convexity will be useful in Chapters 3 and 4, notably when defining the Gâteaux and the Fréchet derivatives of the nonlinear operator f, whenever they exist.

Theorem 2.2.2 *Let us consider (2.2.1) where the nonlinear operator $f : D(f) \subsetneq H \to H$ where $D(f)$ is a closed and convex subset of H and assume that*

- *The domain $D(f)$ is $(T(t))_{t\geq 0}$ invariant, i.e. $T(t)D(f) \subset D(f)$, for all $t \geq 0$;*
- *The relation*
$$\lim_{h\to 0^+} \frac{1}{h} d(x + hf(x); D(f)) = 0 \tag{2.2.7}$$
holds for all $x \in D(f)$, where the distance d between the point x and the subset V of the Hilbert space H is defined as $d(x,V) = \inf_{v\in V}\{\|x-v\|_H\}$;
- *The nonlinear operator f is Lipschitz continuous on $D(f)$ and there exists a nonnegative constant l_f such that the operator $f - l_f I$ is dissipative on $D(f)$.*

Then, (2.2.1) has a unique mild solution $x(t)$ on $[0,\infty)$, for all $x_0 \in D(f)$. Moreover, if $(S(t))_{t\geq 0}$ is defined on $D(f)$ by $S(t)x_0 = x(t)$, for all $t \geq 0$ and $x_0 \in D(f)$, it is a nonlinear semigroup on $D(f)$ with $A + f$ as infinitesimal generator.

By a nonlinear semigroup on $D(f)$, we mean a family of nonlinear operators $(S(t))_{t\geq 0}$ defined from $D(f)$ into $D(f)$ that satisfies the following properties

- $S(0) = I$;
- $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$;
- For any $x \in D(f), \lim_{t\to 0^+} \|S(t)x - x\| = 0$.

Moreover, the semigroup $(S(t))_{t\geq 0}$ is called a contraction if

$$\|S(t)x - S(t)\tilde{x}\| \leq \|x - \tilde{x}\|,$$

for every $t \geq 0$ and all $x, \tilde{x} \in D(f)$, see e.g. (Aksikas, 2005, Definition 4.1.1). The infinitesimal generator of $(S(t))_{t\geq 0}$, denoted by $\mathscr{A}$, is defined by

$$\mathscr{A}x = \lim_{t\to 0^+} \frac{S(t)x - x}{t}, \tag{2.2.8}$$

for all $x \in D(\mathscr{A}) =: \{x \in D(f),\ (2.2.8) \text{ exists}\}$.

2.3 Nonlinear plug-flow tubular reactor with axial dispersion

The existence and uniqueness of solutions for the homogeneous part of (1.2.1) with the boundary conditions (1.2.3) is studied here. Note that only the temperature and the reactant concentration are considered as variables, that is, we consider the following set of PDEs with associated boundary conditions.

$$
\begin{cases}
\frac{\partial T}{\partial \tau}(\zeta,\tau) = -v\frac{\partial T}{\partial \zeta}(\zeta,\tau) + \frac{\lambda_{ea}}{\rho C_p}\frac{\partial^2 T}{\partial \zeta^2}(\zeta,\tau) - \frac{\Delta H}{\rho C_p}k_0 C_A(\zeta,\tau)e^{-\frac{E}{RT(\zeta,\tau)}} \\
\qquad\qquad\qquad + \frac{4h}{\rho C_p d}(1_{[0,L]}(\zeta)T_w(\tau) - T(\zeta,\tau)), \\
\frac{\partial C_A}{\partial \tau}(\zeta,\tau) = -v\frac{\partial C_A}{\partial \zeta}(\zeta,\tau) + D_{ma}\frac{\partial^2 C_A}{\partial \zeta^2}(\zeta,\tau) - k_0 C_A(\zeta,\tau)e^{-\frac{E}{RT(\zeta,\tau)}}, \\
\frac{\lambda_{ea}}{\rho C_p}\frac{\partial T}{\partial \zeta}(0,\tau) = v(T(0,\tau)-T_{in}), D_{ma}\frac{\partial C_A}{\partial \zeta}(0,\tau) = v(C_A(0,\tau)-C_{in}), \\
\frac{\lambda_{ea}}{\rho C_p}\frac{\partial T}{\partial \zeta}(L,\tau) = 0, D_{ma}\frac{\partial C_A}{\partial \zeta}(L,\tau) = 0.
\end{cases}
\tag{2.3.1}
$$

We shall use a dimensionless model. Let us define the dimensionless and rescaled variables

$$
x_1 = \frac{T-T_{in}}{T_{in}}, x_2 = \frac{C_{in}-C}{C_{in}}, x_w = \frac{T_w - T_{in}}{T_{in}}, t = \tau\frac{v}{L}, z = \frac{1}{L}\zeta. \tag{2.3.2}
$$

This change of coordinates allows us to write (2.3.1) as

$$
\begin{cases}
\frac{\partial x_1}{\partial t}(z,t) = -\frac{\partial x_1}{\partial z}(z,t) + \frac{1}{Pe_h}\frac{\partial^2 x_1}{\partial z^2} + \alpha\delta(1-x_2(z,t))e^{\frac{\mu x_1(z,t)}{1+x_1(z,t)}} \\
\qquad\qquad\qquad -\gamma(x_1(z,t) - 1_{[0,1]}(z)x_w(t)) \\
\frac{\partial x_2}{\partial t}(z,t) = -\frac{\partial x_2}{\partial z}(z,t) + \frac{1}{Pe_m}\frac{\partial^2 x_2}{\partial z^2} + \alpha(1-x_2(z,t))e^{\frac{\mu x_1(z,t)}{1+x_1(z,t)}} \\
\frac{\partial x_1}{\partial z}(0,t) = Pe_h x_1(0,t), \frac{\partial x_2}{\partial z}(0,t) = Pe_m x_2(0,t), \\
\frac{1}{Pe_h}\frac{\partial x_1}{\partial z}(1,t) = 0, \frac{1}{Pe_m}\frac{\partial x_2}{\partial z}(1,t) = 0,
\end{cases}
\tag{2.3.3}
$$

where μ, α, γ and δ are given by

$$
\mu = \frac{E}{RT_{in}}, \alpha = \frac{k_0 L}{v}e^{-\mu}, \gamma = \frac{4hL}{\rho C_p d v}, \delta = -\frac{\Delta H}{\rho C_p}\frac{C_{in}}{T_{in}}, \tag{2.3.4}
$$

respectively. Moreover the two dimensionless numbers $Pe_h := \frac{vL\rho C_p}{\lambda_{ea}}$ and $Pe_m := \frac{vL}{D_{ma}}$ are the thermal and the mass Peclet numbers. These represent the ratio between the convection transfer and the conduction transfer and the ratio between the convection transfer and the diffusion transfer, respectively. The physical constraints (1.2.2) in the dimensionless variables become

$$
-1 \le x_1(z,t), 0 \le x_2(z,t) \le 1, \tag{2.3.5}
$$

for $z \in [0,1]$ and $t \in [0,+\infty)$. In order to write (2.3.3) with $x_w \equiv 0$ in an abstract way, let us define the product spaces $X := L^2([0,1];\mathbb{R}) \times L^2([0,1];\mathbb{R})$ and $\mathscr{X} := H^2([0,1];\mathbb{R}) \times H^2([0,1];\mathbb{R})$. Therefore one may write the PDEs (2.3.3) as the abstract ODE

$$
\dot{x}(t) = Ax(t) + f(x(t)), x(0) = x_0 \in X,
$$

where the (unbounded) linear operator A is defined as

$$
A := \begin{pmatrix} -\frac{d}{dz} + \frac{1}{Pe_h}\frac{d^2}{dz^2} - \gamma I & 0 \\ 0 & -\frac{d}{dz} + \frac{1}{Pe_m}\frac{d^2}{dz^2} \end{pmatrix} =: \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{2.3.6}
$$

on the domain

$$D(A) := \left\{ x := (x_1 \quad x_2) \in \mathscr{X}, \frac{dx_1}{dz}(0) = Pe_h x_1(0), \frac{dx_2}{dz}(0) = Pe_m x_2(0), \right.$$
$$\left. \frac{1}{Pe_h}\frac{dx_1}{dz}(0) = 0, \frac{1}{Pe_m}\frac{dx_2}{dz}(0) = 0 \right\}. \tag{2.3.7}$$

Note that $D(A)$ may also be seen as the product space $D(A) = D(A_1) \times D(A_2)$ where

$$D(A_1) = \left\{ x_1 \in H^2([0,1];\mathbb{R}), \frac{dx_1}{dz}(0) = Pe_h x_1(0), \frac{1}{Pe_h}\frac{dx_1}{dz}(1) = 0 \right\}$$

and

$$D(A_2) = \left\{ x_2 \in H^2([0,1];\mathbb{R}), \frac{dx_2}{dz}(0) = Pe_m x_2(0), \frac{1}{Pe_m}\frac{dx_2}{dz}(1) = 0 \right\}.$$

The nonlinear operator $f : D \subset X \to X$ is given by

$$f(x_1,x_2) := \left(\alpha\delta(1-x_2)e^{\frac{\mu x_1}{1+x_1}} \quad \alpha(1-x_2)e^{\frac{\mu x_1}{1+x_1}} \right)^T \tag{2.3.8}$$

where $x := (x_1 \quad x_2)^T$ is in the invariant, closed and convex subset

$$D(f) := \{x \in X, -1 \leq x_1, 0 \leq x_2 \leq 1\}, \tag{2.3.9}$$

that takes the constraints (2.3.5) into account. First observe that according to Example 2.1.5 the operator A is the infinitesimal generator of a C_0-semigroup of contractions on the product Hilbert space X. It comes from the fact that the operators A_1 and A_2 are generators of contraction C_0-semigroups. Then according to (Laabissi et al., 2001, Theorem 5.1) or (Martin, 1987, Chapter 8, Theorem 5.1), the operator $A + f$ is the infinitesimal generator of a nonlinear semigroup, meaning that (2.3.3) possesses a unique mild solution on the time interval $[0,+\infty)$ for any $x_0 \in X$, where Theorem 2.2.2 has been used. The invariance of $D(f)$ with respect to the linear semigroup generated by the operator A is shown in (Laabissi et al., 2001, Proposition 5.2) while the tangential condition (2.2.7) is proved in (Laabissi et al., 2001, Lemma 3.1).

Chapter 3

Linear stability of equilibria for infinite-dimensional systems

Contents

This chapter is more application oriented, trying to give an answer to the question of the existence of equilibria of (2.3.3) as well as their linear stability. By linear stability, we mean here exponential stability of a particular linearization of the nonlinear PDEs (2.3.3). For this, we need some definitions of linearization for nonlinear operators.

3.1 Different types of linearization

As we are working in infinite-dimensional spaces, linearization depends strongly on the norm whose state space is equipped with. The following concept, named as the Gâteaux derivative, generalizes the notion of directional derivative for scalar-valued functions.

Definition 3.1.1 *Let $f : D(f) \subseteq H \to H$ be a nonlinear operator on a Banach space. The operator f is Gâteaux differentiable at $x_0 \in D(f)$ if there exists a* *linear operator* $df(x_0) : H \to H$ *such that*

$$\lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon h) - f(x_0)}{\varepsilon} = df(x_0)h, \tag{3.1.1}$$

for every $h \in D(f)$ *such that* $x_0 + \varepsilon h \in D(f)$ *for all* ε *sufficiently small. The operator* $df(x_0)$ *is called the Gâteaux derivative of the operator* f *at* x_0.

Nevertheless, remark that the directional derivative for scalar-valued functions may not be linear in h. The definition 3.1.1 entails a linear derivative in h. Note that, for the particular example of the plug-flow tubular reactor, see Chapter 2, Section 2.3, the domain of f, see (2.3.9), is a closed and convex subset of H. This means that if there exists $\varepsilon_0 > 0$ such that $x_0 + \varepsilon_0 h \in D(f)$ then the function $x_0 + \varepsilon h \in D(f)$ for any $\varepsilon < \varepsilon_0$. Indeed by convexity it holds that $ax_0 + (1-a)(x_0 + \varepsilon_0 h) \in D(f)$ for all $a \in [0,1]$, which reads in a different way as $x_0 + (1-a)\varepsilon_0 h \in D(f)$. The conclusion follows by noting that $a \in [0,1]$.

For sufficiently smooth nonlinear operators, it is in general not so difficult to show that Gâteaux differentiability holds since the ε in the limit (3.1.1) is a scalar and goes to 0 independently of anything related to the space H. In that case, the Gâteaux differentiability relies on the existence of partial derivatives of the nonlinear operator. A much stronger concept that relies on H and its norm is the Fréchet differentiability and is presented in the following definition. It will be useful notably in Chapter 4 in order to make the link between the stability properties of a linearized model and the stability of the corresponding nonlinear system.

Definition 3.1.2 *The operator* $f : D(f) \subseteq H \to H$ *is said to be Fréchet differentiable at* $x_0 \in D(f)$ *if there exists a bounded linear operator* $Df(x_0) : H \to H$ *such that*

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)h\|_H}{\|h\|_H} = 0, \tag{3.1.2}$$

where h *is such that* $x_0 + h \in D(f)$. *Equivalently, this means that for all* $h \in H$ *such that* $x_0 + h \in D(f), f(x_0 + h) - f(x_0) = Df(x_0)h + w(x_0, h)$ *where* w *satisfies* $\lim_{h \to 0} \frac{\|w(x_0,h)\|_H}{\|h\|_H} = 0$.

Note that, in the case where the operator f is Fréchet differentiable at x_0, it is also Gâteaux differentiable at x_0 and the relation $Df(x_0) = df(x_0)$ holds true. This concept of differentiability is in general difficult to prove and even impossible if the operator f is not bounded. The difficulty comes mainly from the space H since the norms are not equivalent in infinite dimensions.

Consider for instance the nonlinear operator f defined by $f(x) = \sqrt{x^2 + 1}$ for functions $x \in C([0,1];\mathbb{R}) =: H$, which is equipped with the supremum norm $\|x\|_\infty = \sup_{z \in [0,1]} |x(z)|$. Note that this operator is well-defined from H into H. It can be shown that the Gâteaux derivative of f at $x_0 = 1$ is given by the linear and bounded operator $df(1) = \frac{1}{\sqrt{2}} I$. In the next proposition, it is shown that the Gâteaux derivative is actually a Fréchet derivative.

Proposition 3.1.1 *Let H be the space of continuous real-valued functions defined on $[0,1]$ and let $f: H \to H$ be defined as $f(x) = \sqrt{x^2+1}$. The operator f is Fréchet differentiable at $x_0 = 1$ with Fréchet derivative $Df(1) = \frac{1}{\sqrt{2}}I$.*

Proof. Let us take $h \in H$. The following holds

$$\begin{aligned}
&\left\|\sqrt{(h+1)^2+1}-\sqrt{2}-\frac{1}{\sqrt{2}}h\right\|_H = \left\|\frac{(h+1)^2+1-2}{\sqrt{(h+1)^2+1}+\sqrt{2}}-\frac{1}{\sqrt{2}}h\right\|_H \\
&= \left\|\frac{((h+1)^2-1)\sqrt{2}-h(\sqrt{(h+1)^2+1}+\sqrt{2})}{\sqrt{2}(\sqrt{(h+1)^2+1}+\sqrt{2})}\right\|_H \\
&\le \left\|\frac{1}{\sqrt{2}(\sqrt{(h+1)^2+1}+\sqrt{2})}\right\|_H \left\|(h^2+2h)\sqrt{2}-h\sqrt{(h+1)^2+1}-\sqrt{2}h\right\|_H \\
&\le \left\|\sqrt{2}h^2+\sqrt{2}h-h\sqrt{(h+1)^2+1}\right\|_H \\
&\le \sqrt{2}\left\|h^2\right\|_H + \left\|\sqrt{2}h-h\sqrt{(h+1)^2+1}\right\|_H \\
&\le \sqrt{2}\left\|h\right\|_H^2 + \left\|h\right\|_H \left\|\frac{2-(h+1)^2-1}{\sqrt{2}+\sqrt{(h+1)^2+1}}\right\|_H \\
&\le \sqrt{2}\left\|h\right\|_H^2 + \left\|h\right\|_H \left\|\frac{1}{\sqrt{2}+\sqrt{(h+1)^2+1}}\right\|_H \left\|h^2+2h\right\|_H \\
&\le \sqrt{2}\left\|h\right\|_H^2 + \left\|h\right\|_H^2 (2+\left\|h\right\|_H)
\end{aligned}$$

Consequently, the estimate

$$\frac{\left\|\sqrt{(h+1)^2+1}-\sqrt{2}-\frac{1}{\sqrt{2}h}\right\|_H}{\left\|h\right\|_H} \le \left\|h\right\|_H \left(\sqrt{2}+2+\left\|h\right\|_H\right) \tag{3.1.3}$$

holds true. Hence

$$\lim_{\|h\|_H \to 0} \frac{\left\|\sqrt{(h+1)^2+1}-\sqrt{2}-\frac{1}{\sqrt{2}h}\right\|_H}{\left\|h\right\|_H} = 0.$$

□

Note that the norm inequalities that have been useful to prove the existence of a Fréchet derivative are due to the fact that the space H is a multiplicative algebra, i.e. the product of two functions in H is in H and the inequality $\|xy\|_H \le \|x\|_H \|y\|_H$ holds for any $x, y \in H$. This result would suggest that, if an operator is Gâteaux and Fréchet differentiable at some point, then the corresponding derivatives coincide. This can be found in (Lebedev and Vorovich, 2006, Theorem 3.1.1).

Let us now have a look at the same operator but defined on the space $H := L^2([0,1];\mathbb{R})$ equipped with the norm $\|x\|_H^2 = \int_0^1 x^2(z)dz$. The operator f is well-defined on H since

$$\|f(x)\|_H^2 = \int_0^1 (x^2(z)+1)dz = \|x\|_H^2 + 1,$$

for any $x \in H$. Consider the sequence of $H-$functions $\{h_n\}_{n\in\mathbb{N}}$ defined by

$$h_n(z) = \frac{1}{n}1_{[0,1-\frac{1}{n})}(z) + 1_{[1-\frac{1}{n},1]}(z),$$

for $z \in [0,1]$ and $n \in \mathbb{N}$. A simple calculation reveals that the $H-$norm of h_n is given as

$$\|h_n\|_H^2 = \frac{n^2+n-1}{n^3}, n \in \mathbb{N}.$$

Moreover, it holds that $\|\sqrt{(h_n+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h_n\|_H^2 =$

$$\begin{aligned}
&\int_0^1 (\sqrt{(h_n(z)+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h_n(z))^2 dz \\
&= \int_0^{1-\frac{1}{n}} (\sqrt{(\frac{1}{n}+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}n})^2 dz + \int_{1-\frac{1}{n}}^1 (\sqrt{(1+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}})^2 dz \\
&= (1-\frac{1}{n})\left(\sqrt{\frac{(n+1)^2}{n^2}+1} - \sqrt{2} - \frac{1}{\sqrt{2}n}\right)^2 + \frac{1}{n}\left(\sqrt{5} - \sqrt{2} - \frac{\sqrt{2}}{2}\right)^2 \\
&= \frac{n-1}{n}\left(\frac{1}{n}\sqrt{2n^2+2n+1} - \frac{2n+1}{\sqrt{2}n}\right)^2 + \frac{1}{n}\left(\frac{19-6\sqrt{10}}{2}\right) \\
&= \frac{n-1}{2n^3}\left(\sqrt{4n^2+4n+2} - (2n+1)\right)^2 + \frac{1}{n}\left(\frac{19-6\sqrt{10}}{2}\right).
\end{aligned}$$

Computing the quantity $\frac{\|\sqrt{(h_n+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h_n\|_H^2}{\|h_n\|_H^2}$ gives

$$\begin{aligned}
&\frac{n^3}{n^2+n-1}\left[\frac{n-1}{2n^3}\left(\sqrt{4n^2+4n+2} - (2n+1)\right)^2 + \frac{1}{n}\left(\frac{19-6\sqrt{10}}{2}\right)\right] \\
&\qquad = \frac{n-1}{2(n^2+n-1)}\left(\frac{1}{\sqrt{4n^2+4n+2}+2n+1}\right)^2 + \frac{n^2}{n^2+n-1}\left(\frac{19-6\sqrt{10}}{2}\right),
\end{aligned}$$

which tends towards $\frac{19-6\sqrt{10}}{2}$ when n tends to ∞. This proves that the nonlinear operator f is not Fréchet differentiable at 1 on $H = L^2([0,1];\mathbb{R})$. This example shows also that even if an operator f is Gâteaux differentiable and uniformly Lipschitz continuous, its Fréchet differentiability is not guaranteed. The key point that breaks down the Fréchet differentiability of f is due to the fact that the $L^\infty([0,1];\mathbb{R})-$norm of h_n is equal to 1 for any $n \in \mathbb{N}$. The convergence to 0 of the $H-$norm of h_n is not sufficient here, which means that stronger conditions should have been required to ensure Fréchet differentiability on H.

3.2 Plug-flow tubular reactor

Here we go back to the plug-flow tubular reactor with axial dispersion in order to characterize its equilibria and their linear stability. Note that adiabatic conditions are considered here, meaning that we assume no heat exchange between the inside of the reactor and the environment outside, i.e. $\gamma = 0$, where γ is the parameter introduced in (2.3.4).

3.2.1 Equilibria and related properties

As is it common for PDEs, finding equilibria relies generally on a differential equation. More particularly, the kind of differential equations that will be of interest here are ODEs with boundary conditions, also named two point boundary value problems. In other words, writing the equations (2.3.3) in adiabatic form at the equilibrium yields the following set of ODEs with boundary conditions

$$\begin{cases} \frac{1}{Pe_h}\frac{d^2x_1^e}{dz^2} - \frac{dx_1^e}{dz} + \alpha\delta(1-x_2^e)e^{\frac{\mu x_1^e}{1+x_1^e}} = 0, \frac{dx_1^e}{dz}(0) = Pe_h x_1^e(0), \frac{1}{Pe_h}\frac{dx_1^e}{dz}(1) = 0, \\ \frac{1}{Pe_m}\frac{d^2x_2^e}{dz^2} - \frac{dx_2^e}{dz} + \alpha(1-x_2^e)e^{\frac{\mu x_1^e}{1+x_1^e}} = 0, \frac{dx_2^e}{dz}(0) = Pe_m x_2^e(0), \frac{1}{Pe_m}\frac{dx_2^e}{dz}(1) = 0, \end{cases} \tag{3.2.1}$$

where x_1^e and x_2^e denote the dimensionless temperature and concentration at steady-state, respectively. Due to the boundary conditions, we are not dealing with a Cauchy problem when looking at (3.2.1). For this reason, existence and uniqueness properties of a solution pair (x_1^e, x_2^e) are not guaranteed. Moreover, an additional difficulty relies on the link between the two Peclet numbers. We shall consider them equal here with the notation $Pe_h = Pe_m =: Pe = \frac{v}{D}$ where v and D stand for the superficial velocity and the diffusion coefficient, respectively. We want to emphasize the fact that this equality is done for mathematical purposes but it has no real physical meaning. In that case, the change of variables $\chi = x_1^e - \delta x_2^e$ simplifies the problem quite strongly since it allows to decouple the equations and to express one solution as a function of the other like $x_1^e = \delta x_2^e$, see e.g. (Hastir et al., 2020, Appendix A). For that case, the remaining equilibrium equation is expressed as

$$\frac{1}{Pe}\frac{d^2x_1^e}{dz^2} - \frac{dx_1^e}{dz} + \alpha(\delta - x_1^e)e^{\frac{\mu x_1^e}{1+x_1^e}} = 0, \frac{dx_1^e}{dz}(0) = Pe x_1^e(0), \frac{1}{Pe}\frac{dx_1^e}{dz}(1) = 0. \tag{3.2.2}$$

In particular, it has been proven in that case that, depending on the diffusion coefficient, (3.2.1) possesses one or three solutions. Formally, it reads as follows

Proposition 3.2.1 *For some values of the parameters μ, δ, α, there exists D^* large enough, v_1^* and v_2^* such that for all $D \geq D^*$, (3.2.1) has either*

- *at least three solutions if $v \in (v_*, v^*)$, or*
- *at least one solution, otherwise,*

where $v_* := \min\{v_1^*, v_2^*\}$ *and* $v^* := \max\{v_1^*, v_2^*\}$.

This result is based on perturbation theory, see Hoppensteadt (2013) for instance, where a small parameter, defined as $1/D$ here, is considered. The perturbation theory allows to deduce existence, uniqueness and approximation of the solution of (3.2.1) by looking at the same problem in the limit with $\frac{1}{D}$ going to 0. In that way, for D sufficiently large, the following result holds for characterizing the solution of (3.2.2).

Proposition 3.2.2 *Taking into account Proposition 3.2.1, a solution of (3.2.2), denoted by* $x_1^e(z)$*, is given by*

$$x_1^e(z) = a - \frac{k_0 L(\delta - a)e^{-\frac{\mu}{1+a}}}{2D}(1-z)^2 + \mathscr{O}\left(\frac{1}{D^2}\right) =: x_1^\star(z) + \mathscr{O}\left(\frac{1}{D^2}\right), \quad (3.2.3)$$

where a *is a solution of the equation*

$$k_0 L(\delta - a)e^{\frac{-\mu}{1+a}} - va = 0 \quad (3.2.4)$$

and where $\mathscr{O}$ *stands for the Landau notation (see Definition 3.2.1).*

Note that the function $x_1^\star(z)$ will be called an approximated solution of (3.2.2) in what follows. Equation (3.2.4) helps in understanding Proposition 3.2.1 intuitively since the number of equilibria is determined by the parameter a. More particular, for a fixed value of v, the number of equilibria is given by the number of roots of (3.2.4) or equivalently the number of values of a that can reached the fixed value of v. For this, the function v is expressed as

$$v = \frac{k_0 L(\delta - a)e^{\frac{-\mu}{1+a}}}{a} =: v(a, 1/D), \quad (3.2.5)$$

see (3.2.4). A study of $v(a)$ as a function of a leads to Proposition 3.2.1. This is valid for a large diffusion coefficient D (according to perturbation theory in which a small parameter, namely $1/D$, is introduced). The particular form of (3.2.3) is explained later thanks to Theorem 3.2.4.

As an illustration, v as a function of a is depicted in Figure 3.1 in the case of 3 equilibria ($\mu = 10, \delta = 1$). It is clear in this figure that, for a fixed value of v between the values v_1^* and v_2^*, three different values of a can reached the fixed value of the volicity. The chosen value of v is $1.1e-3$.

From the modeling point of view, v is a (fixed) parameter, but here it is interpreted as a function of the parameter a for analysis purposes. It is indeed well known in chemical engineering (Levenspiel (1999)) that the convection-diffusion-reaction (CDR) model is an intermediate model between the plug-flow reactor model (PFTR) (when the diffusion coefficients are equal to zero) and the continuous stirred tank reactor (CSTR) model (described by ODEs) (when these coefficients tend to $+\infty$). As it is highlighted in Varma and Aris (1977), the plug-flow reactor can generate only one equilibrium profile, since the latter is the solution of a set of first-order differential equations with fixed initial values. And, at the other extreme, the CSTR can

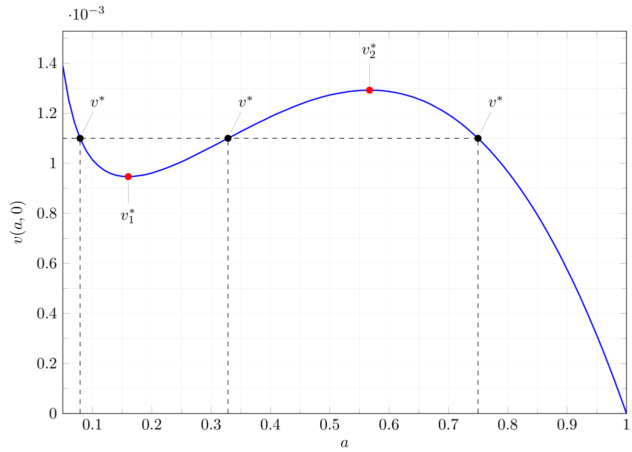

Figure 3.1 – Illustration of the multiplicity of the equilibria for equal Peclet numbers ($\mu = 10, \delta = 1$).

PFTR		**CDR**		**CSTR**
1 equilibrium		1 equilibrium $\rightarrow$ 3 equilibria		3 equilibria
$D = 0$	$\rightarrow$	$D > D^*$	$\rightarrow$	$D = +\infty$

Table 3.1 – Intuitive schematic view of the multiplicity of equilibria.

exhibit three different equilibrium points. Therefore, one could conclude intuitively that there should be a value of the diffusion coefficients above which the tubular reactor model can exhibit multiple equilibrium profiles (and below which there is only one equilibrium profile), see the schematic representation in Table 3.1. In this context, it is important to note that the (dimensionless) Peclet numbers allow us to evaluate the relative importance of convection (characterized by v) versus diffusion (characterized by D). Thus, if there are $a_1 = a_2$ such that $v(a_1) = v(a_2)$, (3.2.2) has at least two solutions. To reach this goal, perturbation theory (Hoppensteadt, 2013, Regular perturbation theorem) is used, which consists of disturbing the equations with a small parameter, $1/D$ here. Then, if a solution can be found to the disturbed equations with $1/D = 0$ (see (Hoppensteadt, 2013, Section 5.2.1., Hypothesis H2)), perturbation theory guarantees that the system has a solution for small $1/D$ (this is the reason why we consider a large diffusion coefficient (D^* large enough)), under a few assumptions, especially continuity conditions (see (Hoppensteadt, 2013, Section 5.2.1., Hypothesis H3)). Furthermore, the solution can be identified to its Taylor expansion of powers of $1/D$, see Section 3.2.2 hereafter.

3.2.2 Linear stability of equilibria

Here we aim at studying the stability of a linearized model corresponding to (2.3.3) in adiabatic form, around any equilibrium pair satisfying (3.2.1), in the case of equal Peclet numbers. Note that the asymptotic stability of such types of systems has already been studied in Dochain (2018), Varma and Aris (1977) or in Luss and Amundson (1967), wherein bistability of the equilibria is established. In other words, when the system exhibits one equilibrium, the latter is asymptotically stable while when three equilibria are highlighted, the pattern "asymptotically stable - unstable - asymptotically stable" holds. Furthermore, numerical methods to deduce stability of equilibria for (2.3.3) have been developed in Nishimura and Matsubara (1969), Lefèvre et al. (2000) or McGowin and Perlmutter (1970). The techniques that have been used are based on the Galerkin Residuals Method which is quite well-suited for parabolic PDEs. The latter consists in a finite dimensional model reduction that allows to conclude on stability on the basis of the dominant eigenvalue of some matrix approximation of the differential operators in (2.3.3).

The approach that will be followed here is quite different in the sense that exponential stability is considered instead of the asymptotic one. To this end, we shall first linearize (2.3.3) around an equilibrium solution of (3.2.1). Then, after the introduction of some properties related to equilibria of the adiabatic form of (2.3.3), the well-posedness of the linearized model in terms of existence and uniqueness of solutions will be studied. A Lyapunov-based method will then be used to conclude on exponential stability of some particular equilibria.

Let us return to the definition of the nonlinear operator f introduced in (2.3.8). For the sake of simplicity in what follows, let us write the operator f as $f := (f_1 \quad f_2)^T$ with $f_1 = \delta f_2$. In order to build a linearized model of (2.3.3) around an equilibrium pair (x_1^e, x_2^e) solution of (3.2.1), let us consider the following lemma.

Lemma 3.2.1 *The nonlinear operator f_1 resulting from (2.3.8) is Gâteaux differentiable at $(x_1^e, x_2^e) \in D(f)$ solution of (3.2.1) and its Gâteaux derivative is given by the linear operator $df_1(x_1^e, x_2^e) : X \to L^2([0,1];\mathbb{R})$ defined for $\begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \in X$ by*

$$df_1(x_1^e, x_2^e)\begin{pmatrix} x_1 & x_2 \end{pmatrix}^T = \tilde{\alpha}\frac{\delta\mu\,(1-x_2^e)}{\left(1+x_1^e\right)^2} e^{\frac{-\mu}{1+x_1^e}} x_1 - \tilde{\alpha}\delta e^{\frac{-\mu}{1+x_1^e}} x_2, \tag{3.2.6}$$

where $\tilde{\alpha} = \frac{k_0 L}{v}$.

Proof. The proof is a straightforward consequence of the existence of the partial derivatives of f_1 with respect to x_1 and x_2 evaluated at the equilibrium (x_1^e, x_2^e). □

In order to ensure that the Gâteaux derivative of f_1 is a bounded linear operator from X into $L^2([0,1];\mathbb{R})$, we need some properties on an exact equilibrium solution (x_1^e, x_2^e) of (3.2.1). Let us start with properties on the approximated form expressed from (3.2.3).

Proposition 3.2.3 *The approximated form $x_1^*(z)$ of a temperature equilibrium profile for the nonisothermal axial dispersion tubular reactor (2.3.3) in adiabatic condition is such that $-1+\eta < x_1^*(z) < \delta$, a.e. on $[0,1]$, for some positive constant η, whenever the diffusion coefficient D is sufficiently large.*

Proof. According to (3.2.3), $x_1^\star(z)$ is given as $x_1^\star(z) = a - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{2D}(1-z)^2$ and the approximated form of the velocity is $v^\star = k_0L(\delta-a)e^{\frac{-\mu}{1+a}}/a$. Since positive velocities and positive values of the parameter a are considered, it follows that $\delta > a > 0$. Moreover, $0 \le (1-z)^2 \le 1$, which yields

$$a - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{2D} \le a - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{2D}(1-z)^2 \le a. \qquad (3.2.7)$$

Consequently, $a - k_0L(\delta-a)e^{\frac{-\mu}{1+a}}/2D \le x_1^\star(z) \le a < \delta$. Furthermore, observe that, since $0 < e^{\frac{-\mu}{1+a}} < e^{\frac{-\mu}{1+\delta}}$ for all $a > 0$, it holds

$$\begin{aligned} x_1^\star(z) &\ge a - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+\delta}}}{2D} \\ &> \frac{2aD - k_0L\delta e^{\frac{-\mu}{1+\delta}} + 2D - 2D}{2D} > -1 + \frac{2D - k_0L\delta e^{\frac{-\mu}{1+\delta}}}{2D} := -1+\eta. \end{aligned} \qquad (3.2.8)$$

To ensure that $\eta > 0$, D has to be large enough, i.e. there must exist some $D^\star > 0$ sufficiently large such that $D \ge D^\star$. By "D large enough", we mean $D^\star > k_0L\delta e^{\frac{-\mu}{1+\delta}}/2$. Note that $\eta \to 1$ as $D \to +\infty$, i.e. as the diffusion is sufficiently dominant. □

Before characterizing some bounds on the exact equilibrium profile, denoted $x_1^e(z)$, let us take into account the following definitions and theorem. Given a function $f : \mathscr{H} \to \mathbb{R}^2$ and an initial condition $\Theta \in \mathbb{R}^2$, we consider the initial value problem

$$\frac{dx}{dz} = f(z,x,\varepsilon), x(0) = \Theta(\varepsilon), \qquad (3.2.9)$$

where ε is a parameter. Assume that the following conditions are satisfied :

Assumption 3.2.1 *For $\varepsilon = 0$, (3.2.9) has a unique solution on $0 \le z \le 1$, denoted by $x_0(z)$. Hence the latter satisfies the equations $\frac{dx_0}{dt} = f(z,x_0,0), x_0(0) = \Theta(0)$.*

Assumption 3.2.2 *The functions f and Θ are smooth functions of their variables for $0 \le z \le 1$, x near x_0 and ε near 0. Specifically, we suppose that f and Θ are $\mathbf{n}+1$ times continuously differentiable in all their variables, so that Θ admits the decomposition $\sum_{k=0}^{\mathbf{n}} \Theta_k\varepsilon^k + \mathscr{O}(\varepsilon^{n+1})$.*

The regular perturbation theorem is expressed as follows.

Theorem 3.2.4 *Let Assumptions 3.2.1 and 3.2.2 hold. Then, for sufficiently small ε, the perturbed problem (3.2.9) has a unique solution, which is $\mathbf{n}+1$ times differentiable with respect to ε. Moreover this solution admits a Taylor expansion $x(z,\varepsilon) = x_0(z) + x_1(z)\varepsilon + \cdots + x_{\mathbf{n}}(z)\varepsilon^{\mathbf{n}} + \mathscr{O}(\varepsilon^{\mathbf{n}+1})$, where the error estimate holds as $\varepsilon \to 0$ uniformly for $0 \leq z \leq 1$.*

Thanks to what preceeds, we are now able to explain an approximated equilibrium profile with an exact one. By the Regular Perturbation Theorem, it holds, in our case,

$$x_1^e(z,\varepsilon) = x_1^\star(z,\varepsilon) + \mathscr{O}(\varepsilon^2), \tag{3.2.10}$$

where $\varepsilon = 1/D$, see (Hastir et al., 2020, Section IV. A.) for detailed arguments concerning the computation of $x_1^\star(z,\varepsilon)$. In order to bound the exact form of the temperature equilibrium profile, let us consider the following definition, see (Hoppensteadt, 2013, Section 5.1.1, Gauge Functions).

Definition 3.2.1 *Suppose that f and g are smooth functions of ε for ε near 0, say $0 < \varepsilon < \varepsilon_0$ for some ε_0 sufficiently small. We say that $f(\varepsilon) = \mathscr{O}(g(\varepsilon))$ as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ is bounded for all sufficiently small ε. Thus, there is a constant $K > 0$ and a sufficiently small constant $\varepsilon^\star > 0$ such that $|f(\varepsilon)| \leq K|g(\varepsilon)|$ holds for all $0 < \varepsilon \leq \varepsilon^\star$.*

This leads to consider the following bounds on an exact temperature equilibrium profile $x_1^e(z)$ of (2.3.3).

Theorem 3.2.5 *The exact form of a temperature equilibrium profile for (2.3.3) satisfies*

$$-1 + \tilde{\eta} < x_1^e(z) < \tilde{\delta}, \tag{3.2.11}$$

when the diffusion coefficient D is sufficiently large, where $\tilde{\eta} = \eta - \frac{K}{D^2} > 0$ and $\tilde{\delta} = \delta + \frac{K}{D^2}$ for some positive constant K.

Proof. By taking (3.2.10) into account and the characterization of $\mathscr{O}$ in Definition 3.2.1, there exist $K > 0$ and $\varepsilon^\star > 0$ such that for a.e. $z \in [0,1]$, $|x_1^e(z,\varepsilon) - x_1^\star(z,\varepsilon)| \leq K\varepsilon^2$, for $\varepsilon \leq \varepsilon^\star$. This holds for a.e. $z \in [0,1]$ since the error estimate is valid uniformly for $0 \leq z \leq 1$. Consequently we have

$$x_1^\star(z,\varepsilon) - K\varepsilon^2 \leq x_1^e(z,\varepsilon) \leq x_1^\star(z,\varepsilon) + K\varepsilon^2. \tag{3.2.12}$$

By using Lemma 3.2.3, it follows that $-1 + \eta - K/D^2 < x_1^e(z,1/D) < \delta + K/D^2$, where we have been using the fact that $\varepsilon = 1/D$, which is equivalent to $-1 + \tilde{\eta} < x_1^e(z,1/D) < \tilde{\delta}$, where $\tilde{\eta} := \eta - K/D^2$ and $\tilde{\delta} := \delta + K/D^2$. The positivity of $\tilde{\eta}$ is guaranteed provided that D is large enough. □

3.2.2.1 Well-posedness of the linearized model

According to Lemma 3.2.1, a Gâteaux linearized version of (2.3.3) around the equilibrium (x_1^e, x_2^e) with $\gamma = 0$ is written as

$$\dot{\xi}(t) = A_{|\gamma=0}\xi(t) + \begin{pmatrix} df_1(x_1^e, x_2^e) \\ df_2(x_1^e, x_2^e) \end{pmatrix} \xi(t), \xi(0) = \xi_0, \tag{3.2.13}$$

where $df_2(x_1^e, x_2^e) = \frac{1}{\delta} df_1(x_1^e, x_2^e), df_1(x_1^e, x_2^e)$ being introduced in (3.2.6) and $\xi(t) = x(t) - x^e$. Note that the notation $A_{|\gamma=0}$ stands for the operator A defined in (2.3.6) with domain (2.3.7) in which γ has been set to 0. By using the notations $p(x_1^e, x_2^e) := \tilde{\alpha}\frac{\mu(1-x_2^e)e^{\frac{-\mu}{1+x_1^e}}}{(1+x_1^e)^2}$ and $r(x_1^e, x_2^e) := -\tilde{\alpha}e^{\frac{-\mu}{1+x_1^e}}$, (3.2.13) may be rewritten as

$$\dot{\xi}(t) = A_{|\gamma=0}\xi(t) + \begin{pmatrix} \delta p(x_1^e, x_2^e) & \delta r(x_1^e, x_2^e) \\ p(x_1^e, x_2^e) & r(x_1^e, x_2^e) \end{pmatrix} \xi(t) =: A_{lin}\xi(t). \tag{3.2.14}$$

In that way, a sufficient condition that ensures the well-posedness of (3.2.14) in terms of existence and uniqueness of a mild solution is the boundedness of the operator $\begin{pmatrix} \delta p(x_1^e, x_2^e) & \delta r(x_1^e, x_2^e) \\ p(x_1^e, x_2^e) & r(x_1^e, x_2^e) \end{pmatrix}$ defined on the Hilbert space $X = L^2([0,1];\mathbb{R}) \times L^2([0,1];\mathbb{R})$. This will be achieved in the following lemma thanks to Theorem 3.2.5.

Lemma 3.2.2 *The operator A_{lin} given in (3.2.14) on the domain $D(A_{lin}) = D(A)$ is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X.*

Proof. Since the operator $A_{|\gamma=0}$ is the infinitesimal generator of a C_0-semigroup, the operator A_{lin} possesses the same property provided that the operator

$$\begin{pmatrix} \delta p(x_1^e, x_2^e) & \delta r(x_1^e, x_2^e) \\ p(x_1^e, x_2^e) & r(x_1^e, x_2^e) \end{pmatrix}$$

is bounded on X, see (Engel and Nagel, 2006, Bounded Perturbation Theorem). By taking $\xi = (\xi_1 \quad \xi_2)^T \in X$, one has that

$$\begin{aligned} \| \begin{pmatrix} \delta p(x_1^e, x_2^e) & \delta r(x_1^e, x_2^e) \\ p(x_1^e, x_2^e) & r(x_1^e, x_2^e) \end{pmatrix} \xi \|_X^2 &= (\delta^2+1)\|p(x_1^e, x_2^e)\xi_1 + r(x_1^e, x_2^e)\xi_2\|_{L^2}^2 \\ &\leq 2(\delta^2+1)\left[\|p(x_1^e, x_2^e)\xi_1\|_{L^2}^2 + \|r(x_1^e, x_2^e)\xi_2^e\|_{L^2}^2\right]. \end{aligned}$$

Moreover, using Theorem 3.2.5 and the relation $x_1^e = \delta x_2^e$, the following estimates hold

$$\begin{aligned} &\| \begin{pmatrix} \delta p(x_1^e, x_2^e) & \delta r(x_1^e, x_2^e) \\ p(x_1^e, x_2^e) & r(x_1^e, x_2^e) \end{pmatrix} \xi \|_X^2 \\ &\leq 2(\delta^2+1)\tilde{\alpha}^2 \left[\frac{\mu^2}{\tilde{\eta}^2}\max\{|1-\frac{\tilde{\delta}}{\delta}|, |\frac{1-\tilde{\eta}}{\delta}+1|\}^2\|\xi\|_{L^2}^2 + \|\xi_2\|_{L^2}^2\right] \\ &\leq 2(\delta^2+1)\tilde{\alpha}^2 \max\left\{\frac{\mu^2}{\tilde{\eta}^2}\max\{|1-\frac{\tilde{\delta}}{\delta}|, |\frac{1-\tilde{\eta}}{\delta}+1|\}^2, 1\right\}\|\xi\|_X^2, \end{aligned}$$

where we have been using the inequalities $\|1/(1+x_1^e)\|_\infty \leq \frac{1}{\tilde{\eta}}$, $\|e^{\frac{-\mu}{1+x_1^e}}\|_\infty \leq 1$ and $\|1-x_2^e\|_\infty \leq \max\{|1-\frac{\tilde{\delta}}{\delta}|, |\frac{1-\tilde{\eta}}{\delta}+1|\}$. This concludes the boundedness of the operator $\begin{pmatrix} \delta p(x_1^e,x_2^e) & \delta r(x_1^e,x_2^e) \\ p(x_1^e,x_2^e) & r(x_1^e,x_2^e) \end{pmatrix}$ and consequently the well-posedness of (3.2.13). □

3.2.2.2 Exponential bistability of the equilibria

In this section, we aim at characterizing the exponential decay of the state trajectory of (3.2.14) by means of a Lyapunov based approach. In particular, sufficient conditions that guarantee exponential stability of (3.2.14) are derived on the system parameters. Then, these conditions are tested on the different cases of equilibria, see Proposition 3.2.1. Before beginning, remember that the case of equal Peclet numbers is considered here, i.e. $Pe_h = Pe_m := \frac{v}{D}$.

Let us start by considering the following change of variables on (3.2.14): $\hat{\xi}_1(z,t) = e^{\frac{-Pe}{2}z}\xi_1(z,t), \hat{\xi}_2(z,t) = e^{\frac{-Pe}{2}z}\xi_2(z,t)$. This entails that (3.2.14) becomes

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_1 + \delta p(x_1^e,x_2^e)\hat{\xi}_1 + \delta r(x_1^e,x_2^e)\hat{\xi}_2, \\ \frac{\partial \hat{\xi}_2}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_2}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_2 + p(x_1^e,x_2^e)\hat{\xi}_1 + r(x_1^e,x_2^e)\hat{\xi}_2, \\ \frac{\partial \hat{\xi}_1}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_1(0,t),\ \frac{\partial \hat{\xi}_1}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_1(1,t), \\ \frac{\partial \hat{\xi}_2}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_2(0,t),\ \frac{\partial \hat{\xi}_2}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_2(1,t). \end{cases} \tag{3.2.15}$$

Now consider the operator matrix $U = \begin{pmatrix} I & 0 \\ I & -\delta I \end{pmatrix}$. The similarity state transformation $\begin{pmatrix} \hat{\xi}_1 \\ \chi \end{pmatrix} = U \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix}$ implies that (3.2.15) takes the form

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_1 - r(x_1^e,x_2^e)\chi + r(x_1^e,x_2^e)\hat{\xi}_1 + \delta p(x_1^e,x_2^e)\hat{\xi}_1, \\ \frac{\partial \chi}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \chi}{\partial z^2} - \frac{Pe}{4}\chi, \\ \frac{\partial \hat{\xi}_1}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_1(0,t),\ \frac{\partial \hat{\xi}_1}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_1(1,t), \\ \frac{\partial \chi}{\partial z}(0,t) = \frac{Pe}{2}\chi(0,t),\ \frac{\partial \chi}{\partial z}(1,t) = -\frac{Pe}{2}\chi(1,t). \end{cases} \tag{3.2.16}$$

Let us consider the following proposition that will help in characterizing the asymptotic behavior of the variable χ.

Proposition 3.2.6 *The component χ in (3.2.16) is expressed as*

$$\chi(z,t) = \sum_{n=1}^{+\infty} \psi_n\, \phi_n(z)\, e^{-(\beta_n^2 + \frac{Pe}{4})t}, \tag{3.2.17}$$

for $z \in [0,1]$ and $t \geq 0$, where $\psi_n = \int_0^1 f(z)\phi_n(z)dz$ with $f(z)$ denoting the initial condition to $\hat{\xi}_1 - \delta\hat{\xi}_2$. The set $\{\phi_n\}_{n\geq 1}$ contains the eigenfunctions of the linear operator $\frac{1}{Pe}\frac{d^2}{dz^2} - \frac{Pe}{4}I$, defined by $\phi_n(z) = K_n[\beta_n\sqrt{Pe}\cos(\beta_n\sqrt{Pe}z) + \frac{Pe}{2}\sin(\beta_n\sqrt{Pe}z)]$.

Note that $\{K_n\}_{n\geq 1}$ is a set of normalization constants expressed as

$$K_n = \left(\frac{2}{\beta_n^2 Pe + Pe + Pe^2/4}\right)^{\frac{1}{2}}$$

and $\{\beta_n\}_{n\geq 1}$ are solutions of the resolvent equation $\tan(\beta\sqrt{Pe}) = \frac{4\beta\sqrt{Pe}}{4\beta^2 - Pe}$, *see e.g. (Delattre et al., 2003).*

Proof. Let us define the linear operator A_c by $A_c\chi = \frac{1}{Pe}\frac{d^2\chi}{dz^2} - \frac{Pe}{4}\chi$ on the domain $D(A_c) = \{\chi \in H^2([0,1];\mathbb{R}), \frac{d\chi}{dz}(0) = \frac{Pe}{2}\chi(0), \frac{d\chi}{dz}(1) = -\frac{Pe}{2}\chi(1)\}$. By defining the functions $P(z) = \frac{1}{Pe}$ and $Q(z) = \frac{Pe}{4}, z \in [0,1]$, it is easy to see that the operator $-A_c$ admits the representation (2.1.18) with the functions p and q replaced by P and Q and with the function ρ given as $\rho(z) = 1, z \in [0,1]$. Moreover $D(-A_c) = D(A_c)$ has the form (2.1.17) with $(\alpha_0, \beta_0) = (1, -\frac{Pe}{2})$ and $(\alpha_1, \beta_1) = (1, \frac{Pe}{2})$. Consequently, according to Lemma 2.1.4, the operator A_c is a Riesz-spectral operator on the space $L^2([0,1];\mathbb{R})$. Moreover, it can be noted that the operator A_c is self-adjoint, which entails with its Riesz-spectral property that it admits the following decomposition

$$A_c x = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle_{L^2} \phi_n(z),$$

where $x \in D(A_c)$ and the sets $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ containing the eigenvalues and the eigenfunctions of the operator A_c, respectively. Thanks to Theorem 2.1.4, the C_0-semigroup whose operator A_c is the infinitesimal generator, is expressed as

$$T_{A_c}(t)\chi_0 = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle \chi_0, \phi_n \rangle_{L^2} \phi_n(z).$$

Since the component χ solution of (3.2.16) is expressed as $\chi(z,t) = (T_{A_c}(t)\chi_0)(z)$, the end of the proof is a consequence of the calculations of the eigenvalues and the eigenfunctions of the operator A_c, see e.g. Varma and Aris (1977). $\square$

Note that U defines a similarity transformation, i.e. a Hilbert space isomorphism on X. Then (3.2.15), which involves the variables $\begin{pmatrix}\hat{\xi}_1\\ \hat{\xi}_2\end{pmatrix}$, is equivalent to (3.2.16) written in the variables $\begin{pmatrix}\hat{\xi}_1\\ \chi\end{pmatrix}$. Consequently, studying exponential stability of (3.2.15) is equivalent to studying exponential stability of (3.2.16). Denoting by $\hat{A}$ the linear operator describing the dynamics of (3.2.15), exponential stability can be viewed by noting that the semigroups generated by the operators $\hat{A}$ and $U\hat{A}U^{-1}$ have the same growth bounds. Since $\{\phi_n\}_{n\in\mathbb{N}}$ is a Riesz-basis for $L^2([0,1];\mathbb{R})$, it follows from (3.2.17) that $\|\chi(\cdot,t)\|_{L^2} \leq e^{-(\beta_*^2+Pe/4)t}\|f(\cdot)\|_{L^2}$, where $-(\beta_*^2 + Pe/4) = \sup_{n\in\mathbb{N}}\{-(\beta_n^2 + Pe/4)\}$ is the growth constant of the semigroup generated by the Riesz-spectral operator A_c. This means that $\|\chi(\cdot,t)\|_{L^2}$ converges exponentially fast to 0 as t tends to $+\infty$. Moreover, by using the reaction invariant $x_1^e - \delta x_2^e = 0$, the stability analysis of the linearized model corresponding to the plug-flow tubular reactor is based on the following parabolic PDE

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - q(z)\hat{\xi}_1, \\ \frac{\partial \hat{\xi}_1}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_1(0,t), \ \frac{\partial \hat{\xi}_1}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_1(1,t), \end{cases} \tag{3.2.18}$$

where $\hat{\xi}_1(z,t) = e^{\frac{-Pe}{2}z}(x_1(z,t) - x_1^e(z))$ and

$$q(z) = \frac{Pe}{4} + \frac{k_0 L}{v} e^{\frac{-\mu}{1+x_1^e(z)}} - \delta \frac{k_0 L}{v} \frac{\mu(1-x_2^e(z))}{(1+x_1^e(z))^2} e^{\frac{-\mu}{1+x_1^e(z)}}. \tag{3.2.19}$$

An estimation of the exponential behavior of $\|\hat{\xi}(\cdot,t)\|_{L^2}$ as a function of time is given in the next proposition.

Proposition 3.2.7 *The state trajectory $\hat{\xi}_1$ solution of the PDE (3.2.18) satisfies the following estimate*

$$\|\hat{\xi}_1(\cdot,t)\|_{L^2} \leq e^{-\left(\frac{\pi^2}{\pi^2+4Pe}+q(c)\right)t} \|\hat{\xi}_1(\cdot,0)\|_{L^2}, \ t \geq 0, \tag{3.2.20}$$

for some $c \in (0,1)$.

Before going into the proof of that proposition, we need two auxiliary results, the first being an extension of the Wirtinger's inequality and the second standing for a generalization of the mean value theorem for integrals.

Lemma 3.2.3 *For any continuously differentiable function w defined on $[0,1]$, the inequalities*

$$-\frac{1}{2}w^2(0) \leq -\frac{1}{4\Lambda}\int_0^1 w^2(z)dz + \frac{2}{\pi^2(2\Lambda-1)}\int_0^1 \left(\frac{dw}{dz}\right)^2 (z)dz \tag{3.2.21}$$

and

$$-\frac{1}{2}w^2(1) \leq -\frac{1}{4\Lambda}\int_0^1 w^2(z)dz + \frac{2}{\pi^2(2\Lambda-1)}\int_0^1 \left(\frac{dw}{dz}\right)^2 (z)dz \tag{3.2.22}$$

hold for all $\Lambda > \frac{1}{2}$.

Proof. By (Chung-Fen et al., 2004, Corollary 9), it holds that

$$\int_0^1 (w(z) - w(0))^2 dz \leq \frac{4}{\pi^2}\int_0^1 \left(\frac{dw}{dz}\right)^2 (z)dz, \tag{3.2.23}$$

or equivalently

$$\int_0^1 w^2(z)dz \leq -w^2(0) + \int_0^1 2w(0)w(z)dz + \frac{4}{\pi^2}\int_0^1 \left(\frac{dw}{dz}\right)^2 (z)dz.$$

By using the Generalized Young's Inequality, see e.g. Krstic and Smyshlyaev (2008), it follows that

$$\left(1-\frac{1}{2\Lambda}\right)\int_0^1 w^2(z)dz \leq (-1+2\Lambda)\,w^2(0)+\frac{4}{\pi^2}\int_0^1\left(\frac{dw}{dz}\right)^2(z)dz, \qquad (3.2.24)$$

for some $\Lambda > 0$. To ensure the positivity of $1-\frac{1}{2\Lambda}$, we have to assume that $\Lambda > \frac{1}{2}$. In this way, one may write (3.2.24) as

$$\int_0^1 w^2(z)dz \leq 2\Lambda w^2(0)+\frac{8\Lambda}{\pi^2\,(2\Lambda-1)}\int_0^1\left(\frac{dw}{dz}\right)^2(z)dz.$$

The inequality (3.2.22) can be deduced by applying similar arguments on the function $\tilde{w}$ defined by $\tilde{w}(z) = w(1-z), z \in [0,1]$. □

Lemma 3.2.4 *Let $a < b$ be two real numbers. If f is a continuous function on $[a,b]$ and g is integrable on $[a,b]$ and either $g(z) \geq 0$ or $g(z) \leq 0$ for all $z \in [a,b]$, then there exists $c \in [a,b]$ such that $\int_a^b f(z)g(z)dz = f(c)\int_a^b g(z)dz$.*

Proof. See (Neuser, 1970, Theorem 3.3). □

Proof. **(Proposition 3.2.7)** Let us choose as Lyapunov functional candidate the function $V : L^2([0,1]:\mathbb{R}) \to \mathbb{R}$, defined by $V(\hat{\xi}_1) = \frac{1}{2}\|\hat{\xi}_1\|^2_{L^2}$. By differentiating V w.r.t. t along the state trajectories corresponding to (3.2.18), one gets

$$\dot{V}(\hat{\xi}_1) = \frac{1}{2}\frac{d}{dt}\int_0^1 \hat{\xi}_1^2 dz = \int_0^1 \hat{\xi}_1\left(\frac{1}{Pe}\frac{d^2\hat{\xi}_1}{dz^2}-q(z)\hat{\xi}_1\right)dz.$$

Integration by parts yields that

$$\dot{V}(\hat{\xi}_1) = -\frac{1}{2}\hat{\xi}_1^2(1)-\frac{1}{2}\hat{\xi}_1^2(0)-\frac{1}{Pe}\int_0^1\left(\frac{d\hat{\xi}_1}{dz}\right)^2 dz-\int_0^1 q(z)\hat{\xi}_1^2 dz.$$

According to Lemma 3.2.4, $\dot{V}$ takes the form

$$\dot{V}(\hat{\xi}_1) = -\frac{1}{2}\hat{\xi}_1^2(1)-\frac{1}{2}\hat{\xi}_1^2(0)-\frac{1}{Pe}\int_0^1\left(\frac{d\hat{\xi}_1}{dz}\right)^2 dz-q(c)\int_0^1 \hat{\xi}_1^2 dz \qquad (3.2.25)$$

for some $c \in (0,1)$. Applying Lemma 3.2.3 to (3.2.25) allows to bound $\dot{V}$ as

$$\dot{V}(\hat{\xi}_1) \leq \left(\frac{-1}{2\Lambda}-q(c)\right)\int_0^1 \hat{\xi}_1^2 dz+\left(\frac{4}{\pi^2(2\Lambda-1)}-\frac{1}{Pe}\right)\int_0^1\left(\frac{d\hat{\xi}_1}{dz}\right)^2 dz.$$

We shall now choose Λ in such a way that $\frac{4}{\pi^2(2\Lambda-1)} - \frac{1}{Pe} = 0$, which yields $\Lambda = \frac{1}{2} + \frac{2Pe}{\pi^2} > \frac{1}{2}$. Consequently,

$$\frac{1}{2}\frac{d}{dt}\|\hat{\xi}_1\|_{L^2}^2 \leq -\left(\frac{\pi^2}{\pi^2+4Pe} + q(c)\right)\int_0^1 \hat{\xi}_1^2 dz = -\left(\frac{\pi^2}{\pi^2+4Pe} + q(c)\right)\|\hat{\xi}_1\|_{L^2}^2. \tag{3.2.26}$$

By applying Gronwall's Lemma, see (Curtain and Zwart, 2020, Lemma A.5.30), one gets

$$\|\hat{\xi}_1(\cdot,t)\|_{L^2} \leq e^{-\left(\frac{\pi^2}{\pi^2+4Pe}+q(c)\right)t}\|\hat{\xi}_1(\cdot,0)\|_{L^2}. \tag{3.2.27}$$

□

The estimate (3.2.27) leads to the following sufficient condition in order to characterize the exponential stability of equilibria of (2.3.3).

Theorem 3.2.8 *A sufficient condition for an equilibrium profile of (2.3.3) to be exponentially stable for the linearized system (3.2.18) is that*

$$\mu \leq \tilde{h}^e(a), \tag{3.2.28}$$

where the function $\tilde{h}^e(a)$ *is defined as* $\frac{\left(1-\tilde{f}^e(a)\right)^2(r(\delta-a-al_D)+a)}{\left(\delta+\tilde{f}^e(a)\right)a}$ *with*

$$\tilde{f}^e(a) = \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{2D} - a + \frac{K}{D^2},$$

$$r = \frac{\pi^2}{\pi^2+4Pe} + \frac{Pe}{4}$$

and

$$l_D = \frac{\frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a+\frac{K}{D^2}}}}{a} - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{a}}{k_0Le^{\frac{-\mu}{1+a+\frac{K}{D^2}}}}, \tag{3.2.29}$$

where K is the positive constant introduced in (3.2.11). Note that in the case where inequality (3.2.28) holds, exponential stability is obtained for all $D \geq D^\star$ where $D^\star > 0$ is sufficiently large, which is the same condition for existence and multiplicity of the equilibria.

Proof. Starting from inequality (3.2.28), it holds that

$$\frac{a\left(\mu\delta + \mu\tilde{f}^e(a) - \left(1-\tilde{f}^e(a)\right)^2\right)}{\delta - a - al_D} \leq r\left(1-\tilde{f}^e(a)\right)^2. \tag{3.2.30}$$

It follows from inequalities (3.2.7) and (3.2.12) that $a - \frac{k_0L(\delta-a)e^{\frac{-\mu}{1+a}}}{2D} - \frac{K}{D^2} < x_1^e(z) <$

$a + \frac{K}{D^2}$, a.e. on $[0,1]$, which can be written as

$$-\tilde{f}^e(a) < x_1^e(z) < a + K/D^2. \tag{3.2.31}$$

Since $x_1^e(z) < a + K/D^2$ a.e. on $[0,1]$, it follows that

$$e^{-\mu/(1+x_1^e(z))} < e^{-\mu/(1+a+\frac{K}{D^2})} \tag{3.2.32}$$

a.e. on $[0,1]$. Combining (3.2.30), (3.2.31) and the previous inequality yields the estimate

$$\frac{k_0 L e^{\frac{-\mu}{1+x_1^e(z)}}}{k_0 L e^{\frac{-\mu}{1+a+\frac{K}{D^2}}}} \frac{a\left(\mu\delta - \mu x_1^e(z) - (1+x_1^e(z))^2\right)}{\delta - a - a l_D} < r\,(1+x_1^e(z))^2,$$

where the positivity of $1-\tilde{f}^e(a)$ and $\delta - a - al_D$ have been used thanks to (3.2.8) and the fact that D is sufficiently large. Equivalently one has that

$$\frac{k_0 L e^{\frac{-\mu}{1+x_1^e(z)}}\left(\mu\delta - \mu x_1^e(z) - (1+x_1^e(z))^2\right)}{\frac{k_0 L(\delta-a) e^{\frac{-\mu}{1+a+\frac{K}{D^2}}}}{a} - k_0 L e^{\frac{-\mu}{1+a+\frac{K}{D^2}}} l_D} < r\,(1+x_1^e(z))^2.$$

By plugging the expression of l_D (3.2.29) in the previous inequality, it follows that

$$\frac{k_0 L e^{\frac{-\mu}{1+x_1^e(z)}}}{v}\left(\mu\delta - \mu x_1^e(z) - (1+x_1^e(z))^2\right) < r\,(1+x_1^e(z))^2,$$

where relation (3.2.4) has been taken into account. By dividing the last inequality by $(1+x_1^e(z))^2$, one gets

$$\frac{k_0 L e^{\frac{-\mu}{1+x_1^e(z)}}}{v} \frac{\mu\delta(1-x_2^e(z))}{(1+x_1^e(z))^2} - \frac{k_0 L e^{\frac{-\mu}{1+x_1^e(z)}}}{v} < r,$$

which can be rewritten as $-q(z) + Pe/4 < r$ by using the definition of the function q, see (3.2.19), where we have been considering the reaction invariant relation $x_1^e(z) = \delta x_2^e(z)$. Hence $\frac{\pi^2}{\pi^2+4Pe} + q(z) > 0$ for a.e. $z \in [0,1]$. In particular, $\frac{\pi^2}{\pi^2+4Pe} + q(c) > 0$ for $c \in (0,1)$. By using the fact that (3.2.27) holds for all $t > 0$, see Proposition 3.2.7, it follows that the equilibrium x_1^e is exponentially stable. □

Note that the parameter a on which the function $\tilde{h}^e$ depends plays the role of a switch for the different equilibria. As explained above, the equilibria are characterized by the values of a that make the function $v - k_0 L(\delta - a)e^{\frac{-\mu}{1+a}}/a$ zero, see (3.2.4). Consequently, for any of these values (depending on the multiplicity of the equilibria), condition (3.2.28) has to be tested to determine if the equilibrium is exponentially stable. Therefore, let us consider the following corollary which takes two different situations into account, a first one involving the case where only one equilibrium is exhibited and a second one dealing with the case of three equilibria.

Corollary 3.2.9 *In the case where the nonisothermal axial dispersion tubular reactor (2.3.3) admits only one equilibrium profile, there exists $D^\star$ sufficiently large such that this equilibrium profile is exponentially stable for all $D \geq D^\star$. Moreover, when three equilibria are exhibited, bistability is established, i.e. the pattern "exponentially stable - unstable - exponentially stable" is depicted.*

Proof. The proof is divided into two parts, depending on the number of equilibria.

Part 1: one equilibrium

Remember that the velocity v is given by $k_0 L(\delta - a)e^{\frac{-\mu}{1+a}}/a$ and remember also that for a fixed value of v the number of equilibria is determined as the number of values of a that reach v, see (Hastir et al., 2020, Section IV). Moreover, by differentiating v w.r.t. a, it yields that

$$\frac{dv}{da} = \frac{k_0 L e^{\frac{-\mu}{1+a}}}{a^2(1+a)^2}\left[-(\mu+\delta)a^2 + \delta(\mu-2)a - \delta\right]. \tag{3.2.33}$$

Hence, in the case where only one equilibrium is exhibited, the first order derivative of v w.r.t. a does not change sign on the interval $]0,\delta[$. This is characterized by the two following possibilities[(1)]

$$\mu\delta\,(\mu\delta - 4\delta - 4) < 0, \tag{3.2.34}$$

$$\mu\delta\,(\mu\delta - 4\delta - 4) = 0. \tag{3.2.35}$$

These two options characterize the fact that only one value of a can reach a fixed value of v, see Hastir et al. (2020). In the first situation, i.e. (3.2.34), the first order derivative of $v(a)$ with respect to a has no root, meaning that v has no extremum and is strictly decreasing, see (Hastir et al., 2020, Section IV.A). Only one value of a can reach a fixed value v (one equilibrium).

In the second case, i.e. (3.2.35), the first order derivative of v with respect to a vanishes only one time in the interval $]0,\delta[$, see (Hastir et al., 2020, Section IV.A) and in particular (Hastir et al., 2020, Equation (15)), wherein it is shown that the point where $\frac{dv}{da}(a)$ vanishes is a point of inflexion, meaning that v is monotone on $]0,\delta[$ (only one value of a can reach a fixed value of v).

To get exponential stability, by Theorem 3.2.28, one has to check that $\mu \leq \tilde{h}^e(a)$. It could be seen as challenging to show that inequality by looking at the expression of the function $\tilde{h}^e(a)$. Therefore we shall look at an approximation of the problem thanks to the Taylor expansion of the functions $\tilde{h}^e(a), \tilde{f}^e(a), r$ and l_D as functions of $\frac{1}{D}$ near $\frac{1}{D} = 0$. Only the first term in the expansion will be considered. Hence, by introducing the parameter $\varepsilon = 1/D$, the problem of finding values of a such that $\mu \leq \tilde{h}^e(a)$ will be approached by a similar one by taking $\varepsilon = 0$, i.e.[(2)] $\mu \leq \frac{\delta(1+a)^2}{a(\delta-a)} =: h(a)$. Equivalently

[(1)]The quantity $\mu\delta\,(\mu\delta - 4\delta - 4)$ represents the discriminant of the second order polynomial in (3.2.33), i.e. the discriminant of $-(\mu+\delta)a^2 + \delta(\mu-2)a - \delta$.

[(2)]Setting $1/D$ to 0 implies that $l_{D_{|\frac{1}{D}=0}} = 0, r_{|\frac{1}{D}=0} = 1$ and $\tilde{f}^e(a)_{|\frac{1}{D}=0} = -a$, which leads to $\tilde{h}^e(a)_{|\frac{1}{D}=0} = \frac{\delta(1+a)^2}{a(\delta-a)}$.

we have

$$(\mu+\delta)\,a^2-\delta\,(\mu-2)\,a+\delta\geq 0. \tag{3.2.36}$$

Let us look at the case for which (3.2.34) holds. The discriminant of the second order polynomial in (3.2.36), denoted by ρ, is equal to $\mu\delta\,(\mu\delta-4\delta-4)$, which is assumed to be negative here. Hence (3.2.36) is satisfied for all values of $a\in]0,\delta[$ and the corresponding equilibrium profile is exponentially stable by Theorem 3.2.28.

In the second case, see (3.2.35), the polynomial in (3.2.36) vanishes only one time when $a=\delta/(2+\delta):=a^\star$. For other values of the parameter a, it is positive, leading to exponential stability of the related equilibria.

Part 2: three equilibria

Similar arguments to those presented before in the proof are used here. In the case where the reactor can exhibit three equilibria, three values of a can reach a fixed value of $v^\star$ meaning that $v^\star(a)$ has two extrema in the interval $]0,\delta[$. Mathematically there holds

$$\mu\delta\,(\mu\delta-4\delta-4)>0, \tag{3.2.37}$$

see (Hastir et al., 2020, Section IV.A) for the reasoning and calculation details. Once more, exponential stability is obtained if (3.2.36) is satisfied. Since (3.2.37) holds, the second order polynomial in (3.2.36) possesses two roots that are given by

$$a_1^\star=\frac{\delta(\mu-2)}{2(\mu+\delta)}-\frac{1}{2(\mu+\delta)}\sqrt{\mu\delta(\mu\delta-4\delta-4)}$$

and

$$a_2^\star=\frac{\delta(\mu-2)}{2(\mu+\delta)}+\frac{1}{2(\mu+\delta)}\sqrt{\mu\delta(\mu\delta-4\delta-4)}.$$

Hence values of a in the interval $]0,a_1^\star[$ or in $]a_2^\star,\delta[$ lead to exponentially stable equilibria. Since $a_1^\star$ and $a_2^\star$ are also roots of the first order derivative of v, the intervals $]0,a_1^\star[$, $[a_1^\star,a_2^\star]$ and $]a_2^\star,\delta[$ denote the three zones where the first, the second and the third equilibrium profiles are located, respectively, see the expression and the shape of $v(a)$. It follows that the first and the third equilibria are exponenially stable. For establishing the instability of the second equilibrium, we refer to (Varma and Aris, 1977, Section 2.5.2.). $\square$

Note that, when an equilibrium profile is exponentially stable, the function q defined in (3.2.19) gives a hint on the exponential decay to 0 of the $L^2([0,1];\mathbb{R})-$ norm of the state trajectory $\hat{\xi}_1$. Let us have a look at q in the limit where $\frac{1}{D}$ tends to 0. This has the consequence that $x_1^e(z)$ behaves like a and $x_2^e(z)$ like $\frac{1}{\delta}a$, see Proposition 3.2.2 and relation (3.2.10). According to the expression of the velocity as a function of a, see (3.2.4), this entails that $q(z)$ behaves like

$$q(z)_{|\frac{1}{D}=0}=\frac{a}{\delta-a}-\frac{\mu a}{(1+a)^2}. \tag{3.2.38}$$

This asymptotic expression of q yields that the nearer a is to δ, the larger $q(z)$ is, for any $z\in[0,1]$. At the other side of the interval $]0,\delta[$, the nearer a is to 0, the closer q

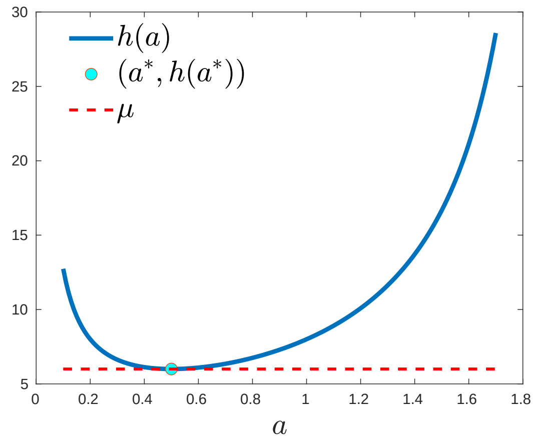

Figure 3.2 – Function h in the case of one equilibrium ($\mu = 6, \delta = 2$).

is to 0. Thanks to Proposition 3.2.7, for an equilibrium characterized by a value of a that is close to 0, the $L^2([0,1];\mathbb{R})-$ norm of the state trajectory $\hat{\xi}_1$ has more chance to converge slowly to 0 than in the case of an equilibrium that is characterized by a value of a that is close to δ. By looking at (3.2.4), a value of a that is close to 0 means that the velocity v is large while a value of a that is close to δ entails that the velocity is low.

The function $h(a)$ is depicted in Figures 3.2 and 3.3 in the cases where one or three equilibria are exhibited. The parameters that are chosen for the case of one equilibrium are as follows: $\mu = 6, \delta = 2$ ($\rho = 0$). For the case where three equilibria are highlighted, the values of μ and δ have been fixed to 10 and 1, respectively ($\rho = 20$). It can be seen that the condition (3.2.28) is always satisfied in the case of one equilibrium. For the case with three equilibria, the condition is satisfied for the first and the third equilibria, which implies bistability.

We shall end this chapter by looking at an intuitive manner to see that the exponential bound found in (3.2.20) is tight in the case where the diffusion coefficient is large. As explained above the function q takes the form (3.2.38) in the limit case where D is large. Hence for D large enough, (3.2.18) behaves like

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - q_a \hat{\xi}_1, \\ \frac{\partial \hat{\xi}_1}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_1(0,t), \ \frac{\partial \hat{\xi}_1}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_1(1,t), \end{cases} \tag{3.2.39}$$

which is a Riesz-spectral system whose operator dynamics $A_r := \frac{1}{Pe}\frac{d^2}{dz^2} - q_a I$ with domain $D(A_r) = \{\hat{\xi}_1 \in H^2([0,1];\mathbb{R}), \frac{d\hat{\xi}_1}{dz}(0) = \frac{Pe}{2}\hat{\xi}_1(0), \frac{d\hat{\xi}_1}{dz}(1) = -\frac{Pe}{2}\hat{\xi}_1(1)\}$ is a Riesz-

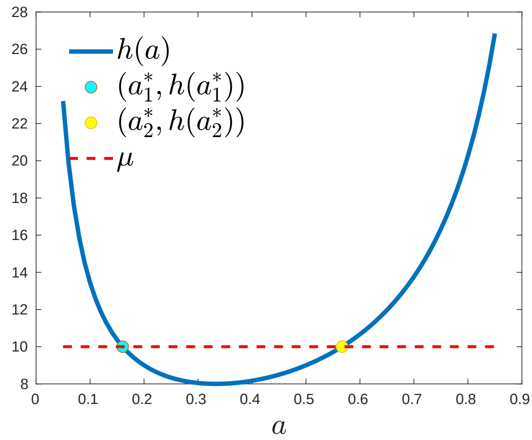

Figure 3.3 – Function h in the case of three equilibria ($\mu = 10, \delta = 1$).

spectral operator with eigenvalues $-s_n^2 - q_a$ where $\{s_n\}_{n\in\mathbb{N}}$ are the solutions of the resolvent equation

$$\tan(s\sqrt{Pe}) = \frac{4s\sqrt{Pe}}{4s^2 - Pe}, s > 0. \tag{3.2.40}$$

Note that the notation $q_a = q(z)_{|\frac{1}{D}=0}$ has been used, see (3.2.38) for the expression of $q(z)_{|\frac{1}{D}=0}$. Let us have a look at the resolvent equation (3.2.40) and let us try to classify its solutions. Obviously 0 is a solution but not interesting here since only positive solutions are considered. A large value of D entails that Pe is small, which has the consequence that the resolvent equation behaves like

$$\tan(s\sqrt{Pe}) = \frac{4s\sqrt{Pe}}{4s^2} = \frac{\sqrt{Pe}}{s}, s \geq 0. \tag{3.2.41}$$

When its argument is small, the tangent may be approximated by its argument, which entails that the next solution after 0 is $s_1 = 1$ when looking at (3.2.41). According to Dehaye (2015), the solutions of (3.2.40) are increasing and possess the property that $\lim_{n\to\infty}\left[s_n - \frac{(n-1)\pi}{\sqrt{Pe}}\right] = 0$. Hence the largest eigenvalue of the operator A_r is $\lambda^* = -1 - q_a$. According to Theorem 2.1.4, the growth bound of the semigroup generated by A_r is equal to λ^*. When this growth bound is negative, it dictates the speed at which the solution of (3.2.18) decreases exponentially to 0. Consider now the exponent found in the exponential of (3.2.20) with the Lyapunov-based approach. This exponent is given by $-\frac{\pi^2}{\pi^2+4Pe} - q(c) =: \overline{\lambda}$ for some $c \in (0,1)$. Considering the limit when $\frac{1}{D}$ is close to 0 yields that $\overline{\lambda} \to -1 - q_a = \lambda^*$. This proves that the exponential rate found

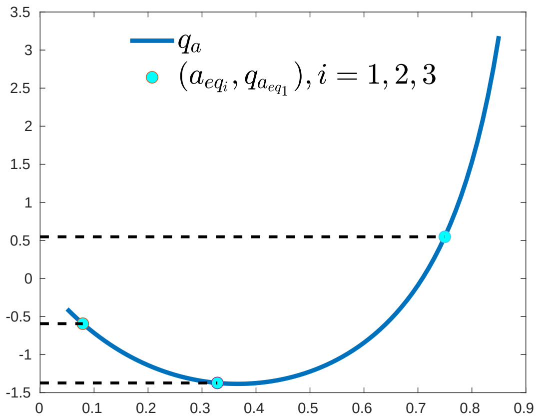

Figure 3.4 – Function q_a in the case of three equilibria ($\mu = 10, \delta = 1$).

in (3.2.20) is quite tight and gives an accurate estimation of the growth bound of the semigroup generated by the operator $\frac{1}{Pe}\frac{d^2}{dz^2} - q(z)I$ on the domain $D(A_r)$.

The representation of the function q_a is depicted in Figure 3.4 in the case of $\mu = 10, \delta = 1, \nu = 0.0011$ and $D = 10$. The three values of a that characterize the three equilibria are illustrated on the Figure. These three values of a are given by $a_{eq_1} = 0.0791, a_{eq_2} = 0.3284$ and $a_{eq_3} = 0.7498$, respectively. This has the consequence that the corresponding values of q_a are $q_{a_{eq_1}} = -0.5935, q_{a_{eq_2}} = -1.3720$ and $q_{a_{eq_3}} = 0.5476$. Plugging these values in the parameter λ^* entails that $\lambda^*_{a=0.0791} = -0.4065, \lambda^*_{a=0.3284} = 0.3720$ and $\lambda^*_{a=0.7498} = -1.5476$, which tells that the first and the third equilibria are exponentially stable while the second is unstable.

Chapter 4

A perturbation method to analyze the nonlinear stability of equilibria

Contents

The topic that is developed in this chapter is based on the concepts of differentiability introduced in Chapter 3. The key point is to try to deduce the nonlinear stability of an equilibrium of a distributed parameter system (locally) on the basis of the stability analysis of the equilibrium for a corresponding linearized system.

4.1 Existing theories to deduce exponential stability

The type of differentiability which will play an important role here is Fréchet differentiability which has been defined in Definition 3.1.2. Let us start by recalling the following result from the finite-dimensional setting, known as the Lyapunov's Indirect Theorem, see e.g. (Haddad and Chellaboina, 2008, Theorem 3.19).

Theorem 4.1.1 *Let us consider the nonlinear finite-dimensional system*

$$\dot{x}(t) = f(x(t)), x(0) = x_0, t \geq 0, \tag{4.1.1}$$

where the nonlinear function $f : D \to \mathbb{R}^n$ is continuously differentiable and D is an open set that contains the origin. Moreover, let the origin be an equilibrium point of (4.1.1) and let $A = \frac{\partial f}{\partial x}_{|x=0}$ be the Jacobian matrix of f at the equilibrium point 0 with $\{\lambda_n\}_{n=1,\ldots,n}$ being the set of its eigenvalues. Then the following statements hold:

- *If $\sup\limits_{i=1,\ldots,n} \mathfrak{Re}(\lambda_n) < 0$ then the equilibrium point 0 is locally exponentially stable;*
- *If there exists $n^* \in \mathbb{N}, 1 \leq n^* \leq n$ such that $\mathfrak{Re}(\lambda_{n^*}) > 0$ then the equilibrium point 0 is unstable.*

The central idea in the proof of the theorem is the continuous differentiability of f. This argument is the most responsible for the stability property of the nonlinear system as well as the local aspect of the stability. Thanks to that result, it is intuitively natural to ask for a same kind of concept when moving to infinite dimension. This argument is the Fréchet differentiability of the nonlinear semigroup whose nonlinear operator dynamics is the infinitesimal generator. We report hereafter some results on the link between the local stability of a nonlinear system and the stability of a corresponding linearized version of it. This is called linearized stability.

The concept of linearized stability is discussed in Henry (1981) wherein some assumptions on the nonlinear operator dynamics are stated like continuous Fréchet differentiability, local Hölder continuity in time and also local Lipschitz continuity. As long as the linearized system around a chosen equilibrium is exponentially stable, (Henry, 1981, Theorem 5.1.1) guarantees that the same conclusion holds for the original system, locally around that equilibrium. In Smoller (1983), the same conclusion on the nonlinear semigroup is shown to hold provided that the latter, generated by the nonlinear operator dynamics, is Fréchet differentiable at the considered equilibrium point, see (Smoller, 1983, Theorem 11.22). It is expressed as follows.

Theorem 4.1.2 *Let X be a Hilbert space. Consider the dynamical system*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)), \\ x(0) = x_0 \in X, \end{cases} \tag{4.1.2}$$

where the assumptions of Theorem 2.2.2 are supposed to hold. Let $(S(t))_{t\geq 0}$ be the nonlinear semigroup whose $A + f$ is the infinitesimal generator and let $(\overline{T}(t))_{t\geq 0}$ be the linear semigroup generated by the operator $A + Df(x^e)$, the Fréchet derivative of $A + f$ at x^e, where x^e is an equilibrium of (4.1.2). Under the assumption that $(S(t))_{t\geq 0}$ is Fréchet differentiable[a] at x^e with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative, it holds that:

- *If* $(\overline{T}(t))_{t\geq 0}$ *is exponentially stable then the equilibrium* x^e *is locally exponentially stable for (4.1.2);*

- *If* $(\overline{T}(t))_{t\geq 0}$ *is unstable then it is locally unstable for (4.1.2).*

[a] As the semigroup $(S(t))_{t\geq 0}$ is indexed by time, by Fréchet differentiability of $(S(t))_{t\geq 0}$, we mean Fréchet differentiability of $S(t)$ for any $t \geq 0$.

Sufficient conditions on the nonlinear operator are often stated to get Fréchet differentiability of the generated nonlinear semigroup, see e.g. (Smoller, 1983, Theorems 11.17 and 11.18). It is for instance asked that the nonlinear operator f is Fréchet differentiable at the equilibrium point while it has to be locally Lipschitz continuous. Two years later, Webb showed the same result in (Webb, 1985, Theorem 4.12) with other assumptions and applied it to the age-dependent population problem. Fréchet differentiability of nonlinear semigroups is also studied in Temam (1997) for semilinear systems of the form (2.2.1). Conditions are required on both linear and nonlinear parts of the dynamics, i.e. on the operators A and f, respectively. For instance, the linear operator has to be closed, negative and self-adjoint while the nonlinear one has to satisfy some appropriate decomposition, see e.g. (Temam, 1997, Section 8). This result has been presented again in (Al Jamal et al., 2014, Theorem 3.8) where it is stated that the conditions that have to be fulfilled are quite restrictive. In Kato (1995), it is shown that exponential stability of an equilibrium of a nonlinear system is guaranteed as long as the same behavior holds for a Gâteaux linearized version of the nonlinear operator describing the dynamics, as long as the nonlinear operator is Fréchet differentiable. Other assumptions like the Lipschitz continuity of the Fréchet derivative in the operator norm are also needed.

More recently in Al Jamal et al. (2014), similar results are established and applied to the Kuramoto-Sivashinski nonlinear partial differential equations. In that paper the limitations of the theory are also discussed. Indeed the assumptions that are required on both the linearized system and the nonlinearity could be hard to verify for many models of nonlinear PDEs. The main difficulty is due to the assumptions needed on the nonlinear operator dynamics that ensure Fréchet differentiability of the nonlinear semigroup. This is the reason why one often works on the semigroup instead of the generator, see for instance (Al Jamal, 2013; Al Jamal and Morris, 2018). In general a case-by-case study has to be performed.

4.2 From the classical Fréchet differentiability to an adapted concept of differentiability

In this section we show how the difficulties reported previously to get Fréchet differentiability may be overcome. We shall introduce a new concept of differentiability for nonlinear operators. A new framework is built around this new concept to be able to characterize the local stability of an equilibrium for a nonlinear system of the form (4.1.2).

The systems that are considered are of the form (4.1.2) where we suppose that the assumptions of Theorem 2.2.2 hold, that is,

- the domain $D(f)$ is $(T(t))_{t\geq 0}$ invariant, i.e. $T(t)D(f) \subset D(f)$, for all $t \geq 0$;
- the relation
$$\lim_{h\to 0^+} \frac{1}{h} d(x+hf(x);D(f)) = 0$$
holds for all $x \in D(f)$;
- the nonlinear operator f is Lipschitz continuous on $D(f)$ and there exists a nonnegative constant l_f such that the operator $f - l_f I$ is dissipative on $D(f)$.

Let x^e be an equilibrium solution of (4.1.2), i.e. $Ax^e + f(x^e) = 0$. In addition, we consider the following assumption on the operator f.

Assumption 4.2.1 *The nonlinear operator $f : D(f) \subseteq X \to X$ is Gâteaux differentiable at the equilibrium x^e. Its Gâteaux derivative is denoted by $df(x^e) : X \to X$ and is assumed to be bounded on X, i.e. $df(x^e) \in \mathscr{L}(X)$.*

Let us perform the change of variables $\xi = x - x^e$ in such a way that the new equilibrium is 0. Consequently, (4.1.2) becomes

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + f(\xi(t) + x^e) - f(x^e), \\ \xi(0) = x_0 - x^e =: \xi_0. \end{cases} \tag{4.2.1}$$

The assumptions of Theorem 2.2.2 that guarantee well-posedness of (4.1.2) imply that (4.2.1) is still well-posed and that for any initial condition in the shifted domain $D^e(\tilde{f})$ of the operator $\tilde{f}(\cdot) := f(\cdot + x^e) - f(x^e)$, i.e. $D^e(\tilde{f}) := D(f) - x^e$, the corresponding state trajectory $\xi(t)$ solution to (4.2.1) lies in $D^e(\tilde{f})$. Indeed, in terms of semigroups, the change of variable $\xi(t) = x(t) - \xi^e$ is expressed as

$$S^e(t)\xi_0 = S(t)x_0 - x^e, \tag{4.2.2}$$

where $(S(t))_{t\geq 0}$ and $(S^e(t))_{t\geq 0}$ denote the semigroups generated by $A + f$ and by $A + f(\cdot + x^e) - f(x^e)$, respectively. Note that the initial conditions x_0 and ξ_0 are chosen in $D(f)$ and $D^e(\tilde{f})$, respectively. The $D(f)-$invariance of the semigroup $(S(t))_{t\geq 0}$ implies that $S^e(t)\xi_0 \in D^e(\tilde{f})$, which proves that $(S^e(t))_{t\geq 0}$ is $D^e(\tilde{f})-$invariant.

As it will be needed later, the following lemma characterizes some type of continuity of the state trajectory $\xi(t)$ of (4.2.1) on a finite time interval $[0,t_0], t_0 > 0$.

Lemma 4.2.1 *Let us consider ξ, the mild solution of (4.2.1) with ξ_0 as initial condition. Suppose that $f(\xi(\cdot)) \in L^p([0,t_0];X)$ for some $p \geq 1$, and that $\xi_0 \in D(A) \cap D^e(\tilde{f})$. Then $\xi(\cdot)$ and $f(\xi(\cdot) + x^e) - f(x^e) - df(x^e)\xi(\cdot)$ lie in $L^\infty([0,t_0];X)$.*

Proof. Let us consider $t \in [0,t_0]$. By (Curtain and Zwart, 1995, Lemma 3.1.5) and (Temam, 1997, Theorem II.3.4), the state trajectory $\xi(\cdot) \in L^\infty([0,t_0];X)$ since $f(\xi(\cdot)) \in L^p([0,t_0];X)$ for some $p \geq 1$. Moreover

$$\begin{aligned}
&\|f(\xi+x^e)-f(x^e)-df(x^e)\xi\|_{L^\infty([0,t_0];X)} \\
&= \sup_{t\in[0,t_0]} \|f(\xi(t)+x^e)-f(x^e)-df(x^e)\xi(t)\|_X \\
&\leq \sup_{t\in[0,t_0]} (\|f(\xi(t)+x^e)-f(x^e)\|_X + \|df(x^e)\|_{op}\|\xi(t)\|_X),
\end{aligned}$$

where $\|\cdot\|_{op}$ denotes the appropriate operator norm and where the boundedness of $df(x^e)$ has been used. Since f is Lipschitz continuous on the invariant subset $D(f)$, one gets that

$$\begin{aligned}
\|f(\xi+x^e)-f(x^e)-df(x^e)\xi\|_{L^\infty([0,t_0];X)} &\leq (l_f + \|df(x^e)\|_{op}) \sup_{t\in[0,t_0]} \|\xi(t)\|_X \\
&= (l_f + \|df(x^e)\|_{op})\|\xi\|_{L^\infty([0,t_0];X)} < \infty,
\end{aligned}$$

where l_f denotes a Lipschitz constant of f. □

The new concept of differentiability needed to overcome the verifiability of the classical Fréchet differentiability is introduced in the following definition.

Definition 4.2.1 *Let Y be a (possibly Banach) infinite-dimensional space continuously embedded in X, i.e. $Y \subseteq X$ and $\|h\|_X \leq \sigma\|h\|_Y$ for any $h \in Y$ and some $\sigma > 0$. It is also required that $D(A) \subset Y$ and that $D(A)\cap D^e(\tilde{f}) \neq \emptyset$. The operator f is said to be $(Y,X)-$Fréchet differentiable at x^e if there exists a bounded and linear operator $Df(x^e) : X \to X$ such that for all $h \in D(A)\cap D^e(\tilde{f})$ there holds $f(x^e+h)-f(x^e) = Df(x^e)+R(x^e,h)$ where*

$$\lim_{\|h\|_Y\to 0} \frac{\|R(x^e,h)\|_X}{\|h\|_X} = 0. \tag{4.2.3}$$

Equivalently, it holds that

$$\lim_{\|h\|_Y\to 0} \frac{\|f(x^e+h)-f(x^e)-Df(x^e)h\|_X}{\|h\|_X} = 0, \tag{4.2.4}$$

that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $h \in D(A)\cap D^e(\tilde{f})$, $\|h\|_Y < \delta$ implies that

$$\frac{\|f(x^e+h)-f(x^e)-Df(x^e)h\|_X}{\|h\|_X} < \varepsilon.$$

Note that convergence in Y implies convergence in X since $\|h\|_X \leq \sigma\|h\|_Y$ for all $h \in D(A)\cap D^e(\tilde{f})$.

Note that when Y coincides with X, this new definition is the same as the classical Fréchet differentiability introduced in Definition 3.1.2, see for instance Al Jamal and Morris (2018). This will be called the $X-$Fréchet differentiability here.

According to the new concept of Fréchet differentiability we shall need new concepts of local exponential stability in order to characterize the latter for the nonlinear system (4.2.1). We start with the classical definition of a globally exponentially stable equilibrium, see e.g. (Al Jamal, 2013, Definition 3.1.1).

Definition 4.2.2 *The equilibrium x^e of (4.2.1) is said to be globally exponentially stable if there exist $\alpha, \beta > 0$ such that for all $x_0 \in X$, it holds $\|x(t) - x^e\|_X \leq \alpha e^{-\beta t}\|x_0 - x^e\|_X, t \geq 0$, or equivalently, $\|\xi(t)\|_X \leq \alpha e^{-\beta t}\|\xi_0\|_X$ for all $\xi_0 \in X$.*

According to our new concept of differentiability, see Definition 4.2.1, let us consider the following new definition of local exponential stability.

Definition 4.2.3 *The equilibrium x^e of (4.1.2) is $(Y,X)-$locally exponentially stable if there exist $\delta, \alpha, \beta > 0$ such that, for all $\xi_0 \in D(A) \cap D^e(\tilde{f})$ with $\|\xi_0\|_Y < \delta$, there holds $\|\xi(t)\|_X \leq \alpha e^{-\beta t}\|\xi_0\|_X, t \geq 0$.*

The main difference from a classical local condition is that the $Y-$norm of the initial condition has to be small instead of its $X-$norm, restricting the set of allowed initial conditions because of the continuous embedding of Y into X. A same adaptation is introduced in the next definition for local stability and instability.

Definition 4.2.4 *The equilibrium x^e of (4.1.2) is said to be $(Y,X)-$locally stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\xi_0 \in D(A) \cap D^e(\tilde{f}), \|\xi_0\|_Y < \delta$ implies $\|\xi(t)\|_X < \varepsilon, t \geq 0$. The equilibrium x^e is $(Y,X)-$(locally) unstable if it is not $(Y,X)-$stable.*

In order to be allowed to consider the $Y-$norm of the state trajectory $\xi(t)$ one has to ensure that (4.2.1) is well-posed on the space Y. This requirement with other regularity conditions are stated in the following assumption.

Assumption 4.2.2 *The nonlinear abstract Cauchy problem (4.2.1) is well-posed on Y. Moreover, it is assumed that the Gâteaux derivative $df(\xi^e)$ of f is bounded on Y. This has the consequence that, provided that the linear operator A generates a C_0-semigroup on Y, the linearized dynamics corresponding to (4.2.1), i.e. the operator $A + df(x^e)$, is well-posed on Y, see e.g. (Engel and Nagel, 2006, Bounded Perturbation Theorem). The nonlinear C_0- semigroup $(S^e(t))_{t\geq 0}$ is also assumed to be $Y-$Fréchet differentiable at* 0.

Let us state the following assumption that will be of great importance in deducing the Fréchet differentiability of the semigroup $(S^e(t))_{t\geq 0}$ on the basis of the Fréchet differentiability of the nonlinear operator $\tilde{f}$.

Assumption 4.2.3 *Let $\xi(t)$ be the mild solution of (4.2.1) for the initial condition ξ_0, where $0 \leq t \leq t_0$. Let us consider a space $(Y, \|\cdot\|_Y)$ that satisfies $D(A) \subset Y \subseteq X$. It is assumed that the nonlinear operator f is $(Y,X)-$Fréchet differentiable at*[a] *x^e and that ξ is continuously dependent of the initial condition*

ξ_0 *on X and on Y at zero in the sense that the inequalities*

$$\|\xi(t)\|_X \le \gamma_t^X \|\xi_0\|_X, \tag{4.2.5}$$

$$\|\xi(t)\|_Y \le \gamma_t^Y \|\xi_0\|_Y \tag{4.2.6}$$

hold for some positive γ_t^X, γ_t^Y *that may depend on time.*

[a] This is equivalent to assume that the nonlinear operator $\tilde{f} := f(\cdot + x^e) - f(x^e)$ is $(Y,X)-$Fréchet differentiable at 0.

According to this assumption, one may characterize the behavior of the ratio $\frac{\|f(\xi+x^e)-f(x^e)-Df(x^e)\xi\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X}$ as the $Y-$norm of the initial condition ξ_0 goes to 0, see the following lemma. The notation $Df(x^e)$ stands for the $(Y,X)-$Fréchet derivative of f at x^e. Note that the latter coincides with the Gâteaux derivative when restricted on Y, i.e. $Df(x^e)h = df(x^e)h$ for any $h \in Y$.

Lemma 4.2.2 *Let us consider* $\xi(t)$*, the solution of the abstract differential equation (4.2.1), where* $t \in [0,t_0]$ *for some positive* t_0 *and where* $f(\xi(\cdot))$ *is assumed to be in* $L^p([0,t_0];X)$ *for some* $p \ge 1$*. Then, under Assumptions 4.2.1 and 4.2.3, the condition*

$$\lim_{\|\xi_0\|_Y \to 0} \frac{\|f(\xi + x^e) - f(x^e) - Df(x^e)\xi\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X} = 0 \tag{4.2.7}$$

holds for any initial condition $\xi_0 \in D(A) \cap D^e(\tilde{f})$.

Proof. First note that, if $f \in L^p([0,t_0];X)$ for some $p \ge 1$, it follows that ξ and $f(\xi+x^e) - f(x^e) - df(x^e)\xi$ are in $L^\infty([0,t_0];X)$ and are continuous, see Lemma 4.2.1. Then observe that the function $f(\xi(\cdot)+x^e) - f(x^e) - Df(x^e)\xi(\cdot)$ is time–continuous on the interval $[0,t_0]$. Hence there exists $t^* \in [0,t_0]$ such that

$$\begin{aligned}\sup_{t\in[0,t_0]} &\|f(\xi(t)+x^e) - f(x^e) - Df(x^e)\xi(t)\|_X \\ &= \|f(\xi(t^*)+x^e) - f(x^e) - Df(x^e)\xi(t^*)\|_X.\end{aligned} \tag{4.2.8}$$

Moreover, according to (4.2.5), the estimate

$$\frac{1}{\|\xi_0\|_X} \le \gamma_{t^*}^X \frac{1}{\|\xi(t^*)\|_X} \tag{4.2.9}$$

holds true. Combining (4.2.8) and (4.2.9) yields

$$\begin{aligned}&\frac{\|f(\xi + x^e) - f(x^e) - Df(x^e)\xi\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X} \\ &\qquad = \frac{\|f(\xi(t^*)+x^e) - f(x^e) - Df(x^e)\xi(t^*)\|_X}{\|\xi_0\|_X}\end{aligned}$$

$$\leq \gamma_{t^*}^X \frac{\|f(\xi(t^*)+x^e)-f(x^e)-Df(x^e)\xi(t^*)\|_X}{\|\xi(t^*)\|_X}.$$

According to the inequality (4.2.5), imposing that $\|\xi_0\|_Y$ converges to 0 implies that so does $\|\xi(t^*)\|_Y$. Moreover, since the initial condition ξ_0 has been chosen in $D(A)\cap D^e(\tilde{f})$, the solution $\xi(t)$ of (4.2.1) is actually a classical solution, and hence, lies in $D(A)$. Consequently, by the assumption of $(Y,X)-$Fréchet differentiability of f at x^e, we have

$$\lim_{\|\xi_0\|_Y\to 0} \frac{\gamma_{t^*}^X\|f(\xi(t^*)+x^e)-f(x^e)-Df(x^e)\xi(t^*)\|_X}{\|\xi(t^*)\|_X}=0.$$

In view of the inequality above, it follows that

$$\lim_{\|\xi_0\|_Y\to 0} \frac{\|f(\xi+x^e)-f(x^e)-Df(x^e)\xi\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X}=0.$$

□

In order to be able to link the stability of a linearization of (4.2.1) with the stability of (4.2.1) let us consider the following lemma that enables the link between the Fréchet differentiability of f and the Fréchet differentiability of the C_0-semigroup $(S^e(t))_{t\geq 0}$. Let us start by considering the following linear system

$$\begin{cases} \dot{\overline{\xi}}(t)=A\overline{\xi}(t)+df(x^e)\overline{\xi}(t), \\ \overline{\xi}(0)=\xi_0, \end{cases} \tag{4.2.10}$$

that corresponds to a Gâteaux linearization of (4.2.1) around 0. Under the assumption of boundedness of the operator $df(x^e)$ both viewed as an operator defined on X or on Y, the linear operator $A+df(x^e)$ is the infinitesimal generator of a C_0-semigroup on X and on Y. The C_0-semigroup whose operator A is the infinitesimal generator is denoted by[(1)] $(T(t))_{t\geq 0}$.

Lemma 4.2.3 *Let us consider a space $(Y,\|\cdot\|_Y)$ satisfying $D(A)\subset Y\subseteq X$ and for which $\|h\|_X\leq\sigma\|h\|_Y$ for all $h\in Y$. Under Assumptions 4.2.1 and 4.2.3, the nonlinear C_0-semigroup $(S^e(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at* 0 *and its Fréchet derivative is given by the linear C_0-semigroup $(\overline{T}(t))_{t\geq 0}$ whose infinitesimal generator is $A+df(\xi^e)$, that corresponds to the Gâteaux derivative of $A+f(\cdot+x^e)-f(x^e)$ at* 0.

Proof. Take $\xi_0\in D(A)\cap D^e(\tilde{f})$ and $t\in[0,t_0]$. The mild solutions associated to (4.2.1) and (4.2.10) are given by

$$\xi(t)=S^e(t)\xi_0=T(t)\xi_0+\int_0^t T(t-s)[f(\xi(s)+x^e)-f(x^e)]ds$$

(1) According to (Curtain and Zwart, 2020, Theorem 2.1.7), the estimate $\|T(t)\|\leq M_\omega e^{\omega t}$ holds for all $\omega>\omega_0$ and some M_ω that depends on the choice of ω, where ω_0 denotes the growth bound of $(T(t))_{t\geq 0}$.

and

$$\overline{\xi}(t) = \overline{T}(t)\xi_0 = T(t)\xi_0 + \int_0^t T(t-s)df(x^e)\overline{\xi}(s)ds,$$

respectively. Obviously, it holds that

$$\begin{aligned}
&\|S^e(t)\xi_0 - \overline{T}(t)\xi_0\|_X = \|\int_0^t T(t-s)[f(\xi(s)+x^e) - f(x^e) - df(x^e)\overline{\xi}(s)]ds\|_X \\
&= \|\int_0^t T(t-s)[f(\xi(s)+x^e) - f(x^e) - df(x^e)\xi(s) + df(x^e)\xi(s) - df(x^e)\overline{\xi}(s)]ds\|_X \\
&\leq M_\omega \int_0^t e^{\omega(t-s)}\|f(\xi(s)+x^e) - f(x^e) - df(x^e)\xi(s)\|_X ds \\
&\qquad\qquad + M_\omega \int_0^t e^{\omega(t-s)}\|df(x^e)(\xi(s) - \overline{\xi}(s))\|_X ds.
\end{aligned}$$

By using the boundedness of the Gâteaux derivative of f at x^e on the space X, one gets that

$$\begin{aligned}
\|e^{-\omega t}(S^e(t)\xi_0 - \overline{T}(t)\xi_0)\|_X \leq & M_\omega \int_0^t e^{-\omega s}\|f(\xi(s)+x^e) - f(x^e) - df(x^e)\xi(s)\|_X ds \\
& + M_\omega \|df(x^e)\|_{\mathscr{L}(X)} \int_0^t \|e^{-\omega s}(S^e(s)\xi_0 - \overline{T}(s)\xi_0)\|_X ds.
\end{aligned}$$

It follows by using Gronwall's lemma that

$$\|S^e(t)\xi_0 - \overline{T}(t)\xi_0\|_X \leq M_\omega e^{(\omega+\eta)t} k_0 \int_0^t \|R(\xi(s),x^e)\|_X ds \tag{4.2.11}$$

where $R(\xi,x^e)$ stands for $f(\xi+x^e) - f(x^e) - df(x^e)\xi, \eta := M_\omega \|df(x^e)\|_{\mathscr{L}(X)}$ and $k_0 = \max\{1, e^{-\omega t_0}\}$. Consequently, this leads to the following estimate

$$\|S^e(t)\xi_0 - \overline{T}(t)\xi_0\|_X \leq M_\omega e^{(|\omega|+\eta)t_0} k_0 t_0 \|R(\xi,x^e)\|_{L^\infty([0,t_0];X)}.$$

Since (4.2.7) holds by Lemma 4.2.2, the nonlinear semigroup $(S^e(t))_{t\geq 0}$ is $(Y,X)-$ Fréchet differentiable at 0 with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative. □

Note that the previous lemma is valid also if X is a Banach space. No arguments related to the Hilbert spaces setting are used.

4.3 Deducing local exponential stability

Here we aim at constructing the link between the exponential stability/instability of the semigroup $(\overline{T}(t))_{t\geq 0}$ with the local exponential stability/instability of $(S^e(t))_{t\geq 0}$ by using the new concept of differentiability introduced in the previous section.

Before going into the main theorem of this section, let us introduce the following assumption on the exponential stability of the semigroup $(\overline{T}(t))_{t\geq 0}$ on Y when the latter satisfies the same property on X.

Assumption 4.3.1 *In the case where $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on X, it is supposed that $(\overline{T}(t))_{t\geq 0}$ is also exponentially stable on Y, that is*

$$\|\overline{T}(t)\xi_0\|_Y \leq \overline{\eta}e^{-\theta t}\|\xi_0\|_Y, t \geq 0, \forall \xi_0 \in Y,$$

for some $\overline{\eta} \geq 1, \theta > 0$.

In the following remark we illustrate a consequence of the exponential stability of the linearized model (4.2.10) on the space Y thanks to Theorem 4.1.2.

Remark 4.3.1 *Under Assumptions 4.2.2 and 4.3.1, the estimate*

$$\|S^e(t)\xi_0\|_Y \leq M\|\xi_0\|_Y, t \geq 0, \text{ for } \|\xi_0\|_Y < \delta \tag{4.3.1}$$

holds for some $M > 0$ and $\delta > 0$ that may depend on M. Indeed, the $Y-$Fréchet differentiability of $(S^e(t))_{t\geq 0}$ in Assumption 4.2.2 combined with Assumption 4.3.1 yields that $(S^e(t))_{t\geq 0}$ is locally exponentially stable on Y, see Theorem 4.1.2. In other words

$$\|S^e(t)\xi_0\|_Y \leq Me^{-\beta t}\|\xi_0\|_Y, t \geq 0, \|\xi_0\|_Y < \delta,$$

for some $M, \beta, \delta > 0$, see e.g. Al Jamal and Morris (2018). This implies (4.3.1).

Hereafter an alternative function that measures the elements on Y is introduced and shown to be locally equivalent with $\|\cdot\|_Y$ around the equilibrium x^e. A specific property of the nonlinear semigroup $(S^e(t))_{t\geq 0}$ is illustrated by using this new manner of measuring.

Remark 4.3.2 *Let us consider the function*

$$|||x||| := \sup_{t\geq 0}\|S^e(t)x\|_Y$$

for $x \in Y, \|x\|_Y < \delta$, $\delta > 0$ sufficiently small. The quantities $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent around the equilibrium x^e, that is

$$\|x\|_Y \leq |||x||| \leq M\|x\|_Y,$$

for some $M > 0$ and $\|x\|_Y < \delta$. This is valid since (4.3.1) is satisfied and since the relation $\sup_{t\geq 0}\|S^e(t)x\|_Y \geq \|S^e(0)x\|_Y = \|x\|_Y$ holds. Moreover, $(S^e(t))_{t\geq 0}$ satisfies a contraction property when evaluated with $|||\cdot|||$ on Y. For this, take any $x \in Y$ and any $t_0 > 0$ and observe that

$$\begin{aligned}|||S^e(t_0)x||| := \sup_{t\geq 0}\|S^e(t)S^e(t_0)x\|_Y = \sup_{t\geq 0}\|S^e(t+t_0)x\|_Y &= \sup_{s\geq t_0}\|S^e(s)x\|_Y \\ &\leq \sup_{s\geq 0}\|S^e(s)x\|_Y =: |||x|||.\end{aligned}$$

The following theorem generalizes Theorem 4.1.2 in which the classical notion of Fréchet differentiability is needed for the semigroup $(S(t))_{t\geq 0}$. Here we extend the latter by considering our setting that takes the adapted concept of Fréchet differentiability into account.

Theorem 4.3.1 *Let us consider Assumptions 4.2.1 to 4.3.1. If* 0 *is a globally exponentially stable equilibrium of the linearized model (4.2.10), then it is a* $(Y,X)-$ *locally exponentially stable equilibrium of (4.2.1). Conversely, if* 0 *is a* $(Y,X)-$ *unstable equilibrium of (4.2.10), it is* $(Y,X)-$ *locally unstable for the nonlinear system (4.2.1).*

Proof. Let us start the proof with the case where the semigroup $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on X. Let us consider $\xi_0 \in D(A)\cap D^e(\tilde{f})$. Thanks to Assumptions 4.2.1 and 4.2.3, it holds that the semigroup $(S^e(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0 by Lemma 4.2.3, i.e. $S^e(t)\xi_0 = \overline{T}(t)\xi_0 + r(t,x^e,\xi_0)$, where

$$\lim_{\|\xi_0\|_Y\to 0} \frac{\|r(t,x^e,\xi_0)\|_X}{\|\xi_0\|_X} = 0, \tag{4.3.2}$$

where the above limit holds uniformly in t on any compact time interval $[0,t_0], t_0 > 0$. As the functions $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent, see Remark 4.3.2, it holds that

$$\lim_{|||\xi_0|||\to 0} \frac{\|r(t,x^e,\xi_0)\|_X}{\|\xi_0\|_X} = 0.$$

In other words, for any $t_0 > 0$ and $\varepsilon > 0$, there exists $\delta(t_0,\varepsilon) > 0$ such that, if $|||\xi_0||| < \delta(t_0,\varepsilon)$,

$$\frac{\|r(t,x^e,\xi_0)\|_X}{\|\xi_0\|_X} < \varepsilon,$$

for any $t \in [0,t_0]$. By the strong continuity in t of the semigroups $(S^e(t))_{t\geq 0}$ and $(\overline{T}(t))_{t\geq 0}$, the function $r(t,x^e,\xi_0)$ is also continuous in t. Since 0 is a globally exponentially stable equilibrium of (4.2.10) by assumption, there exist $\alpha \geq 1$ and $\beta > 0$ such that for all $\xi_0 \in D(A)\cap D^e(\tilde{f})$

$$\|\overline{T}(t)\xi_0\|_X \leq \alpha e^{-\beta t}\|\xi_0\|_X, t \geq 0. \tag{4.3.3}$$

Hence for any $t_0 < +\infty$ and any $\varepsilon > 0$, it holds that

$$\begin{aligned} \|S^e(\tau)\xi_0\|_X &\leq \|\overline{T}(\tau)\xi_0\|_X + \|r(\tau,x^e,\xi_0)\|_X \\ &\leq \alpha e^{-\beta\tau}\|\xi_0\|_X + \varepsilon\|\xi_0\|_X =: C\|\xi_0\|_X, \end{aligned} \tag{4.3.4}$$

for $\tau \in [0,t_0]$ and any ξ_0 such that $|||\xi_0||| < \delta(t_0,\varepsilon)$, where $C := \alpha e^{-\beta\tau} + \varepsilon$. Let us choose $t_0 = \frac{\ln(4\alpha)}{\beta} > 0$. The relation (4.3.3) with t replaced by t_0 gives $\|\overline{T}(t_0)\xi_0\|_X \leq \frac{1}{4}\|\xi_0\|_X$. Moreover, there holds

$$\lim_{|||\xi_0|||\to 0} \frac{\|S^e(t_0)\xi_0 - \overline{T}(t_0)\xi_0\|_X}{\|\xi_0\|_X} = 0,$$

that is, there exists $\delta > 0$ such that, if $|||\xi_0||| < \delta$, then $\|S^e(t_0)\xi_0 - \overline{T}(t_0)\xi_0\|_X \leq \frac{1}{4}\|\xi_0\|_X$. Consequently,

$$\begin{aligned}\|S^e(t_0)\xi_0\|_X &= \|S^e(t_0)\xi_0 - \overline{T}(t_0)\xi_0 + \overline{T}(t_0)\xi_0\|_X \\ &\leq \|S^e(t_0)\xi_0 - \overline{T}(t_0)\xi_0\|_X + \|\overline{T}(t_0)\xi_0\|_X \leq \frac{1}{2}\|\xi_0\|_X = e^{-\ln 2}\|\xi_0\|_X.\end{aligned}$$

Let $k > 0$ be an integer. The semigroup property for $(S^e(t))_{t\geq 0}$ and the fact that $(S^e(t_0))^k$ maps $D(A)\cap D^e(\tilde{f})$ into $D(A)\cap D^e(\tilde{f})$ for every $k \in \mathbb{N}$ entails that for every $t_0 \geq 0$, one gets

$$\begin{aligned}\|S^e(kt_0)\xi_0\|_X =& \|(S^e(t_0))^k\xi_0\|_X = \|S^e(t_0)(S^e(t_0))^{k-1}\xi_0\|_X \\ &\leq e^{-\ln 2}\|(S^e(t_0))^{k-1}\xi_0\|_X \leq e^{-(\ln 2)k}\|\xi_0\|_X,\end{aligned} \tag{4.3.5}$$

where we have been using recursively the fact that if $|||\xi_0||| < \delta$, then $|||S^e(t_0)\xi_0||| < \delta$ too, see Remark 4.3.2. For $t > 0$, let[(2)] $k = \lfloor \frac{t}{t_0} \rfloor$ and $\tau = t - kt_0 \in [0, t_0]$. By using the semigroup property, (4.3.4) and (4.3.5), one may deduce the relations

$$\begin{aligned}\|S^e(t)\xi_0\|_X &= \|S^e(\tau + kt_0)\xi_0\|_X = \|S^e(\tau)S^e(kt_0)\xi_0\|_X \\ &\leq C\|S^e(kt_0)\xi_0\|_X \leq Ce^{-(\ln 2)k}\|\xi_0\|_X = \tilde{C}e^{-\gamma t}\|\xi_0\|_X\end{aligned}$$

for $\gamma = \frac{\ln 2}{t_0}$ and $\tilde{C} = Ce^{\ln 2 \frac{\tau}{t_0}}$. Hence 0 is a $(Y,X)-$locally exponentially stable equilibrium for (4.2.1).

In order to prove the second part of the theorem, let 0 be a $(Y,X)-$locally stable equilibrium to the nonlinear system (4.2.1). One has

$$S^e(t)\xi_0 = \overline{T}(t)\xi_0 + r(t, x^e, \xi_0). \tag{4.3.6}$$

Since 0 is $(Y,X)-$locally stable, it follows that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\xi_0\|_Y < \delta$, then $\|S^e(t)\xi_0\|_X \leq \frac{\varepsilon}{2}$, for all $t \geq 0$. From (4.3.2) and since $\|\xi_0\|_X \leq \sigma\|\xi_0\|_Y$, it follows that

$$\lim_{\|\xi_0\|_Y\to 0} \frac{\|r(t, x^e, \xi_0)\|_X}{\|\xi_0\|_Y} \leq \lim_{\|\xi_0\|_Y\to 0} \sigma\frac{\|r(t, x^e, \xi_0)\|_X}{\|\xi_0\|_X} = 0.$$

Hence, $\|r(t, x^e, \xi_0)\|_X$ has to converge to 0 when so does $\|\xi_0\|_Y$. Consequently, there exists δ^* with $0 < \delta^* < \delta$ such that, if $\|\xi_0\|_Y < \delta^*$, then $\|r(t, x^e, \xi_0)\|_X \leq \frac{\varepsilon}{2}$. Since $\|\xi_0\|_Y < \delta^* < \delta$, it follows from (4.3.6) and from the last inequality that

$$\|\overline{T}(t)\xi_0\|_X \leq \|r(t, x^e, \xi_0)\|_X + \|S^e(t)\xi_0\|_X \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

[(2)] The notation $\lfloor\cdot\rfloor$ stands for the integer part.

Remark 4.3.3 *Here we want to report some differences between the new approach, i.e. mainly Theorem 4.3.1 and (Al Jamal et al., 2014, Theorem 3.3). The definition of "locally" means here that* $\|\xi_0\|_Y$ *is assumed to converge to* 0 *instead of* $\|\xi_0\|_X$*, because of the* $(Y,X)-$*Fréchet differentiability of the nonlinear semigroup* $(S^e(t))_{t\geq 0}$*. This somehow restricts the set of initial conditions that are allowed to be considered in our approach. This leads to additional technical difficulties, notably when we need to apply successively the property that* $\|S^e(t_0)\xi_0\|_X \leq e^{-ln\,2}\|\xi_0\|_X$ *whenever* $\|\xi_0\|_Y < \delta$*, on higher composition orders of the nonlinear semigroup, i.e. on* $(S^e(t_0))^k\xi_0, k \in \mathbb{N}, k > 1$*. This is possible due to Assumptions 4.2.2 and 4.3.1 that allow to use a locally equivalent function to* $\|\cdot\|_Y$*, namely*[a] $|||\cdot|||$*. This technical detail is not needed in (Al Jamal et al., 2014) because* $\|S^e(t_0)\xi_0\|_X \leq e^{-ln\,2}\|\xi_0\|_X$ *implies directly that* $\|S^e(t_0)\xi_0\|_X \leq \delta$ *whenever* $\|\xi_0\|_X \leq \delta$*.*

[a] It can be shown that the function $|||\cdot|||$ would define a norm if $(S^e(t))_{t\geq 0}$ was a linear C_0-semigroup, see e.g. (Engel and Nagel, 2006, Lemma 3.10).

The alternative space Y has to be chosen e.g. in order to avoid limitations in the manipulations of norm-inequalities. Good choices are in general L^∞, Sobolev spaces of integer orders ($H^p, p \in \mathbb{N}_0$) which are all multiplicative algebras[(3)] or even the domain of the operator A equipped with the graph norm $\|\cdot\|_A^2 := \|A\cdot\|_X^2 + \|\cdot\|_X^2$. Hence, they allow for example to split the norm of the product of two functions into the product of the norms, which is not permitted in L^p-spaces, $1 \leq p < \infty$, in which Hölder inequality has to be applied.
In order to give a systematic view and what could be called an "algorithmic" view of the method, the following scheme is proposed, see Figure 4.1.

It can be summarized as follows: the objective is to deduce exponential stability or instability of equilibrium profiles for nonlinear distributed parameter systems, where the state space is called X. First, a Gâteaux linearized version of the nonlinear system is built and its exponential stability is studied. Then, after the choice of the alternative space Y, the nonlinear semigroup is proved to be $Y-$Fréchet differentiable. In addition, its linearization has to be exponentially stable on Y when it is exponentially stable on X. Next, the new concept of $(Y,X)-$Fréchet differentiability plays its role to make the connection between Y and X to deduce exponential stability or instability of the equilibria for the nonlinear system (by using $X-$norms). The specificity here is that local means that the $Y-$norm of the initial condition is small instead of its $X-$norm.

4.4 Applications

The aim of this section is to motivate the use of the approach developed previously thanks to the $(Y,X)-$Fréchet differentiability.

[(3)] Note that there are no canonical choices for a given problem.

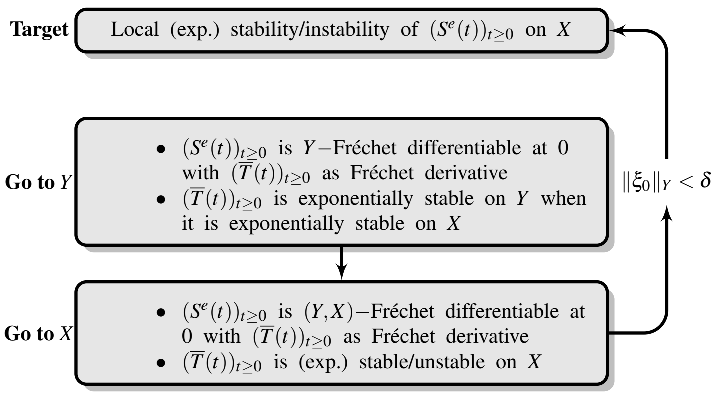

Figure 4.1 – Schematic view of the new methodology developed based on an adapted concept of Fréchet differentiability.

4.4.1 A nonlinear heat equation

We start by looking at the nonlinear operator $f : L^2([0,1];\mathbb{R}) \to L^2([0,1];\mathbb{R})$ defined as $f(x) = \sqrt{x^2+1}$ and introduced in Chapter 3. It has been proved there that it is not Fréchet differentiable at 1 by considering $X := L^2([0,1];\mathbb{R})$ as state space. With the standard approach where classical Fréchet differentiability is needed, this would have bring us to the conclusion that nothing could be said about the stability of an equilibrium of a system wherein f was attached to the dynamics. However, the $(Y,X)-$Fréchet differentiability enables us to tell a bit more about stability of such a nonlinear system. First observe that, by considering $Y = C([0,1];\mathbb{R})$, one may conclude that the nonlinear operator f is $(Y,X)-$Fréchet differentiable at 1 by looking at the proof of Proposition[(4)] 3.1.1. This is equivalent to the fact that $\tilde{f}(x) := f(x+1) - f(1)$ is $(Y,X)-$Fréchet differentiable at 0. Now consider the nonlinear system

$$\begin{cases} \dfrac{\partial \xi}{\partial t} = \dfrac{\partial^2 \xi}{\partial z^2} - \dfrac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \displaystyle\int_0^1 \xi \, dz + f(\xi+1) - f(1), \\ \frac{\partial \xi}{\partial z}(0,t) = 0 = \frac{\partial \xi}{\partial z}(1,t), \end{cases} \tag{4.4.1}$$

[(4)]The arguments that ensure $(Y,X)-$Fréchet differentiability are almost the same as the one presented in Proposition 3.1.1 and are allowed because the function h that has to be considered in the proof has to lie in Y. By performing the same kind of computations, it yields that (3.1.3) takes the form $\frac{\left\| \sqrt{(h+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h \right\|_X}{\|h\|_X} \leq \|h\|_Y \left(\sqrt{2}+2+\|h\|_Y\right)$, which ensures $(Y,X)-$Fréchet differentiability.

which can be seen as a nonlinear heat equation with an integral term. The interpretation will be made later. It can be easily written in an abstract way as

$$\dot{\xi}(t) = A\xi(t) + f(\xi(t)+1) - f(1), \xi(0) = \xi_0, \tag{4.4.2}$$

with A being defined by

$$A\xi = \frac{d^2\xi}{dz^2} - \frac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \int_0^1 \xi(z)dz =: A_d\xi - \frac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \int_0^1 \xi(z)dz$$

for $\xi \in D(A)$ expressed as

$$D(A) = \left\{ \xi \in H^2([0,1];\mathbb{R}), \frac{d\xi}{dz}(0) = 0 = \frac{d\xi}{dz}(1) \right\}. \tag{4.4.3}$$

According to Example 2.1.2 the linear operator A_d is the infinitesimal generator of a C_0-semigroup on X. Since the perturbation $-\frac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \int_0^1 \xi(z)dz$ defines a linear and bounded operator on X the operator A is still the infinitesimal generator of a C_0-semigroup on X. Similar arguments may be used to conclude that A generates a C_0-semigroup on Y and may be found in (Hundertmark et al., 2013, Lecture 4) or (Engel and Nagel, 2006, Chapter 2, Section 2.11). Moreover, let us observe that the scalar valued function $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ defined as $\mathfrak{f}(x) = \sqrt{(x+1)^2+1} - \sqrt{2}$ satisfies $\sup_{x\in\mathbb{R}} |\mathfrak{f}'(x)| = 1$ which implies that the nonlinear operator $\tilde{f}$ is uniformly Lipschitz continuous both on X and on Y. This guarantees that (4.4.1) possesses a unique mild solution on X and on Y, which can even be a classical solution provided that the initial condition is taken in[(5)] $D(A)$, see Theorem 2.2.1.

It can be seen that the state 0 is an equilibrium of (4.4.1). The question one may ask is whether that steady-state is stable for the nonlinear system or not. Therefore, let us take the Gâteaux linearization of (4.4.1) at 0. This yields the linear PDE

$$\begin{cases} \frac{\partial \overline{\xi}}{\partial t} = \frac{\partial^2 \overline{\xi}}{\partial z^2} - \frac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \int_0^1 \overline{\xi} dz + \frac{\sqrt{2}}{2}\overline{\xi}, \\ \frac{\partial \overline{\xi}}{\partial z}(0,t) = 0 = \frac{\partial \overline{\xi}}{\partial z}(1,t). \end{cases} \tag{4.4.4}$$

This linear system has been shown exponentially stable on X in Chapter 2 as an illustration of Datko's Lemma, see Lemma 2.1.5. In order to apply Theorem 4.3.1 it remains to show that the linear operator governing (4.4.4) generates an exponentially stable C_0-semigroup on the space Y. Let us therefore recall that the linear operator

$$A_{lin}\overline{\xi} := \frac{d^2\overline{\xi}}{dz^2} + \frac{\sqrt{2}}{2}\overline{\xi} - \frac{\sqrt{2}+\sqrt{6}}{2} 1_{[0,1]}(z) \int_0^1 \overline{\xi} dz$$

defined on $D(A_{lin}) = D(A)$ is a Riesz-spectral operator with simple eigenvalues

$$\{-\frac{\sqrt{6}}{2}\} \cup \{-n^2\pi^2 + \frac{\sqrt{2}}{2}\}_{n\in\mathbb{N}_0}$$

(5) When the state space X is considered, the domain $D(A)$ is expressed as (4.4.3) while it is given by $\{\xi \in C^2([0,1];\mathbb{R}), \frac{d\xi}{dz}(0) = 0 = \frac{d\xi}{dz}(1)\}$ when the state space Y is chosen.

and normalized eigenfunctions given by

$$\{1_{[0,1]}(z)\} \cup \{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}_0},$$

see Chapter 2, Section 2.1.3. Consequently, the corresponding C_0-semigroup, denoted by $(T(t))_{t\geq 0}$, has the form

$$\begin{aligned}&(T(t)\overline{\xi}_0)(z)\\&\quad= e^{\frac{-\sqrt{6}}{2}t}\langle\overline{\xi}_0, 1_{[0,1]}\rangle_X 1_{[0,1]}(z) + 2\sum_{n=1}^{\infty} e^{(-n^2\pi^2+\frac{\sqrt{2}}{2})t}\langle\overline{\xi}_0, \cos(n\pi z)\rangle_X \cos(n\pi z),\end{aligned}$$

for $\overline{\xi}_0 \in Y, t \geq 0$ and $z \in [0,1]$. A sufficient condition for exponential stability of $(T(t))_{t\geq 0}$ on Y is the convergence of the following integral

$$\int_0^{\infty} \|T(t)\overline{\xi}_0\|_Y^p dt,$$

for some $p \in [1,\infty)$ and every $\overline{\xi}_0 \in Y$, see Remark 2.1.1. Let us consider $\overline{\xi}_0 \in Y$ and $p = 1$. An estimation of $\|T(t)\overline{\xi}_0\|_Y^p$ yields that

$$\|T(t)\overline{\xi}_0\|_Y \leq e^{\frac{-\sqrt{6}}{2}t}|\langle\overline{\xi}_0, 1_{[0,1]}\rangle_X| + 2\sum_{n=1}^{\infty} e^{(-n^2\pi^2+\frac{\sqrt{2}}{2})t}|\langle\overline{\xi}_0, \cos(n\pi z)\rangle_X|.$$

Integrating both sides of the previous inequality yields that

$$\begin{aligned}\int_0^{\infty}\|T(t)\overline{\xi}_0\|_Y dt &\leq \frac{2}{\sqrt{6}}|\langle\overline{\xi}_0, 1_{[0,1]}\rangle_X + 2\sum_{n=1}^{\infty}\frac{|\langle\overline{\xi}_0, \cos(n\pi z)\rangle_X|}{n^2\pi^2 - \frac{\sqrt{2}}{2}}\\&\leq \frac{2}{\sqrt{6}}\|\overline{\xi}_0\|_X + \sqrt{2}\|\overline{\xi}_0\|_X\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2 - \frac{\sqrt{2}}{2}},\end{aligned}$$

which is a convergent series. Consequently, thanks to Theorem 4.3.1, the steady-state 0 is locally exponentially stable for system (4.4.1). Only the diffusion operator as linear part would not have been sufficient to have stability of the nonlinear system. Just itself with Neumann boundary condition is unstable since 0 is in its spectrum. The stability comes from the integral term $-\frac{\sqrt{2}+\sqrt{6}}{2}1_{[0,1]}(z)\int_0^1 \xi(z)dz$ which may be interpreted as the optimal control solution of the optimal control problem (2.1.26) for the linearized system (4.4.4), see Chapter 2, Section 2.1.3.

4.4.2 Nonlinear bistability of the equilibria for the plug-flow tubular reactor with axial dispersion

Here we shall apply the results presented in Section 4.3 to determine the nonlinear stability of the equilibria of (2.3.3) in the adiabatic case, i.e. $\gamma = 0$. This will be based on the results that have been obtained in Chapter 3 for the stability of the linearized version of (2.3.3). We shall focus on the case where the two Peclet numbers are equal, i.e. $Pe_h = Pe_m := v/D$ where v and D stand for the superficial velocity of the fluid and the diffusion coefficient, respectively. We start by performing the same change of variables as in (3.2.15), i.e. $\xi_1(z,t) = x_1(z,t) - x_1^e(z), \xi_2(z,t) = x_2(z,t) - x_2^e(z)$ together with $\hat{\xi}_1(z,t) = e^{-\frac{Pe}{2}z}\xi_1(z,t), \hat{\xi}_2(z,t) = e^{-\frac{Pe}{2}z}\xi_2(z,t)$ where (x_1^e, x_2^e) denotes an equilibrium pair of (2.3.3), i.e. a solution of (3.2.1). This entails that the nonlinear PDEs (2.3.3) take the following form

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_1 + \delta e^{\frac{-Pe}{2}z}\hat{f}_1(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e), \\ \frac{\partial \hat{\xi}_2}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \hat{\xi}_2}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_2 + e^{\frac{-Pe}{2}z}\hat{f}(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e), \\ \frac{\partial \hat{\xi}_1}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_1(0,t), \frac{\partial \hat{\xi}_2}{\partial z}(0,t) = \frac{Pe}{2}\hat{\xi}_2(0,t), \\ \frac{\partial \hat{\xi}_1}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_1(1,t), \frac{\partial \hat{\xi}_2}{\partial z}(1,t) = -\frac{Pe}{2}\hat{\xi}_2(1,t), \end{cases} \tag{4.4.5}$$

where $\hat{f}(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e) = f_1(e^{\frac{Pe}{2}z}\hat{\xi}_1 + x_1^e, e^{\frac{Pe}{2}z}\hat{\xi}_2 + x_2^e) - f_1(x_1^e, x_2^e)$ with $f_1(x,y) = \tilde{\alpha}(1-y)e^{\frac{-\mu}{1+x}}$ for $(x,y) \in D(f)$ given in (2.3.9) and $f_1(-1,y) = 0$. It can be easily seen that the dynamics of the variable $\chi(z,t) := \hat{\xi}_1(z,t) - \delta\hat{\xi}_2(z,t)$ is driven by the following PDE

$$\begin{cases} \frac{\partial \chi}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \chi}{\partial z^2} - \frac{Pe}{4}\chi, \\ \frac{\partial \chi}{\partial z}(0,t) = \frac{Pe}{2}\chi(0,t), \frac{\partial \chi}{\partial z}(1,t) = -\frac{Pe}{2}\chi(1,t). \end{cases}$$

Thanks to Proposition 3.2.6 one may observe that χ tends to 0 exponentially fast when t goes to ∞, which allows us to focus on the following PDE in order to study nonlinear stability of the equilibria of (2.3.3)

$$\begin{cases} \frac{\partial \xi}{\partial t} = \frac{1}{Pe}\frac{\partial^2 \xi}{\partial z^2} - \frac{Pe}{4}\xi + e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)], \\ \frac{\partial \xi}{\partial z}(0) = \frac{Pe}{2}\xi(0), \frac{\partial \xi}{\partial z}(1) = -\frac{Pe}{2}\xi(1), \end{cases} \tag{4.4.6}$$

where the notation $\xi := \hat{\xi}_1, x^e := x_1^e$ and $\delta g(x, \frac{1}{\delta}x) = \tilde{\alpha}(\delta - x)e^{\frac{-\mu}{1+x}} =: \tilde{g}(x)$ has been used. We shall choose X as the state space $L^2([0,1];\mathbb{R})$. Let us define the linear operator A by $A\xi = \frac{1}{Pe}\frac{d^2\xi}{dz^2} - \frac{Pe}{4}\xi$ on the domain

$$D(A) = \{\xi \in H^2([0,1];\mathbb{R}), \frac{d\xi}{dz}(0) = \frac{Pe}{2}\xi(0), \frac{d\xi}{dz}(1) = -\frac{Pe}{2}\xi(1)\}, \tag{4.4.7}$$

while the nonlinear operator $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)]$ is defined on the closed and convex subset $\mathscr{D}$ built from $D(f)$, see (2.3.9), and expressed as

$$\left\{\xi \in X | \xi \geq e^{-\frac{Pe}{2}z}(-1 - x^e), -e^{-\frac{Pe}{2}z}x^e \leq \xi - \chi \leq e^{-\frac{Pe}{2}z}(\delta - x^e), \text{ a.e. on } [0,1]\right\}. \tag{4.4.8}$$

Note that the operator $A + e^{-\frac{Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z} \cdot + x^e) - \tilde{g}(x^e)]$ is the infinitesimal generator of a nonlinear C_0-semigroup on $D(A) \cap \mathscr{D}$ denoted by $(S(t))_{t\geq 0}$ and that satisfies $(S(t)\xi_0)(z) = \xi(z,t)$, where ξ_0 denotes the initial condition given to (4.4.6), see e.g. Laabissi et al. (2001). In Aksikas et al. (2007), it is shown that the nonlinear operator $\tilde{f}(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e)$ is Lipschitz continuous in the variables $\hat{\xi}_1$ and $\hat{\xi}_2$ for

$(e^{\frac{Pe_h}{2}z}\hat{\xi}_1(t,z)+x_1^e \quad e^{\frac{Pe_m}{2}z}\hat{\xi}_2(t,z)+x_2^e) \in D(f)$. This is equivalent to the fact that the operator $j(\xi) := e^{-\frac{Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)]$ is Lipschitz continuous when considering $\xi \in \mathscr{D}$. In Chapter 3, the operator $j(\xi)$ has been shown Gâteaux differentiable at 0 with a Gâteaux derivative given by the linear and bounded operator $d\tilde{g}(x^e) : X \to X$ defined as $d\tilde{g}(x^e)\xi = (\tilde{\alpha}\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}} - \tilde{\alpha}e^{\frac{\mu}{1+x^e}})\xi$ for $\xi \in X$. The resulting linear system, given by (3.2.18), has also been proven well-posed and its exponential stability has been characterized in Proposition 3.2.7.

According to Theorem 4.1.2 it should be natural to ask now whether the semigroup generated by the operator $A\cdot+j(\cdot)$ is Fréchet differentiable at the origin on X or not, which could be studied by looking at the Fréchet differentiability of the nonlinear part of the dynamics at 0, i.e. the operator $j(\xi)$. Let us consider therefore an adaptation of the nonlinear $j(\xi)$ given by

$$F : (V := \{\xi \in X, 0 \leq \xi \leq 1\}) \to X, F(\xi) = (1-\xi)e^{\frac{-1}{1+\xi}} - e^{-1} \tag{4.4.9}$$

for $\xi \in V$. This nonlinear operator could be viewed as a simplification of $j(\xi)$ with $Pe = 0$, all other constants equal to 1 and the equilibrium x^e being set to the null function for the ease of calculation. In the following counter-example, it is shown that F is not Fréchet differentiable at 0 on X.

Proposition 4.4.1 *The nonlinear operator F given by (4.4.9) is not Fréchet differentiable at 0 on X.*

Proof. Suppose, for the sake of a contradiction, that F is Fréchet differentiable at 0. Since it is also Gâteaux differentiable at 0 the corresponding derivatives are equal. Remark that the Gâteaux derivative of F at x^e in the direction $h \in L^2(0,1)$ is given by $dF(x^e)h = \left(-e^{\frac{-1}{1+x^e}} + \frac{1-x^e}{(1+x^e)^2}e^{\frac{-1}{1+x^e}}\right)h$. By looking at that derivative for $x^e \equiv 0$, one gets that $dF(0) \equiv 0$. It follows that the corresponding Fréchet derivative $DF(0)$ would be the null operator on $L^2(0,1)$. Because of the Fréchet differentiability of F at 0, the relation

$$\lim_{\|h\|_X \to 0} \frac{\|F(0+h) - F(0) - DF(0)h\|_X}{\|h\|_X} = 0$$

holds for every $h \in V$, i.e.

$$\lim_{\|h\|_X \to 0} \frac{\|(1-h)e^{\frac{-1}{1+h}} - e^{-1}1_{[0,1]}\|_X}{\|h\|_X} = 0, \tag{4.4.10}$$

where $1_{[0,1]}$ denotes the characteristic function of the interval $[0,1]$. Let us consider the sequence of $X-$functions $\{h_n\}_{n\in\mathbb{N}}$ defined by $h_n(z) = \frac{1}{n}1_{[0,1-\frac{1}{n}]}(z) + 1_{[1-\frac{1}{n},1]}(z)$, for $n \in \mathbb{N}$. Remark that $h_n \in V$ for each $n \in \mathbb{N}$. Moreover,

$$\|h_n\|_X^2 = \int_0^{1-\frac{1}{n}} \frac{1}{n^2}dz + \int_{1-\frac{1}{n}}^1 dz = \frac{n^2+n-1}{n^3}. \tag{4.4.11}$$

It is obvious that $\lim_{n\to+\infty} \|h_n\|_X^2 = 0$. Hence (4.4.10) implies that

$$\lim_{n\to+\infty} \frac{\|(1-h_n)e^{\frac{-1}{1+h_n}} - e^{-1}1_{[0,1]}\|_X}{\|h_n\|_X} = 0. \tag{4.4.12}$$

Let us compute $\|(1-h_n)e^{\frac{-1}{1+h_n}} - e^{-1}1_{[0,1]}\|_X^2 =: S_n$. It holds

$$\begin{aligned} S_n &= \int_0^1 \left(e^{\frac{-1}{1+h_n(z)}} - e^{-1}1_{[0,1]}(z) - e^{\frac{-1}{1+h_n(z)}} h_n(z) \right)^2 dz \\ &= \int_0^{1-\frac{1}{n}} \left(e^{\frac{-1}{1+\frac{1}{n}}} - e^{-1} - e^{\frac{-1}{1+\frac{1}{n}}} \frac{1}{n} \right)^2 dz + \int_{1-\frac{1}{n}}^1 \left(e^{\frac{-1}{2}} - e^{-1} - e^{\frac{-1}{2}} \right)^2 dz \\ &= e^{-2} + e^{\frac{-2n}{1+n}} \frac{(n-1)^3}{n^3} - \frac{2e^{-2+\frac{1}{1+n}}(n-1)^2}{n^2}. \end{aligned} \tag{4.4.13}$$

Combining (4.4.13) with (4.4.11) yields

$$\frac{\|(1-h_n)e^{\frac{-1}{1+h_n}} - e^{-1}1_{[0,1]}\|_X^2}{\|h_n\|_X^2} = \frac{n^3e^{-2} + (n-1)^3 e^{\frac{-2n}{1+n}} - 2n(n-1)^2 e^{-2+\frac{1}{1+n}}}{n^2+n-1}.$$

It follows that $\lim_{n\to+\infty} \frac{\|(1-h_n)e^{\frac{-1}{1+h_n}} - e^{-1}1_{[0,1]}\|_X}{\|h_n\|_X} = e^{-1}$, which contradicts (4.4.12). $\square$

As it has been of interest in Section 4.4.1, we shall consider an auxiliary space Y in order to overcome the difficulties encountered on X with the classical definition of Fréchet differentiability. The latter is fixed as being $Y := C([0,1];\mathbb{R})$ equipped with the supremum norm $\|f\|_\infty$ defined as $\sup_{z\in[0,1]} |f(z)|$ for any $f \in Y$. In order to apply the new framework developed in Sections 4.2 and 4.3, we shall check that Assumptions 4.2.1, 4.2.3, 4.2.2 and 4.3.1 are satisfied. First observe that the nonlinear operator $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\cdot + x^e) - \tilde{g}(x^e)]$ is Gâteaux differentiable at 0 on X and that the Gâteaux derivative is a bounded linear operator, see Lemma 3.2.1. This implies that Assumption 4.2.1 is satisfied.

Same arguments as the ones used in Lemma 3.2.1 may be used to show that the nonlinear operator $j(\cdot)$ is Gâteaux differentiable at 0 on Y. Moreover, the Gâteaux linearization yields the same Gâteaux derivative as the one found on X, i.e. (3.2.6). According to Drame et al. (2008), the linear operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on the space Y. This together with the boundedness of the Gâteaux derivative viewed as an operator defined on Y has the consequence that the linearized system corresponding to (4.4.6) is well-posed on Y.

By (Deimling, 1985, Chapter 6, Example 20.4), the PDE (4.4.6) admits a mild solution $\xi(z,t) := (S(t)\xi_0)(z)$ on Y, for $t \geq 0$, for every $\xi_0 \in Y$, which is a classical solution provided that $\xi_0 \in D(A) \cap \mathscr{D}$, wherein X is replaced by Y. That is,

$$S(t)\xi_0 = T(t)\xi_0 + \int_0^t T(t-s)(e^{\frac{-Pe}{2}z}[\tilde{g}(\eta(S(s)\xi_0)) - \tilde{g}(x^e)])ds \tag{4.4.14}$$

for every $\xi_0 \in D(A) \cap \mathscr{D}$, where $(T(t))_{t\geq 0}$ is the C_0-semigroup whose operator A is the infinitesimal generator and where the shortcut of notation $\eta(\cdot) := e^{\frac{Pe}{2}z} \cdot + x^e$ has been used.

Note that the functions $S(\cdot)\xi_0$ and $S(\cdot)\tilde{\xi}_0 : [0,t_0) \to D(A) \cap \mathscr{D}$ are continuous and satisfy identity (4.4.14) with initial states ξ_0 and $\tilde{\xi}_0$, respectively, where $t_0 > 0$. Hence, by (Martin, 1987, Lemma 5.1.),

$$\|S(t)\xi_0 - S(t)\tilde{\xi}_0\|_Y \leq \gamma_t \|\xi_0 - \tilde{\xi}_0\|_Y, \tag{4.4.15}$$

for all $t \in [0,t_0)$, for some positive γ_t that is increasing in t and which is bounded on $[0,t_0)$. Inequality (4.4.15) is commonly called the continuous dependence property of the well-posed system (4.4.6) in the Banach space $Y = C([0,1];\mathbb{R})$. In particular for $\tilde{\xi}_0 = 0$,

$$\|S(t)\xi_0\|_Y \leq \gamma_t \|\xi_0\|_Y, \tag{4.4.16}$$

for all $t \in [0,t_0)$, since $S(t)$ maps the origin to itself, see (Augner, 2019, Remark 2.12). This shows that the semigroup $(S(t))_{t\geq 0}$ depends continuously on the initial condition at 0 on the space Y. Looking at (4.4.16), one observes that

$$\|S(\cdot)\xi_0\|_{L^\infty([0,t_0];Y)} := \sup_{t\in[0,t_0]} \|S(t)\xi_0\|_Y \leq \gamma_{t_0} \|\xi_0\|_Y < \infty, \tag{4.4.17}$$

for all $\xi_0 \in D(A) \cap \mathscr{D}$, where $\gamma_{t_0} = \sup_{t\in[0,t_0]} \gamma_t < \infty$. Let us look at a similar property on X.

Consider the same $t_0 > 0$ as before and $0 \leq t \leq t_0$. Define $V(\xi(t)) = \frac{1}{2}\int_0^1 \xi^2(t)dz$ where ξ denotes the state trajectory of system (4.4.6). Differentiating V w.r.t. t along the state trajectory (4.4.6) yields

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_X^2 \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|_X^2 + \int_0^1 e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi(t))) - \tilde{g}(x^e))\xi(t)dz, \tag{4.4.18}$$

where (3.2.26) has been used. The Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_X^2 \\
&\leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|_X^2 + \|e^{\frac{-Pe}{2}z}(\tilde{g}(\eta(\xi(t))) - \tilde{g}(x^e))\|_X \cdot \|\xi(t)\|_X.
\end{aligned}$$

Since the operator $\tilde{g}$ is Lipschitz continuous, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_X^2 \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|_X^2 + l_{\tilde{g}}\|e^{\frac{Pe}{2}z}\xi(t)\|_X \cdot \|\xi(t)\|_X,$$

where $l_{\tilde{g}} > 0$ denotes a Lipschitz constant associated to $\tilde{g}$. It follows that

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_X^2 \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4} + l_{\tilde{g}}e^{\frac{Pe}{2}}\right)\|\xi(t)\|_X^2.$$

By using Gronwall's Lemma and by denoting the constant $|\frac{-\pi^2}{\pi^2+4Pe}-\frac{Pe}{4}+l_{\tilde{g}}e^{\frac{Pe}{2}}|$ by k, one gets the inequality:

$$\|\xi(t)\|_X \leq e^{kt}\|\xi_0\|_X, \tag{4.4.19}$$

for all $t \in [0,t_0]$, which leads to

$$\|\xi\|_{L^\infty([0,t_0];X)} \leq \sup_{t\in[0,t_0]} e^{kt}\|\xi_0\|_X = e^{kt_0}\|\xi_0\|_X.$$

Hence, continuous dependence is proved on X and the time boundedness of the state trajectories is achieved both on X and on Y thanks to the estimates (4.4.19) and (4.4.17), respectively. In order to conclude on the $(Y,X)-$Fréchet differentiability of the nonlinear C_0-semigroup $(S(t))_{t\geq 0}$, the $(Y,X)-$Fréchet differentiability of the nonlinear operator $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\cdot+x^e)-\tilde{g}(x^e)]$ on 0 has to be studied. Therefore the following lemma characterizes the maximum amplitude of the function η on the domain $\mathscr{D}$ wherein the dynamics of the asymptotic reaction invariant χ is considered.

Lemma 4.4.1 *Let us consider the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, which has been introduced in Proposition 3.2.6. Assume that there exists a positive and sufficiently large constant κ such that the condition*

$$\sum_{n=1}^{+\infty}|\psi_n| \leq \frac{1}{\sqrt{2}}e^{-\frac{Pe}{2}}\left(1-\frac{1}{\kappa}\right) \tag{4.4.20}$$

holds. Then

$$\frac{1}{\kappa}-1 \leq e^{\frac{Pe}{2}z}\xi + x^e(z) \leq \delta + 1 - \frac{1}{\kappa},$$

for all $t\geq 0$, a.e. $z\in[0,1]$ and all $\xi\in\mathscr{D}$.

Proof. From Proposition 3.2.6, let us start by recalling that the asymptotic reaction invariant $\chi(z,t)$ satisfies

$$\chi(z,t) = \sum_{n=1}^{\infty}\psi_n\phi_n(z)e^{-(\beta_n^2+\frac{Pe}{4})t},$$

where the quantities ψ_n, ϕ_n and β_n are given in Proposition 3.2.6. Observe that for $n\in\mathbb{N}$, there holds

$$\begin{aligned}
|\phi_n(z)| &= K_n|\beta_n\sqrt{Pe}\cos(\beta_n\sqrt{Pe}z)+\frac{Pe}{2}\sin(\beta_n\sqrt{Pe}z)| \\
&= K_n\left|\left\langle\begin{pmatrix}\beta_n\sqrt{Pe}\\ \frac{Pe}{2}\end{pmatrix},\begin{pmatrix}\cos(\beta_n\sqrt{Pe}z)\\ \sin(\beta_n\sqrt{Pe}z)\end{pmatrix}\right\rangle_{\mathbb{R}^2}\right| \\
&\leq \left[\frac{2}{\beta_n^2Pe+Pe+\frac{Pe^2}{4}}\right]^{\frac{1}{2}}\left(\beta_n^2Pe+\frac{Pe^2}{4}\right)^{\frac{1}{2}} \leq \sqrt{2},
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used and where $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ denotes the standard inner product of $\mathbb{R}^2$. It follows that

$$|\chi(z,t)| \leq \sum_{n=1}^{+\infty} |\psi_n \phi_n(z)| e^{-(\beta_n^2+\frac{Pe}{4})t} \leq \sqrt{2} \sum_{n=1}^{+\infty} |\psi_n| \leq e^{-\frac{Pe}{2}} \left(1-\frac{1}{\kappa}\right). \tag{4.4.21}$$

Now take $\xi \in \mathscr{D}$. By definition of $\mathscr{D}, \xi$ satisfies the estimate

$$-e^{-\frac{Pe}{2}z} x^e(z) \leq \xi - \chi(z,t) \leq e^{-\frac{Pe}{2}z}(\delta - x^e(z)),$$

for a.e. $z \in [0,1]$ and every $t \geq 0$. Equivalently, there holds

$$e^{\frac{Pe}{2}z}\chi(z,t) \leq e^{\frac{Pe}{2}z}\xi + x^e \leq \delta - x^e + e^{\frac{Pe}{2}z}\chi(z,t).$$

By using the relation (4.4.21) wherein it is considered that κ is such that $1-\frac{1}{\kappa} > 0$, one gets that

$$e^{\frac{Pe}{2}z} e^{-\frac{Pe}{2}} \left(\frac{1}{\kappa}-1\right) \leq e^{\frac{Pe}{2}z}\xi + x^e \leq \delta - x^e + e^{\frac{Pe}{2}z} e^{-\frac{Pe}{2}} \left(1-\frac{1}{\kappa}\right).$$

The relation $0 \leq x^e \leq \delta$ due to the physical constraints (2.3.9) enables us to conclude that

$$\frac{1}{\kappa} - 1 \leq e^{\frac{Pe}{2}z}\xi + x^e(z) \leq \delta + 1 - \frac{1}{\kappa}.$$

□

Observe that, despite that the asymptotic reaction invariant $\chi(z,t)$ depends on the space variable z and on the time variable t, the bounds found on $\eta(\xi)$ are independent of these two variables.

Remark 4.4.1 *The assumption that is needed in Lemma 4.4.1 means that the sequence of the projections of the initial condition of the asymptotic reaction invariant χ, namely $\xi_1 - \delta\xi_2$, on the basis of eigenfunctions of the operator A has to be summable, i.e. $\{\psi_n\}_{n\in\mathbb{N}} \subset l^1(\mathbb{N})$ where $l^1(\mathbb{N})$ denotes the space of absolutely summable sequences equipped with the norm $\|x\|_{l^1} := \sum_{n=1}^{\infty} |x_n|$ with $x = (x_1, x_2, \dots) \in l^1(\mathbb{N})$. Moreover, its $l^1(\mathbb{N})-$norm can not exceed the constant $\frac{1}{\sqrt{2}} e^{-\frac{Pe}{2}}(1-\frac{1}{\kappa})$.*

As an illustration, let us take as initial conditions the functions

$$\xi_{i,0}(z) := \omega_i \left(\sin(\pi z) + \frac{2\pi}{Pe}\right), i = 1,2,$$

where ω_i are constant numbers that have to be chosen in such a way that condition (4.4.20) is satisfied. Note that the subscript i denotes the initial condition given to the $i-$th variable. It is straightforward to see that this kind of initial conditions is in $D(A)$. Here after, we first show that the series $\sum_{n=1}^{+\infty} |\psi_n|$ is convergent with the considered initial conditions. In a second time we shall give the value above which the constant κ has to be chosen such that condition (4.4.20) holds. Since the series is convergent it is always possible to adapt the constants $\omega_i, i = 1,2$ and then to choose κ such that (4.4.20) holds. With the choice of $\xi_{i,0}$, the coefficient $|\psi_n|$ satisfies

$$|\psi_n| \leq |\omega_1 - \delta\omega_2| \cdot \left(\left| \int_0^1 \sin(\pi z)\phi_n(z)dz \right| + \frac{2\pi}{Pe} \left| \int_0^1 \phi_n(z)dz \right| \right).$$

Computing $|\int_0^1 \phi_n(z)dz|$ yields

$$\left| \int_0^1 \phi_n(z)dz \right| = \left| K_n \left[\sin(\beta_n\sqrt{Pe}) - \frac{Pe}{2\beta_n\sqrt{Pe}} \cos(\beta_n\sqrt{Pe}) + \frac{Pe}{2\beta_n\sqrt{Pe}} \right] \right|.$$

By (Dehaye, 2015, Section 4.3.2), the approximation $\beta_n \approx \pi(n-1)/\sqrt{Pe}$ holds for n large. Moreover by considering Pe small and by plugging the asymptotic form of K_n in the previous equality, it implies that

$$\left| \int_0^1 \phi_n(z)dz \right| \approx \left| \frac{\sqrt{2}}{\pi(n-1)} \left(\frac{Pe}{2\pi(n-1)} (1-(-1)^{n+1}) \right) \right| \leq \frac{\sqrt{2}Pe}{\pi^2(n-1)^2}.$$

Moreover, one has

$$\begin{aligned}
\left| \int_0^1 \sin(\pi z)\phi_n(z)dz \right| = \Bigg| & \frac{K_n\beta_n\sqrt{Pe}}{2} \int_0^1 \sin((\pi + \beta_n\sqrt{Pe})z)dz \\
& + \frac{K_n\beta_n\sqrt{Pe}}{2} \int_0^1 \sin((\pi - \beta_n\sqrt{Pe})z)dz + \frac{K_nPe}{4} \int_0^1 \cos((\pi - \beta_n\sqrt{Pe})z)dz \\
& - \frac{K_nPe}{4} \int_0^1 \cos((\pi + \beta_n\sqrt{Pe})z)dz \Bigg|.
\end{aligned}$$

Considering n large enough yields that $K_n \approx \frac{\sqrt{2}}{\pi(n-1)}$, which results in

$$\begin{aligned}
\left| \int_0^1 \sin(\pi z)\phi_n(z)dz \right| \approx & \Bigg| \frac{\sqrt{2}}{2} \int_0^1 \sin(\pi n z)dz + \frac{\sqrt{2}}{2} \int_0^1 \sin((2-n)\pi z)dz \\
& + \frac{\sqrt{2}Pe}{4\pi(n-1)} \int_0^1 \cos((2-n)\pi z)dz - \frac{\sqrt{2}Pe}{4\pi(n-1)} \int_0^1 \cos(n\pi z)dz \Bigg| \\
= & \sqrt{2} \left| \frac{((-1)^n - 1)}{\pi n^2 - 2\pi n} \right| \leq \frac{2\sqrt{2}}{|\pi n^2 - 2\pi n|} = \frac{2\sqrt{2}}{\pi n^2 - 2\pi n}.
\end{aligned}$$

The last equality holds since n is sufficiently large[(6)]. Combining the previous computations gives that

$$\left| \int_0^1 f(z)\phi_n(z)dz \right| = \mathcal{O} \left(|\omega_1 - \delta\omega_2| \cdot \left(\frac{2\sqrt{2}}{\pi n^2 - 2\pi n} + \frac{2\pi}{Pe} \frac{\sqrt{2}Pe}{\pi^2(n-1)^2} \right) \right)$$

[(6)] Without loss of generality, since n is large, one can assume that $n > 2$ which entails that $\pi n^2 - 2\pi n > 0$.

$$= \mathscr{O}\left(|\omega_1 - \delta\omega_2|\frac{4\sqrt{2}}{\pi n^2 - 2\pi n}\right),$$

which is the general term of a convergent series. This proves that $\{\psi_n\}_{n\in\mathbb{N}_0}$ is $l_1(\mathbb{N})-$ summable. In other words, there exists a positive and finite real number Ω such that

$$\Omega := \sum_{n=1}^{\infty}\left|\int_0^1\left(\sin(\pi z) + \frac{2\pi}{Pe}\right)\phi_n(z)dz\right|.$$

Let us adopt the notation $\omega^* := |\omega_1 - \delta\omega_2|$. The inequality (4.4.20) becomes

$$\omega^*\Omega \leq \frac{1}{\sqrt{2}}e^{-\frac{Pe}{2}}(1 - \frac{1}{\kappa}).$$

This can be written equivalently as $\frac{1}{\kappa} \leq 1 - \sqrt{2}e^{\frac{Pe}{2}}\omega^*\Omega$, whose right hand-side is supposed to be positive[(7)]. Hence, the constant κ has to be taken such that

$$\kappa \geq \frac{1}{1 - \sqrt{2}e^{\frac{Pe}{2}}\omega^*\Omega}. \tag{4.4.22}$$

Since Ω is a finite positive real number and ω^* is a degree of freedom in the analysis, there will always exist κ such that (4.4.22) is satisfied, i.e. such that assumption (4.4.20) of Lemma 4.4.1 holds. The procedure is as follows: first the constant ω^* has to be chosen such that $1 - \sqrt{2}e^{\frac{Pe}{2}}\Omega$ is positive and secondly the constant κ is adjusted in such a way that (4.4.22) holds true.

Before going further, let us recall that the reaction invariant, χ, is subject to the following PDE

$$\begin{cases} \frac{\partial\chi}{\partial t} = \frac{1}{Pe}\frac{\partial^2\chi}{\partial z^2} - \frac{Pe}{4}\chi, \\ \frac{\partial\chi}{\partial z}(0,t) = \frac{Pe}{2}\chi(0,t),\ \frac{\partial\chi}{\partial z}(1,t) = -\frac{Pe}{2}\chi(1,t), \end{cases}$$

whose operator dynamics $A_c\chi = \frac{1}{Pe}\frac{d^2\chi}{dz^2} - \frac{Pe}{4}\chi$ is a Riesz-spectral operator, see Proposition 3.2.6. According to Theorem 2.1.4, the domain of the operator A_c may be written as $D(A_c) = \left\{\chi \in X, \sum_{n=1}^{\infty}\lambda_n^2\langle\chi,\phi_n\rangle_X^2 < \infty\right\}$, where $\{\lambda_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 1}$ represent the eigenvalues and the eigenfunctions of the operator A_c, see Proposition 3.2.6 for their expressions. Moreover, there holds

$$\begin{aligned} \sum_{n=1}^{\infty}|\psi_n| = \sum_{n=1}^{\infty}|\langle\chi_0,\phi_n\rangle_X| &= \sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n}\right|\cdot|\lambda_n\langle\chi_0,\phi_n\rangle_X| \\ &\leq \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\sum_{n=1}^{\infty}\lambda_n^2\langle\chi_0,\phi_n\rangle_X^2, \end{aligned}$$

where the initial condition is chosen in $D(A_c)$ and where the Cauchy-Schwarz inequality has been used. On one hand, observe that the series $\sum_{n=1}^{\infty}\lambda_n^2\langle\chi_0,\phi_n\rangle_X^2$ is convergent,

[(7)]This is always possible to render this quantity positive because ω^* can be chosen arbitrarily.

according to the definition of $D(A_c)$. On the other hand, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is convergent too, thanks to the relation $\lambda_n^2 \approx \mathscr{O}(n^4)$. This means that the series $\sum_{n=1}^{\infty} |\psi_n|$ is convergent for any initial condition χ_0 chosen in the domain of the operator A_c.

Let us now consider the following corollary that gives estimations on some functions of elements of the invariant subspace $\mathscr{D}$ when considering the asymptotic reaction invariant χ. This is a direct consequence of Lemma 4.4.1.

Corollary 4.4.2 *The following inequalities hold:*

$$\left\| \alpha(\delta - \eta(\xi)) e^{\frac{-\mu}{1+\eta(\xi)}} \right\|_Y \leq \alpha \left(\delta + 1 - \frac{1}{\kappa} \right) \tag{4.4.23}$$

and

$$\left\| \frac{1}{(1+\eta(\xi))(1+x^e)} \right\|_Y \leq \kappa \tag{4.4.24}$$

for all $t \geq 0$ and any $\xi \in \mathscr{D}$, where the function η is defined as $\eta(\xi) := e^{\frac{Pe}{2}z}\xi + x^e$.

Proof. Let us pick any $\xi \in \mathscr{D}$. By Lemma 4.4.1, it holds that

$$\frac{1}{\kappa} \leq 1 + e^{\frac{Pe}{2}z}\xi + x^e(z) \leq \delta + 2 - \frac{1}{\kappa},$$

for all $t \geq 0$ and a.e. $z \in [0,1]$. Moreover the equilibria x_1^e and x_2^e satisfy the reaction invariant $x_1^e - \delta x_2^e = 0$ which entails that $1 \leq 1 + x_1^e(z) \leq 1 + \delta$ holds for a.e. $z \in [0,1]$, see (2.3.9). These inequalities imply (4.4.24). Moreover Lemma 4.4.1 implies that

$$\frac{1}{\kappa} - 1 \leq \delta - e^{\frac{Pe}{2}z}\xi + x^e(z) \leq \delta + 1 - \frac{1}{\kappa}.$$

This completes the proof of (4.4.23) since $1 + e^{\frac{Pe}{2}z}\xi + x^e(z) \geq \frac{1}{\kappa} > 0$. □

Lemma 4.4.1 together with Corollary 4.4.2 enables us to state the following proposition.

Proposition 4.4.3 *The nonlinear operator $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\cdot + x^e) - \tilde{g}(x^e)]$ is $(Y,X)-$Fréchet differentiable at 0. Its Fréchet derivative is given by the linear and bounded operator $d\tilde{g}(x^e)$ defined as $d\tilde{g}(x^e)\xi = (\tilde{\alpha}\frac{\mu(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} - \tilde{\alpha} e^{\frac{-\mu}{1+x^e}})\xi$ for $\xi \in X$.*

Proof. Let us consider $\xi \in D(A) \cap \mathscr{D}$. Using the definitions of $\tilde{g}$ and of its Gâteaux derivative gives the following:

$$\begin{aligned} &\left\| e^{-\frac{Pe}{2}z}[\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e) e^{\frac{Pe}{2}z}\xi] \right\|_X := \Bigg\| e^{-\frac{Pe}{2}z}(\tilde{\alpha}(\delta - \eta(\xi)) e^{\frac{-\mu}{1+\eta(\xi)}} \\ &- \tilde{\alpha}(\delta - x^e) e^{\frac{-\mu}{1+x^e}}) - \left(\tilde{\alpha} \frac{\mu(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} \xi - \tilde{\alpha} e^{\frac{-\mu}{1+x^e}} \xi \right) \Bigg\|_X \end{aligned}$$

$$= \tilde{\alpha} \left\| e^{-\frac{Pe}{2}z}(\delta - x^e)\left[e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}}\right] - \frac{\mu(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} \xi - \left[e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}}\right]\xi \right\|_X .$$

Thanks to the series decomposition of the exponential function, observe that

$$\begin{aligned}
e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}} &= e^{\frac{-\mu}{1+x^e}} \left(e^{\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}} - 1 \right) \\
&= e^{\frac{-\mu}{1+x^e}} \left(\sum_{n=0}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} - 1 \right) \\
&= e^{\frac{-\mu}{1+x^e}} \left(\sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} + \frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right).
\end{aligned}$$

Consequently, the relations

$$\begin{aligned}
&\left\| e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi) \right\|_X = \\
&\tilde{\alpha} \left\| e^{-\frac{Pe}{2}z}(\delta - x^e) e^{\frac{-\mu}{1+x^e}} \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} + \frac{e^{-\frac{Pe}{2}z}(\delta - x^e) e^{\frac{-\mu}{1+x^e}} \mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right. \\
&\left. - \frac{\mu(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} \xi - e^{\frac{-\mu}{1+x^e}} \sum_{n=1}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!}\xi \right\|_X \\
&\leq \tilde{\alpha} \left\| e^{\frac{-\mu}{1+x^e}} \right\|_Y \cdot \left\| e^{-\frac{Pe}{2}z}(\delta - x^e) \frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \frac{1}{n!} \right. \\
&+ \mu(\delta - x^e)\xi \left[\frac{1}{(1+\eta(\xi))(1+x^e)} - \frac{1}{(1+x^e)^2} \right] \\
&\left. - \sum_{n=1}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!}\xi \right\|_X
\end{aligned}$$

hold. In that way, by using the triangular inequality and the fact that $\|e^{\frac{-\mu}{1+x^e}}\|_Y \leq 1$, one obtains that

$$\begin{aligned}
&\left\| e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi) \right\|_X \\
&\leq \tilde{\alpha} \left\| \mu(\delta - x^e) \frac{1}{(1+\eta(\xi))(1+x^e)} \xi \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \frac{1}{n!} \right\|_X \\
&+ \tilde{\alpha}\mu \|\delta - x^e\|_Y \cdot \left\| \frac{e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)^2} \xi \right\|_X + \alpha \left\| \sum_{n=1}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!}\xi \right\|_X .
\end{aligned}$$

Integration in Lebesgue spaces[8] and Corollary 4.4.2 imply the following:

$$
\begin{aligned}
&\left\|\mu(\delta-x^e)\frac{1}{(1+\eta(\xi))(1+x^e)}\xi\sum_{n=2}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^{n-1}\frac{1}{n!}\right\|_X \\
&\leq \mu\delta\kappa\sum_{n=2}^{+\infty}\frac{1}{n!}\left\|\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^{n-1}\xi\right\|_X \\
&\leq \mu\delta\kappa\sum_{n=2}^{+\infty}\frac{\mu^{n-1}(e^{\frac{Pe}{2}})^{n-1}\kappa^{n-1}}{n!}\left\|\xi^{n-1}\xi\right\|_X \leq \mu\delta\kappa\sum_{n=2}^{+\infty}\frac{(\mu e^{\frac{Pe}{2}}\kappa)^{n-1}\|\xi\|_Y^{n-1}}{n!}\|\xi\|_X
\end{aligned}
$$

Noting that

$$
\sum_{n=2}^{+\infty}\frac{k^{n-1}}{n!}=\frac{1}{k}\sum_{n=2}^{+\infty}\frac{k^n}{n!}=\frac{1}{k}\left(\sum_{n=0}^{+\infty}\frac{k^n}{n!}-k-1\right)=\frac{e^k-k-1}{k}
$$

leads to

$$
\begin{aligned}
&\left\|\mu(\delta-x^e)\frac{1}{(1+\eta(\xi))(1+x^e)}\xi\sum_{n=2}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^{n-1}\frac{1}{n!}\right\|_X \\
&\leq \mu\delta\kappa\frac{e^{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}-\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y-1}{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}\|\xi\|_X\,.
\end{aligned}
$$

By using the relation $\|\delta-x^e\|_Y\leq\delta$ induced by the invariant subspace (2.3.9) and Lemma 4.4.1, it holds that

$$
\begin{aligned}
&\left\|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi))-\tilde{g}(x^e)-d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right\|_X \\
&\leq \tilde{\alpha}\mu\delta\kappa\|\xi\|_X\left(\frac{e^{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}-\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y-1}{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}\right) \\
&+\tilde{\alpha}\mu\delta e^{\frac{Pe}{2}}\|\xi\|_Y\|\xi\|_X+\tilde{\alpha}\|\xi\|_X\left(e^{\mu e^{\frac{Pe}{2}}\kappa\|\xi\|_Y}-1\right).
\end{aligned}
\tag{4.4.25}
$$

As a consequence

$$
\begin{aligned}
&\lim_{\|\xi\|_Y\to 0}\frac{\left\|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi))-\tilde{g}(x^e)-d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right\|_X}{\|\xi\|_X} \\
&\leq \lim_{\|\xi\|_Y\to 0}\left[\tilde{\alpha}\mu\delta\kappa\frac{e^{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}-\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y-1}{\mu\kappa e^{\frac{Pe}{2}}\|\xi\|_Y}\right.
\end{aligned}
$$

[8] It has been used that the space Y is a multiplicative algebra in which the relation $\|fg\|_Y\leq\|f\|_Y\|g\|_Y$ holds for any $f,g\in Y$.

$$+\tilde{\alpha}\mu\delta e^{\frac{Pe}{2}}\|\xi\|_Y+\tilde{\alpha}(e^{\mu e^{\frac{Pe}{2}}\kappa\|\xi\|_Y}-1)\Bigg]=0,$$

which shows the $(Y,X)-$Fréchet differentiability of $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\cdot+x^e)-\tilde{g}(x^e)]$ at 0. □

This proposition mixed with the continuous dependence of the state trajectory of (4.4.6) at 0 on X and Y yields that Assumption 4.2.3 is satisfied, which enables us to conclude that the semigroup $(S(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0, see Lemma 4.2.3.

Very similar arguments also lead to the conclusion that

$$\lim_{\|\xi\|_Y\to 0}\frac{\left\|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi))-\tilde{g}(x^e)-d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right\|_Y}{\|\xi\|_Y}=0, \tag{4.4.26}$$

which has the consequence that the nonlinear semigroup $(S(t))_{t\geq 0}$ is $Y-$Fréchet differentiable at 0. This together with the well-posedness of (4.4.6) and its Gâteaux linearization on Y has the consequence that Assumption 4.2.2 is satisfied.

In order to apply Theorem 4.3.1 it remains to show that the linear semigroup generated by the operator $A+d\tilde{g}(x^e)$ is exponentially stable on Y when it is exponentially stable on X, which would show that Assumption 4.3.1 is satisfied.

Remember that the linearized system corresponding to (4.4.6) around any equilibrium is written as

$$\begin{cases} \frac{\partial\overline{\xi}}{\partial t}=\frac{1}{Pe}\frac{\partial^2\overline{\xi}}{\partial z^2}-q(z)\overline{\xi}, \\ \frac{\partial\overline{\xi}}{\partial z}(0,t)=\frac{Pe}{2}\overline{\xi}(0,t), \frac{\partial\overline{\xi}}{\partial z}(1,t)=-\frac{Pe}{2}\overline{\xi}(1,t), \end{cases} \tag{4.4.27}$$

where the function $q(z)$ is defined in (3.2.19) for $z\in[0,1]$. In the perspective of studying exponential stability, we shall focus on the cases where either the reactor exhibits only one equilibrium profile or three. In order to study exponential stability on the space Y let us consider the approximated linearized system

$$\begin{cases} \frac{\partial\overline{\xi}_a}{\partial t}=\frac{1}{Pe}\frac{\partial^2\overline{\xi}_a}{\partial z^2}-q(c)\overline{\xi}_a, \\ \frac{\partial\overline{\xi}_a}{\partial z}(0,t)=\frac{Pe}{2}\overline{\xi}_a(0,t), \frac{\partial\overline{\xi}_a}{\partial z}(1,t)=-\frac{Pe}{2}\overline{\xi}_a(1,t), \end{cases} \tag{4.4.28}$$

where the subscript a is used to denote the approximated solution and $c\in[0,1]$. According to Lemma 2.1.4, (4.4.28) defines a Riesz-spectral system whose solution can be expressed in a series form as

$$\overline{\xi}_a(z,t)=\sum_{n=1}^{\infty}\psi_n\phi_n(z)e^{-(\beta_n^2+q(c))t}, \tag{4.4.29}$$

for $z\in[0,1], t\geq 0$ and all $c\in[0,1]$, where $\psi_n=\int_0^1\phi_n(z)\overline{\xi}_a(z,0)dz$. Performing the same types of inequalities as in the proof of Proposition 3.2.7 yields that the estimate

$$\|\overline{\xi}_a(\cdot,t)\|_X\leq e^{-\left(\frac{\pi^2}{\pi^2+4Pe}+q(c)\right)t}\|\overline{\xi}_a(\cdot,0)\|_X$$

is satisfied. Moreover, in Corollary 3.2.9 the bistability of the equilibria has been studied, meaning that in the case of one or three equilibria, the constant $-(\frac{\pi^2}{\pi^2+4Pe}+q(c))$ is negative. This has the consequence that the growth bound of the semigroup generated by the operator dynamics of (4.4.28) is negative. As the operator describing the dynamics of the approximated system (4.4.28) is of Riesz-spectral type, the relation $\sup_{n\in\mathbb{N}}\{-\beta_n^2+q(c)\}<0$ holds. Moreover, by the proof of Lemma 4.4.1, one has that

$$\begin{aligned}|\overline{\xi}_a(z,t)| &\leq \sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}|\psi_n||\phi_n(z)| \leq \sqrt{2}\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}\int_0^1 |\overline{\xi}_0^a(z)\phi_n(z)|dz \\ &\leq 2\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}\|\overline{\xi}_0^a\|_{L^1([0,1];\mathbb{R})} \\ &\leq 2\|\overline{\xi}_0^a\|_Y \sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t},\end{aligned}$$

where the notation $\overline{\xi}_0^a(z) := \overline{\xi}_a(z,0)$ has been used. Consequently, it holds that

$$\|\overline{\xi}_a(\cdot,t)\|_Y \leq 2\|\xi_0^a\|_Y \sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}.$$

A straightforward computation leads to

$$\int_0^{+\infty}\|\overline{\xi}_a(\cdot,t)\|_Y dt \leq 2\|\xi_0^a\|_Y \sum_{n=1}^{+\infty}\frac{1}{\beta_n^2+q(c)} \tag{4.4.30}$$

According to (Dehaye, 2015, Section 4.3.2), the estimate $\beta_n \simeq \pi(n-1)/\sqrt{Pe}$ holds for n sufficiently large. Hence (4.4.30) becomes

$$\begin{aligned}\int_0^{+\infty}\|\overline{\xi}_a(z,t)\|_Y dt &\leq 2\nu\|\overline{\xi}_0^a\|_Y\left(\sum_{n=1}^{+\infty}\frac{Pe}{\pi^2(n-1)^2+Pe\,q(c)}\right) \\ &= 2\nu\|\overline{\xi}_0^a\|_Y\eta < +\infty,\end{aligned}$$

for some $\nu>0$ and $\eta>0$. From an extension of Datko's lemma, see e.g. (Buse et al., 2006, Section 1) or Remark 2.1.1, it follows that $\overline{\xi}_a$ is exponentially stable on Y, i.e. there exists $M\geq 1$ and $\alpha>0$ such that

$$\|\overline{\xi}_a(\cdot,t)\|_Y \leq Me^{-\alpha t}\|\overline{\xi}_0^a\|_Y, t\geq 0. \tag{4.4.31}$$

Note that considering the $Y-$norm of $\overline{\xi}_a$ makes sense since $\overline{\xi}_a$ lies in $D(A)$, see (4.4.7), for initial conditions $\overline{\xi}_0^a$ chosen in $D(A)$, which is a subspace of Y. Consider now the system (4.4.27), which can also be expressed as

$$\begin{cases}\frac{\partial\overline{\xi}}{\partial t} = \frac{1}{Pe}\frac{\partial^2\overline{\xi}}{\partial z^2} - q(c)\overline{\xi} + [q(c)-q(z)]\,\overline{\xi}, \\ \frac{\partial\overline{\xi}}{\partial z}(0) = \frac{Pe}{2}\overline{\xi}(0), \frac{\partial\overline{\xi}}{\partial z}(1) = -\frac{Pe}{2}\overline{\xi}(1).\end{cases} \tag{4.4.32}$$

Since the estimate (4.4.31) holds for the approximated system (4.4.28), by (Engel and Nagel, 2006, Bounded Perturbation Theorem), one has that

$$\|\overline{\xi}(\cdot,t)\|_Y \leq Me^{(-\alpha+M\|q(c)-q(\cdot)\|_Y)t}\|\overline{\xi}_0\|_Y,$$

$t \geq 0$, where $\overline{\xi}$ is the solution to (4.4.27) (or equivalently (4.4.32)) with initial condition $\overline{\xi}_0$. By considering large diffusion phenomenon (meaning that Pe is sufficiently small), the equilibrium $x^e \to a$ uniformly for $z \in [0,1]$, where $a \in (0,\delta)$, see relation (3.2.10) together with Proposition 3.2.2 and Theorem 3.2.4. As a consequence, it holds that $\|q(c)-q(\cdot)\|_Y \to 0$ as $Pe \to 0$. Fix $\varepsilon > 0$ such that $M\varepsilon < \alpha$. Then there exists $\delta > 0$ such that $Pe < \delta$ implies that $\|q(c)-q(\cdot)\|_Y < \varepsilon$. Hence

$$-\alpha + M\|q(c)-q(\cdot)\|_Y < -\alpha + M\varepsilon := -\zeta < 0$$

by construction. It follows that the solution of (4.4.27) satisfies

$$\|\overline{\xi}(\cdot,t)\|_Y \leq Me^{-\zeta t}\|\xi_0\|_Y, t \geq 0,$$

which ensures that Assumption 4.3.1 is satisfied. We are now able to state the following theorem that fills the gap between the stability of equilibria for the linearized system (4.4.27) and the local stability of the equilibria for the nonlinear system (4.4.6). Its proof is a direct consequence of Theorem 4.3.1.

Theorem 4.4.4 *Consider the nonlinear PDE (4.4.6) that describes the time evolution of the temperature in a nonisothermal axial dispersion tubular reactor. In the case where the reactor exhibits one equilibrium profile, the latter is $(Y,X)-$ locally exponentially stable for the nonlinear system (4.4.6). In the case of three equilibria the pattern $(Y,X)-$ "locally exponentially stable - locally unstable - locally exponentially stable" is highlighted, which is called bistability.*

Note that, despite that the auxiliary space Y has been chosen as the space of continuous functions for the two applications treated in this chapter, it is not the only possibility. In general, multiplicative algebras should be appropriate, like Sobolev spaces of integer order, $H^p, p \in \mathbb{N}$, for instance.

We end this chapter with the illustration of Theorem 4.4.4. A set of parameters is chosen in order to highlight three equilibria. The initial condition is considered sufficiently small in the $Y-$norm and the diffusion is assumed to be large enough.

We choose $\xi_0(z) = \omega(\sin(\pi z) + \frac{2\pi}{Pe})$ as initial condition for the system (4.4.6), where ω is a weighting parameter that can make the $Y-$norm of ξ_0 as small as desired. Observe that these initial conditions are in $D(A)$ since the required regularity, i.e. $\xi_0 \in H^2([0,1];\mathbb{R})$, and the boundary conditions $\frac{d\xi_0}{dz}(0) = \frac{Pe}{2}\xi_0(0)$ and $\frac{d\xi_0}{dz}(1) = -\frac{Pe}{2}\xi_0(1)$ are satisfied.

An explicit computation of the $Y-$norm of ξ_0 gives

$$\|\xi_0\|_Y := \sup_{z\in[0,1]} |\xi_0(z)| = \frac{\omega}{Pe}(Pe + 2\pi). \tag{4.4.33}$$

By choosing for instance $\omega = \varepsilon Pe, \varepsilon > 0$, (4.4.33) becomes $\|\xi_0\|_Y = \varepsilon Pe + 2\pi\varepsilon$, which can be made as small as desired by considering ε small.

As mentioned in Theorem 4.4.4, in the case of three equilibria, these are alternatively locally exponentially stable with the pattern "stable - unstable - stable". In order to illustrate the latter, the parameters have been chosen as follows: $\mu = 10, \delta = 1, v = 1.1e-3$ and $D = 1e-3$. This implies that $Pe = 1.1$. The parameter ω is fixed to $1.1e-2$ such that $\|\xi_0\|_Y = 0.07383$. The state trajectories $\xi(z,t)$ and their $X-$norm are represented in Figures 4.2, 4.4, 4.6, 4.3, 4.5 and 4.7 for the different equilibrium profiles, respectively, wherein it can be observed that bistability is confirmed. Moreover, observe that the state trajectory associated to the second equilibrium is not diverging, and is going to be stabilized around another equilibrium. This phenomenon is often encountered when dealing with nonlinear systems. The state is moving from the unstable equilibrium to one of the two stable ones.

Remark 4.4.2 *The numerical method that has been used is based on a discretization of the spatial interval* $[0,1]$, *into n equal pieces,* $n = 50$. *By defining the state vector* $\xi_n \in \mathbb{R}^n$ *whose components are given by* $\xi_n^i(t) = \xi((i-1)h,t)$, *the system (4.4.6) has been discretized by means of finite differences, where h is the discretization step (*$h = \frac{1}{n-1}$*). Based on this, a finite-dimensional approximation of the operator* $\frac{d^2}{dz^2}$, *denoted by the matrix* $L_n \in \mathbb{R}^{n\times n}$, *has produced the finite-dimensional approximation of (4.4.6), whose dynamics are given by*

$$\dot{\xi}_n(t) = \frac{1}{Pe} L_n \xi_n(t) - \frac{Pe}{4}\xi_n(t) + E_n\left(G_n(\xi_n(t) + x_n^e) - G_n(x_n^e)\right), \qquad (4.4.34)$$

where the matrix E_n *is given by*

$$E_n = diag(1, e^{-\frac{Pe}{2}h}, e^{-Peh}, \ldots, e^{-\frac{Pe}{2}}),$$

with diag denoting a diagonal matrix whose arguments are on the diagonal. The $i-$*th component of the vectors* x_n^e *and* G_n, $i = 1, \ldots, n$, *are expressed as*

$$x_n^{e,i} = x^e((i-1)h)$$

and

$$G_n^i(\xi_n + x_n^e) = \tilde{g}(\xi_n^i + x_n^{e,i}),$$

respectively, while the $i-$*th component of* $G_n(x^e)$ *is given by*

$$G_n^i(x_n^e) = \tilde{g}(x_n^{e,i}).$$

Then, for any equilibrium, the system (4.4.34) has been numerically integrated via the routine `ode15s` *of Matlab©.*

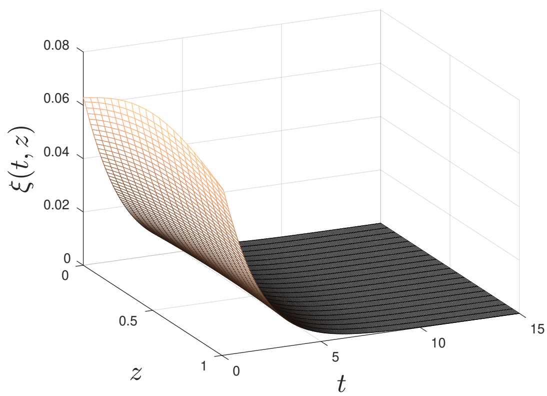

Figure 4.2 – State trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

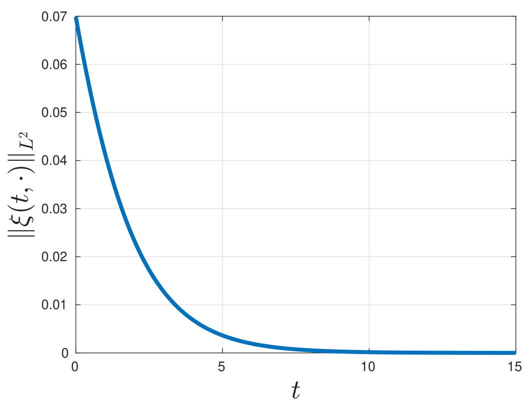

Figure 4.3 – $X-$norm of the state trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

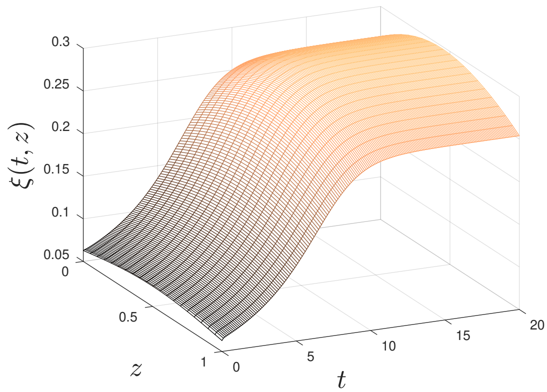

Figure 4.4 – State trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

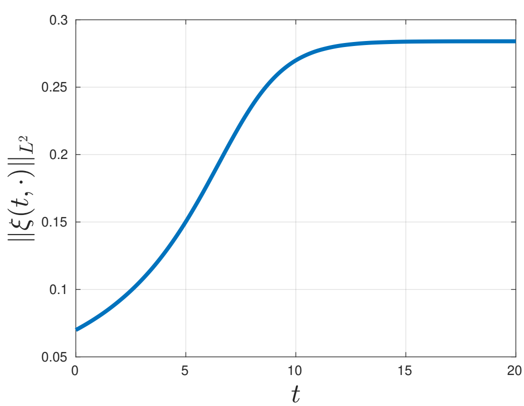

Figure 4.5 – $X-$norm of the state trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

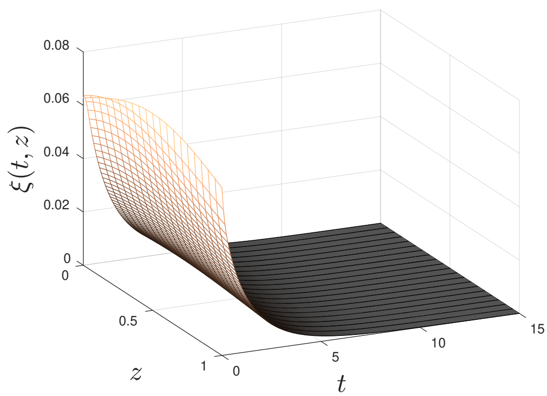

Figure 4.6 – State trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

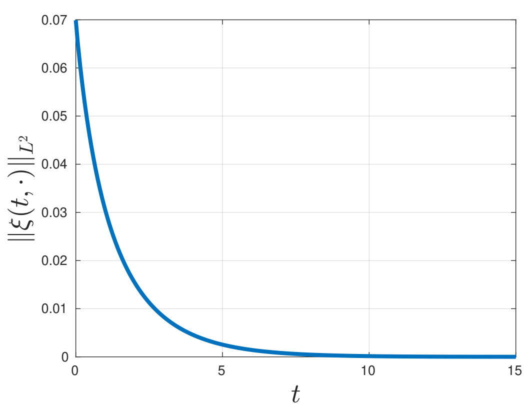

Figure 4.7 – $X-$norm of the state trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

Chapter 5

Local stabilization of nonlinear infinite-dimensional systems with an adapted Fréchet differentiability concept

Contents

The concepts presented in the previous chapter concerning the stability of equilibria are extended in this chapter to the stabilization problem of equilibrium profiles for nonlinear infinite-dimensional systems by means of particular types of control inputs. Regularity assumptions on the control operator are stated in terms of the state space and the auxiliary space. In particular, it is shown how to link the Fréchet differentiability of the nonlinear closed-loop semigroup with the Fréchet differentiability of the open-loop semigroup.

Moreover, a specific class of LQ-optimally controlled systems is presented and is shown to fullfil the required assumptions provided that some spectral conditions are satisfied.

We shall illustrate these new results for the regulation of a nonlinear heat equation with Neumann boundary conditions.

5.1 From stability to stabilization

Let us consider X as being a (separable) Hilbert space. We take the class of nonlinear infinite-dimensional systems presented in (4.2.1) again in which additional inputs are considered. Formally this reads as follows:

$$\begin{cases} \dot{\xi}_{cl}(t) = A\xi_{cl}(t) + f(\xi_{cl}(t) + x^e) - f(x^e) + Bu(t), \\ \xi_{cl}(0) = \xi_0 \in X, \end{cases} \tag{5.1.1}$$

where the operator $B \in \mathscr{L}(U,X)$, U being the input space. The control objective aims at designing the input function $u(t)$ in such a way that the state trajectory corresponding to (5.1.1) with initial condition ξ_0 converges locally exponentially fast to zero. Note that the subscript ”cl” refers to the abbreviation "closed-loop". It is easy to see that $(\xi_{cl}^e, u^e) = (0,0)$ consitutes an equilibrium solution of the steady-state equations associated to (5.1.1).

In what follows we shall consider input functions expressed as state feedbacks, that is, $u(t) := K\xi_{cl}(t)$ where the operator $K : X \to U$ is an appropriate linear and bounded operator that have to be designed accordingly to the control objective described above. With links to what has been constructed in Chapter 4, the main question will consist in determining a stabilizing state feedback K for the controlled linearized dynamics

$$\begin{cases} \dot{\overline{\xi}}_{cl}(t) = A\overline{\xi}_{cl}(t) + df(x^e)\overline{\xi}_{cl}(t) + BK\overline{\xi}_{cl}(t), \\ \overline{\xi}(0) = \xi_0, \end{cases} \tag{5.1.2}$$

where the notations are the same as those used in Chapter 4. Note that we shall restrict ourselves to the case where the domain of the nonlinear operator f is the whole space X.

In that way, in order to be able to use the framework developed in Chapter 4 we shall see how only the Fréchet differentiability of the nonlinear operator $f(\cdot + x^e) - f(x^e)$ at 0 (in the different senses) allows us to deduce the Fréchet differentiability of the nonlinear semigroup generated by the nonlinear operator dynamics of (5.1.1). Let us therefore state the following assumption.

Note that, in what follows, we consider again an auxiliary space Y continuously embedded in X and that satisfies $D(A) \subseteq Y \subseteq X$.

Assumption 5.1.1 *The control operator B is linear and bounded, when viewed as acting from the input space U into the auxiliary space Y, i.e. $B \in \mathscr{L}(U,Y)$. Moreover, the feedback operator $K \in \mathscr{L}(X,U)$.*

As a direct consequence of Assumption 5.1.1, we have the following lemma.

Lemma 5.1.1 *Under Assumption 5.1.1 and due to the continuous embedding of Y into X, it holds that $B \in \mathscr{L}(U,X)$ and $K \in \mathscr{L}(Y,U)$.*

Proof. We begin with the boundedness of B from U in into X. Take any $u \in U$. Thanks to $B \in \mathscr{L}(U,Y)$ it holds that $\|Bu\|_Y \leq \|B\|_{\mathscr{L}(U,Y)}\|u\|_U$. The continuous embedding of Y into X entails the following

$$\|Bu\|_X \leq \sigma\|Bu\|_Y \leq \sigma\|B\|_{\mathscr{L}(U,Y)}\|u\|_U.$$

Now let us consider $\xi \in Y$. By assumption it holds that $\|K\xi\|_U \leq \|K\|_{\mathscr{L}(X,U)}\|\xi\|_X$. The fact that $\|\xi\|_X \leq \sigma\|\xi\|_Y$ ends the proof. □

Before looking at the Fréchet differentiability of the nonlinear semigroup generated by the nonlinear operator dynamics corresponding to (5.1.1) with $\tilde{u}(t) = K\xi_{cl}(t)$, we shall ensure that this semigroup exists and makes sense both on X and on Y.

Lemma 5.1.2 *Assuming that the linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup on X, that the nonlinear operator $f : X \to X$ is globally Lipschitz continuous and that Assumption 5.1.1 holds gives that the nonlinear operator $A + f(\cdot + x^e) - f(x^e) + BK$ is the infinitesimal generator of a nonlinear semigroup on X.*

Proof. Observe that Assumption 5.1.1 implies that $B \in \mathscr{L}(U,X)$ and $K \in \mathscr{L}(X,U)$. Consequently, $BK \in \mathscr{L}(X)$. Hence according to (Engel and Nagel, 2006, Bounded Perturbation Theorem), the linear operator $A + BK$ is the infinitesimal generator of a C_0-semigroup. Since the nonlinear operator $f : X \to X$ is uniformly Lipschitz continuous, so is the operator $f(\cdot + x^e) - f(x^e)$. According to Theorem 2.2.1, the conclusion follows. □

Note that the boundedness assumption 4.2.2 implies that the same conclusion holds true for the space Y where the nonlinear operator is redefined as acting from Y into Y.

In what follows we adopt the following notations: the nonlinear semigroup generated by the operator $A + f(\cdot + x^e) - f(x^e) + BK$ is denoted by $(S_{cl}(t))_{t\geq 0}$ while the linear semigroup generated by the operator $A + df(x^e) + BK$ is denoted by $(\overline{T}_{cl}(t))_{t\geq 0}$. Moreover the linear operator A generates the C_0-semigroup[(1)] $(T(t))_{t\geq 0}$.

The following result characterizes the continuous dependence of the solution of (5.1.1) thanks to Assumption 5.1.1.

Proposition 5.1.1 *Consider the nonlinear controlled system (5.1.1) where*

$$u(t) = K\xi_{cl}(t)$$

for some linear gain operator K. Under Assumption 5.1.1, the mild solution $\xi_{cl}(t)$ to (5.1.1) depends continuously on the initial condition ξ_0 at 0 on X and Y on any compact interval $[0,t_0]$, where $t_0 > 0$.

Proof. Let us take $\xi_0 \in D(A)$ and $t_0 > 0$. We shall prove the continuous dependence on the initial condition on the space X. Similar arguments can be used to prove this property on Y. By assumption, the feedback operator $K \in \mathscr{L}(X,U)$. According to Lemma 5.1.1 the operators B and K satisfy $B \in \mathscr{L}(U,X)$ and $K \in \mathscr{L}(X,U)$, respectively. Now consider the mild solution of (5.1.1). It is expressed as

$$\xi_{cl}(t) = T(t)\xi_0 + \int_0^t T(t-s)[f(\xi_{cl}(s) + x^e) - f(x^e) + BK\xi_{cl}(s)]ds.$$

[(1)]For this semigroup, it holds that for any $\omega > \omega_0$ there exists $M_\omega > 0$ such that $\|T(t)\| \leq M_\omega e^{\omega t}, t \geq 0$.

Taking the $X-$norm of both sides yields

$$\begin{aligned}\|\xi_{cl}(t)\|_X \leq & \|T(t)\xi_0\|_X + \int_0^t \|T(t-s)[f(\xi_{cl}(s)+x^e)-f(x^e)]\|_X ds \\ & + \int_0^t \|T(t-s)BK\xi_{cl}(s)\|_X ds \\ \leq & M_\omega e^{\omega t}\|\xi_0\|_X + \int_0^t Me^{\omega(t-s)}\|f(\xi_{cl}(s)+x^e)-f(x^e)\|_X ds \\ & + \int_0^t M_\omega e^{\omega(t-s)}\|BK\xi_{cl}(s)\|_X ds.\end{aligned}$$

The boundedness of B and K together with the Lipschitz continuity of f imply that

$$\begin{aligned}\|e^{-\omega t}\xi_{cl}(t)\|_X \leq & M_\omega\|\xi_0\|_X + M_\omega l_N \int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds \\ & + M_\omega \|B\|_{\mathscr{L}(U,X)}\|K\|_{\mathscr{L}(X,U)}\int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds \\ = & M_\omega\|\xi_0\|_X + \tilde{\eta}\int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds,\end{aligned}$$

where $\tilde{\eta} := M_\omega l_f + M_\omega\|B\|_{\mathscr{L}(U,X)}\|K\|_{\mathscr{L}(X,U)}$ with l_f being one Lipschitz constant of f. Then Gronwall's inequality, see (Robinson, 2001, Lemma 2.8), yields

$$\|\xi_{cl}(t)\|_X \leq M_\omega e^{(\omega+\tilde{\eta})t}\|\xi_0\|_X,$$

which proves continuous dependence at 0 on X. □

We shall now focus on the $Y-$ and the $(Y,X)-$Fréchet differentiability of the nonlinear semigroup $(S_{cl}(t))_{t\geq 0}$ in order to be able to link the global stability of $(\overline{T}_{cl}(t))_{t\geq 0}$ with the local stability of $(S_{cl}(t))_{t\geq 0}$ thanks to Theorem 4.3.1.

Proposition 5.1.2 *Under Assumption 5.1.1, if the nonlinear operator $f(\cdot+x^e)-f(x^e)$ is $(Y,X)-$ and $Y-$Fréchet differentiable at 0, the nonlinear semigroup $(S_{cl}(t))_{t\geq 0}$ is $(Y,X)-$ and $Y-$Fréchet differentiable at 0 with $\overline{T}_{cl}(t)$ as Fréchet derivative.*

Proof. We shall prove the proposition for the $(Y,X)-$Fréchet differentiability. Similar arguments lead to the conclusion for the $Y-$Fréchet differentiability. First note that according to Proposition 5.1.1, the solution of (5.1.1) in closed-loop with $u(t) = K\xi_{cl}(t)$ depends continuously on the initial condition at 0 on X and on Y. Hence Assumption 4.2.3 is satisfied. Consequently the relation

$$\lim_{\|\xi_0\|_Y\to 0}\frac{\|f(\xi_{cl}+x^e)-f(x^e)-df(x^e)\xi_{cl}\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X} = 0 \tag{5.1.3}$$

is satisfied according to Lemma 4.2.2 for any positive t_0. Then, observe that for $t \in [0,t_0]$, the state trajectories of (5.1.1) and (5.1.2) are given by

$$\xi_{cl}(t) = T(t)\xi_0 + \int_0^t T(t-s)[f(\xi_{cl}(s)+x^e)-f(x^e)+BK\xi_{cl}(s)]ds$$

and

$$\overline{\xi}_{cl}(t) = T(t)\xi_0 + \int_0^t T(t-s)[df(x^e)\overline{\xi}_{cl}(s) + BK\overline{\xi}_{cl}(s)]ds,$$

respectively. As a consequence,

$$\begin{aligned}\|e^{-\omega t}(\xi_{cl}(t) - \overline{\xi}_{cl}(t))\|_X \leq & M_\omega \int_0^t e^{-\omega s}\|f(\xi_{cl}(s)+x^e) - f(x^e) - df(x^e)\xi_{cl}(s)\|_X ds \\ & + M_\omega \int_0^t e^{-\omega s}\|df(x^e)(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds \\ & + M_\omega \int_0^t e^{-\omega s}\|BK(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds.\end{aligned}$$

Using the boundedness of the Gâteaux derivative of f at x^e and the fact that $BK \in \mathscr{L}(X)$, it follows that

$$\begin{aligned}\|e^{-\omega t}(\xi_{cl}(t) - \overline{\xi}_{cl}(t))\|_X \leq & M_\omega \int_0^t e^{-\omega s}\|f(\xi_{cl}(s)+x^e) - f(x^e) - df(x^e)\xi_{cl}(s)\|_X ds \\ & + \lambda \int_0^t \|e^{-\omega s}(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds,\end{aligned}$$

where $\lambda := M_\omega(\|df(x^e)\|_{\mathscr{L}(X)} + \|BK\|_{\mathscr{L}(X)})$. Hence, by Gronwall's lemma,

$$\|\xi_{cl}(t) - \overline{\xi}_{cl}(t)\|_X \leq M_\omega e^{(\omega+\lambda)t} k_0 \int_0^t \|R(\xi_{cl}(s), x^e)\|_X ds,$$

where $R(\xi_{cl}, x^e)$ stands for $f(\xi_{cl} + x^e) - f(x^e) - df(x^e)\xi_{cl}$ and $k_0 := \max\{1, e^{-\omega t_0}\}$. Since (5.1.3) holds, the nonlinear semigroup[(2)] $(S_{cl}(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0 with $(\overline{T}_{cl}(t))_{t\geq 0}$ as Fréchet derivative. □

By looking at Theorem 4.3.1, designing a stabilizing state feedback K for the linearized dynamics (5.1.2) both on X and Y implies that the nonlinear system[(3)] (5.1.1) with the input function $u(t)$ expressed as $u(t) = K\xi_{cl}(t)$ is locally exponentially stable on X, where locally means that initial conditions $\xi_0 \in Y$ having a sufficiently small $Y-$norm have to be considered, see Definition 4.2.3.

5.2 A class of LQ-optimally controlled nonlinear systems

Here we focus our attention on a specific class of single-input systems of the form (5.1.1) defined on a state (Hilbert) space X equipped with the inner product $\langle \cdot, \cdot \rangle_X$. The auxiliary (possibly Banach) space is denoted by Y and satisfies $D(A) \subseteq Y \subseteq X$. The

(2) It holds that $\xi_{cl}(t) = S_{cl}(t)\xi_0$ and $\overline{\xi}_{cl}(t) = \overline{T}_{cl}(t)\xi_0$ for any $t \geq 0$ and $\xi_0 \in X$.
(3) This has the consequence that Assumption 4.3.1 is satisfied.

input space is chosen as being $U=\mathbb{R}$ while the control operator $B:\mathbb{R}\to X$ is defined as the multiplication of the scalar input u by the $Y-$function b, i.e. $Bu=bu, b\in Y$ for all $u\in\mathbb{R}$. In addition to this setting, we consider an operator $C:X\to\mathbb{R}$ being a linear functional, that is, without loss of generality, defined by $C\xi=\langle c,\xi\rangle_X$ for some $c\in X$ and any $\xi\in X$. This linear functional may be interpreted as scalar measurements on the system. According to these considerations, the nonlinear system (5.1.1) has the form

$$\begin{cases} \dot{\xi}_{cl}(t)=A\xi_{cl}(t)+f(\xi_{cl}(t)+x^e)-f(x^e)+bK\xi_{cl}(t), \\ \xi_{cl}(0)=\xi_0, \end{cases} \tag{5.2.1}$$

where the initial condition $\xi_0\in X$. The nonlinear operator $f:X\to X$ is assumed to be uniformly Lipschitz continuous and $Y-$ and $(Y,X)-$Fréchet differentiable at x^e with x^e being a solution of the equation $Ax^e+f(x^e)=0$. Its $(Y,X)-$Fréchet derivative is given by the linear operator $df(x^e)$ which is assumed to be bounded both viewed as acting from X into X or from Y into Y. Linearizing (5.2.1) around the equilibrium 0 yields the linear system

$$\begin{cases} \dot{\overline{\xi}}_{cl}(t)=A\overline{\xi}_{cl}(t)+df(x^e)\overline{\xi}_{cl}(t)+bK\overline{\xi}_{cl}(t), \\ \overline{\xi}_{cl}(0)=\xi_0. \end{cases} \tag{5.2.2}$$

Besides, it is assumed that the pairs $(A+df(x^e),b)$ and $(c,A+df(x^e))$ are exponentially stabilizable and detectable, respectively, see Definition 2.1.10. These two assumptions will be interpreted after having defined the optimal control problem. Note that the condition $b\in Y$ implies that the operator $B\in\mathscr{L}(\mathbb{R},Y)$ which has the consequence that $B\in\mathscr{L}(U,X)$ according to Lemma 5.1.1.

In what follows the operator $A+df(x^e)$ is assumed to be a Riesz-spectral operator whose spectrum is composed of only simple eigenvalues. Hence, it may be represented by the following series expansion

$$(A+df(x^e))\overline{\xi}_{cl}:=\sum_{n=0}^{\infty}\lambda_n\langle\overline{\xi}_{cl},\psi_n\rangle_X\phi_n, \tag{5.2.3}$$

where $\{\lambda_n\}_{n\in\mathbb{N}},\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are the eigenvalues, the eigenfunctions basis of the operator $A+df(x^e)$ and the eigenfunctions basis of its adjoint[(4)] on X, respectively. The set of functions $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ form an biorthogonal basis, i.e. $\langle\phi_n,\psi_m\rangle_X=\delta_{nm}$.

As a quite common assumption on the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $A+df(x^e)$ we ask that there exists a positive and finite real number κ such that

$$\sup_{m\in\mathbb{N}}\{\sum_{\substack{n=0\\n\neq m}}^{\infty}\frac{1}{|\lambda_n-\lambda_m|^2}\}=\kappa \tag{5.2.4}$$

holds true. As mentioned in Section 2.1.2.2 this has the consequence that

$$\inf_{n,m\in\mathbb{N},n\neq m}|\lambda_n-\lambda_m|=\mu,\mu>0. \tag{5.2.5}$$

[(4)]The adjoint operator of A is denoted by A^*.

Let us consider the notation $\mathscr{A} := A + df(x^e)$ in what follows.

As a computation of the feedback operator K, we focus on the following optimal control problem for system (5.1.2). The goal consists in finding a control law $u_o \in L^2([0,\infty);\mathbb{R})$ that minimizes the cost functional

$$J(\xi_0, u) = \int_0^\infty (\rho_1 |C\overline{\xi}_{cl}(t)|^2 + \rho_2 |u(t)|^2) dt, \tag{5.2.6}$$

where the real numbers satisfy $\rho_1 \geq 0$ and $\rho_2 > 0$, subject to the dynamics (5.2.2). The assumption of exponential stabilizability of $(\mathscr{A}, b)$ ensures that the problem is optimizable, that is, there exists at least one control input u that renders the cost (5.2.6) finite. Meanwhile, the exponential detectability of $(c, \mathscr{A})$ implies that the feedback operator K, solution of the optimal control problem, is exponentially stabilizing, see (Curtain and Zwart, 2020, Theorem 9.2.9). Looking at the weighting parameters, they aim at penalizing the effects of the functions $C\overline{\xi}_{cl}(t)$ and $u(t)$ in the cost (5.2.6). The optimal control that minimizes (5.2.6) subject to the dynamics (5.2.2) is given by $u_o(t) = K_o \overline{\xi}_{cl}(t)$, where $K_o := -\frac{1}{\rho_2} B^* Q$ with Q being the unique positive self-adjoint solution of the following operator Riccati equation

$$\mathscr{A}^* Q + Q\mathscr{A} + \rho_1 C^* C - \frac{1}{\rho_2} QBB^* Q = 0 \tag{5.2.7}$$

on $D(\mathscr{A})$ with the condition that $Q(D(A)) \subset D(A^*)$. For more details about the LQ-optimal control problem for infinite- dimensional systems, see for instance Callier and Winkin (1990, 1992); Curtain and Zwart (2020) and references therein.

According to Winkin et al. (2004), the general form of the state feedback $K_o \in \mathscr{L}(X, \mathbb{R})$ is given by $K_o \overline{\xi}_{cl} := \langle k_o, \overline{\xi}_{cl} \rangle$, for any $\overline{\xi}_{cl} \in X$ and some $k_o \in X$. Lemma 5.1.1 implies that the feedback operator K_o is bounded from the auxiliary space Y into the input space $\mathbb{R}$.

By considering (Shun-Hua, 1981, Theorem 2.1) and due to the form of the operator K_o, the linear closed-loop operator $\mathscr{A} + bK_o$ preserves the Riesz-spectral property. Provided that the eigenvalues of the closed-loop operator $\mathscr{A} + bK_o$ are simple, the later admits the spectral decomposition

$$(\mathscr{A} + bK_o)\overline{\xi}_{cl} = \sum_{n=0}^{\infty} \lambda_n^o \langle \overline{\xi}_{cl}, \psi_n^o \rangle_X \phi_n^o, \tag{5.2.8}$$

where $\{\lambda_n^o\}_{n\in\mathbb{N}}, \{\phi_n^o\}_{n\in\mathbb{N}}$ and $\{\psi_n^o\}_{n\in\mathbb{N}}$ are the eigenvalues, the eigenfunctions of the operator $\mathscr{A} + bK_o$ and the eigenfunctions of its adjoint, respectively.

This framework entails that the nonlinear semigroup generated by the nonlinear operator dynamics associated[(5)] to (5.2.1) is $(Y,X)-$ and $Y-$Fréchet differentiable at 0 with the linear semigroup $(\overline{T}_{cl}(t))_{t\geq 0}$ generated by the dynamics of (5.2.2) as Fréchet derivative, see Proposition 5.1.2.

It remains to solve the question of exponential stability of $(\overline{T}_{cl}(t))_{t\geq 0}$ on the space Y. Sufficient conditions that ensures the desired result are proposed in the next proposition.

[(5)] Which exists on X and on Y thanks to Lemma 5.1.2.

Proposition 5.2.1 *Let us consider the linear feedback gain K_o, given by $\langle k_o, \cdot\rangle_X$, that is obtained as the solution of the optimal control problem that consists in minimizing (5.2.6) subject to (5.2.2). Under the following assumption*

$$\sum_{n=0}^{\infty} \frac{\|\psi_n^o\|_X \|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|} < \infty, \tag{5.2.9}$$

the feedback gain K_o exponentially stabilizes the dynamics (5.2.2) around 0 *on the space Y.*

Proof. As mentionned previously in (5.2.8) the closed-loop operator $\mathscr{A} + b\langle k_0, \cdot\rangle$ is a Riesz-spectral operator since so is $\mathscr{A}$ and since (5.2.4) holds true.

Observe that $\sup_{n\in\mathbb{N}}\{\mathfrak{Re}(\lambda_n^o)\} < 0$ since the semigroup generated by $\mathscr{A} + b\langle k_0, \cdot\rangle$ is exponentially stable on X by construction. Take $\xi_0 \in Y$. It holds that

$$\|\overline{\xi}_{cl}(t)\|_Y = \|\sum_{n=0}^{\infty} e^{\lambda_n^o t}\langle \xi_0, \psi_n^o\rangle_X \phi_n^o\|_Y \leq \sum_{n=0}^{\infty} e^{\mathfrak{Re}(\lambda_n^o)t}\|\psi_n^o\|_X \|\phi_n^o\|_Y \|\xi_0\|_X.$$

It is stated in Remark 2.1.1 that, if the condition

$$\int_0^{\infty} \|\overline{\xi}_{cl}(t)\|_Y^p dt < \infty \tag{5.2.10}$$

holds for some $p \in [1, \infty)$ and for any $\xi_0 \in Y$, then the state trajectory $\overline{\xi}_{cl}(t)$ is exponentially stable when evaluated in $Y-$norm. The condition (5.2.10) in this specific context with $p = 1$ follows from

$$\int_0^{\infty} \|\overline{\xi}_{cl}(t)\|_Y dt \leq \int_0^{\infty} \sum_{n=0}^{\infty} e^{\mathfrak{Re}(\lambda_n^o)t}\|\psi_n^o\|_X \|\phi_n^o\|_Y \|\xi_0\|_Y dt = \sum_{n=0}^{\infty} \frac{\|\psi_n^o\|_X \|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|}\|\xi_0\|_X$$

and Assumption (5.2.9). □

5.3 Illustration

In this section we aim at applying the results that have been presented before in order to regulate the state of a nonlinear heat equation with Neumann boundary conditions. The dynamical model that is under consideration is driven by the following PDE

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \sqrt{x^2+1} + b(z)u(t), \\ \frac{\partial x}{\partial z}(0,t) = 0 = \frac{\partial x}{\partial z}(1,t), \end{cases} \tag{5.3.1}$$

where $t \geq 0$ and $z \in [0,1]$ stand for the temporal and the spatial variables, respectively. The function b is defined by $b(z) = 1_{[0,1]}(z)$ for all $z \in [0,1]$. For any $t \geq 0$, it is assumed that the spatially dependent profile $x(\cdot,t)$ lies in the state space $L^2([0,1];\mathbb{R}) =: X$. We want to insist on the fact that (5.3.1) does not correspond to a real physical example but is has been constructed in order to illustrate the theory developed in the previous section. The control objective consists in designing an input $u(t)$ such that the state x reaches the reference profile $r(z) := 1_{[0,1]}(z)$ exponentially fast. Remark that, for $r(z)$ to be an equilibrium solution of (5.3.1), it is necessary that the steady state control input u^e takes the form $u^e = -\sqrt{2}$. Remark also that reaching the profile $r(z)$ is not possible without applying any control since it is not an equilibrium profile of the homogeneous part of (5.3.1) (i.e. with $u(t) \equiv 0$). Therefore let us perform the change of variables $\tilde{u}(t) = u(t) - u^e$ together with $\xi(z,t) = x(z,t) - r(z)$. In this way, (5.3.1) is given by

$$\begin{cases} \frac{\partial \xi}{\partial t} = \frac{\partial^2 \xi}{\partial z^2} + \sqrt{(\xi+1)^2+1} - 1_{[0,1]}(z)\sqrt{2} + b(z)\tilde{u}(t), \\ \frac{\partial \xi}{\partial z}(0,t) = 0 = \frac{\partial \xi}{\partial z}(1,t). \end{cases} \tag{5.3.2}$$

With this change of variables and according to the setting developed in Section 5.1, the control objective may be interpreted as designing the control input $\tilde{u}(t)$ as the state feedback $\tilde{u}(t) = K\xi(t)$ such that the state trajectory of (5.3.2) converges to 0 exponentially fast when t goes to ∞. As it has already been pointed out in Section 3.1, the nonlinear operator $\tilde{f}$ given by $\tilde{f}(\xi) := \sqrt{(\xi+1)^2+1} - \sqrt{2}$ for $\xi \in X$ is not Fréchet differentiable at 0 on the state space X. As it has been proposed in Section 4.4.1 let us consider the Banach space of continuous functions $Y := C([0,1];\mathbb{R})$ as auxiliary space.

First notice that (5.3.2) admits the abstract representation (5.2.1) with the operator $A : D(A) \to X$ being defined by $A\xi = \frac{d^2\xi}{dz^2}$ for $\xi \in D(A)$ whose expression may be found in (4.4.3). The nonlinear operator $f : X \to X$ is given by $f(\xi) = \sqrt{\xi^2+1}$. Moreover, the control operator $B : \mathbb{R} \to X$ takes the form $Bu = b(z)u$ for any $u \in \mathbb{R}$. Owing to the fact that $b \in Y$, the operator $B \in \mathscr{L}(\mathbb{R},Y)$. It is easy to see that the operator A is the infinitesimal generator of a C_0-semigroup on X. The same property holds on Y too, where tools related to the Banach space setting have to be used, see e.g. (Engel and Nagel, 2006, Chapter 2, Section 2.11). Moreover, according to Section 4.4.1, the nonlinear operator $\tilde{f}(\cdot) := f(\cdot + r) - f(r)$ is uniformly Lipschitz continuous on X and on Y with one Lipschitz constant given by $l_{\tilde{f}} := 1$. These facts have the consequence that the nonlinear equation (5.3.2) with $\tilde{u}(t) \equiv 0$ possesses a unique mild solution on X and on Y. Considering control inputs $\tilde{u}$ expressed as state feedbacks $\tilde{u}(t) = K\xi(t)$ where $K \in \mathscr{L}(X,U)$ does not change this property of existence and uniqueness of a mild solution. Furthermore note that the solution may be classical whenever the initial condition is chosen in $D(A)$.

Concerning the continuous dependence of the solution $\xi(t)$ on the initial condition at 0 on X and on Y, we refer to (Martin, 1987, Lemma 5.1). Indeed, as the initial conditions are chosen in the domain of the linear operator A we are dealing with classical solutions, i.e. solutions that are differentiable in time. This implies continuity in time. Hence, continuous dependence is proved.

As the space Y has been introduced, note that according to Proposition 3.1.1 and to Section 4.4.1, the nonlinear operator $\tilde{f}$ is $Y-$ and $(Y,X)-$Fréchet differentiable at 0, with the linear and bounded operator $d\tilde{f}(0)$ expressed as $d\tilde{f}(0) = \frac{1}{\sqrt{2}}I$ as Fréchet derivative. This together with continuous dependence of the solution of (5.3.2) at

0 implies that the nonlinear closed-loop semigroup $(S_{cl}(t))_{t\geq 0}$ satisfying $S_{cl}(t)\xi_0 = \xi(t), t \geq 0$ where ξ_0 stands for the initial condition is $Y-$ and $(Y,X)-$Fréchet differentiable at 0 as well, see Lemma 4.2.3. Its Fréchet derivative is given by the linear semigroup $(\overline{T}_{cl}(t))_{t\geq 0}$ which is generated by the dynamics of the following linear system

$$\begin{cases} \frac{\partial \overline{\xi}}{\partial t} = \frac{\partial^2 \overline{\xi}}{\partial z^2} + \frac{1}{\sqrt{2}}\overline{\xi} + bK\overline{\xi}, \\ \frac{\partial \overline{\xi}}{\partial z}(0,t) = 0 = \frac{\partial \overline{\xi}}{\partial z}(1,t). \end{cases} \tag{5.3.3}$$

Observe that (5.3.3) admits the abstract representation

$$\dot{\overline{\xi}}(t) = \mathscr{A}\overline{\xi}(t) + bK\overline{\xi}(t), \overline{\xi}(0) = \xi_0, \tag{5.3.4}$$

where the linear operator $\mathscr{A} = A + \frac{1}{\sqrt{2}}I$ is defined on $D(\mathscr{A}) = D(A)$. Since $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup on X and Y and since the operator bK is bounded when viewed as acting from X into X or from Y into Y, the abstract Cauchy problem (5.3.4) is well-posed on X and on Y.

The design of a stabilizing state feedback K for the linear dynamics (5.3.4) on X and Y is adressed in the next section.

5.3.1 Resolution of an optimal control problem

As proposed in Section 5.2, the state feedback K will be computed as the solution of the following optimal control problem

$$\begin{cases} u^* = \arg\min J(\xi_0, \tilde{u}) := \int_0^\infty (\rho_1 \langle c, \overline{\xi}(t)\rangle_X^2 + \rho_2 \tilde{u}^2(t))dt, \\ \dot{\overline{\xi}}(t) = \mathscr{A}\overline{\xi}(t) + b\tilde{u}(t), \overline{\xi}(0) = \xi_0, \end{cases} \tag{5.3.5}$$

where we seek for a control input $u^*(t) = K\overline{\xi}(t)$ that is square integrable in infinite horizon, i.e. $u^*(\cdot) \in L^2([0,\infty);\mathbb{R})$. Note that the weights ρ_1 and ρ_2 are supposed to be such that $\rho_1 \geq 0$ and $\rho_2 > 0$. The function c is chosen as $c(z) = 1_{[0,1]}(z)$. In order to ensure that (5.3.5) is feasible, we need to verify that the pairs $(\mathscr{A}, b)$ and $(c, \mathscr{A})$ are exponentially stabilizable and detectable, respectively, see (Curtain and Zwart, 2020, Theorem 9.2.9). For this purpose, observe that the operator $\mathscr{A}$ is of Riesz-spectral type. Its spectrum is composed of only eigenvalues expressed as $\{\lambda_n\}_{n\in\mathbb{N}} = \{-n^2\pi^2 + \frac{1}{\sqrt{2}}\}_{n\in\mathbb{N}}$. The corresponding normalized eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ are given by $\{1_{[0,1]}(z)\} \cup \{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}_0}$ and form an orthonormal basis of X. Observe also that the operator $\mathscr{A}$ is self-adjoint and has a compact resolvent operator. The last feature is obtained since the relation $\lim_{n\to\infty} \frac{1}{\lambda_n} = 0$ holds true, see (Curtain and Zwart, 2020, Lemma 3.2.12). Consequently, the exponential stabilizability of the pair $(\mathscr{A}, b)$ is guaranteed provided that $\langle \phi_n, b\rangle_X \neq 0$ for any $n \in \mathbb{N}$ for which $\mathfrak{Re}(\lambda_n) > 0$, see (Curtain and Zwart, 2020, Theorem 8.2.2). It is easy to see that the only n for which the test has to be performed is $n = 0$. For this n it holds that $\langle \phi_0, b\rangle_X = 1$ which implies exponential stabilizability of $(\mathscr{A}, b)$. Since the functions c and b are the same and $\mathscr{A}$

is self-adjoint, the exponential detectability of $(c, \mathscr{A})$ is equivalent to the exponential stabilizability of $(\mathscr{A}, b)$.

These considerations entail the well-posedness of the optimal control problem (5.3.5) since this means that there exists at least one control input $u(\cdot) \in L^2([0,\infty);\mathbb{R})$ such that the functional J in (5.3.5) is finite. Let us now focus on the solution of (5.3.5). According to e.g. Shun-Hua (1981); Winkin et al. (2004); Curtain and Zwart (2020), the optimal feedback operator $K \in \mathscr{L}(X,\mathbb{R})$ is given by the operator $K := -\frac{1}{\rho_2}B^*Q$ where the positive self-adjoint operator $Q \in \mathscr{L}(X)$ is the unique stabilizing solution of the operator Riccati equation (5.2.7). Therefore we shall use the Riesz-spectral nature of the operator $\mathscr{A}$. Since it satisfies that property, it is isomorphic to a diagonal operator on the space of square summable sequences, $l^2(\mathbb{N}) := \{\{\alpha_n\}_{n\in\mathbb{N}}, \sum_{n=0}^{\infty} \alpha_n^2 < \infty\}$, see (Tucsnak and Weiss, 2009, Proposition 2.6.2). The isomorphism that transforms any element of X into a square summable sequence is denoted by $\mathscr{F} : X \to l^2(\mathbb{N})$ and defined for any $\overline{\xi} \in X$ by

$$\mathscr{F}\overline{\xi} = \{\langle\overline{\xi}, \phi_n\rangle_X\}_{n\in\mathbb{N}}.$$

It satisfies $\mathscr{F} \in \mathscr{L}(X, l^2(\mathbb{N}))$ and its inverse is given by the operator $\mathscr{F}^{-1} : l^2(\mathbb{N}) \to X$ as

$$\mathscr{F}^{-1}\{\alpha_n\}_{n\in\mathbb{N}} = \sum_{n=0}^{\infty} \alpha_n \phi_n,$$

for all $\{\alpha_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. The isomorphism $\mathscr{F}$ transforms the operators $\mathscr{A}, b$ and $C := \langle c, \cdot\rangle_X$ into the operators $\tilde{\mathscr{A}} : D(\tilde{\mathscr{A}}) \subset l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $\tilde{b} : \mathbb{R} \to l^2(\mathbb{N})$ and $\tilde{C} : l^2(\mathbb{N}) \to \mathbb{R}$ as[(6)]

$$\tilde{\mathscr{A}}\alpha = \{\lambda_n \alpha_n\}_{n\in\mathbb{N}}, \tilde{b}u = \{\langle b, \phi_n\rangle_X u\}_{n\in\mathbb{N}}, \tilde{C}\eta = \eta_0,$$

for any $\alpha := \{\alpha_n\}_{n\in\mathbb{N}} \in D(\tilde{\mathscr{A}})$, which can be expressed as

$$D(\tilde{\mathscr{A}}) = \{\alpha \in l^2(\mathbb{N}), \sum_{n=0}^{\infty} (1+\lambda_n^2)|\alpha_n|^2 < \infty\}$$

and $u \in \mathbb{R}$. According to the orthogonality of the function b to all the eigenfunctions $\phi_n, n \geq 1$, the operator $\tilde{b}$ may also be rewritten as

$$\tilde{b}u = (\delta_{1n})_{n\in\mathbb{N}} u,$$

where δ_{1n} is the Kronecker symbol, i.e. $\tilde{b}_0 = 1$ and $\tilde{b}_n = 0, n \geq 1$. Consequently, the isomorphism $\mathscr{F}$ allows to rewrite the optimal control problem (5.3.5) defined on X as the following optimal control problem on $l^2(\mathbb{N})$

$$\begin{cases} u^* = \arg\min J(\xi_0, \tilde{u}) := \displaystyle\int_0^{\infty} (\rho_1 \alpha_0(t)^2 + \rho_2 \tilde{u}^2(t))dt, \\ \dot{\alpha}_0(t) = \frac{1}{\sqrt{2}}\alpha_0(t) + \tilde{u}(t), \\ \dot{\alpha}_n(t) = (-n^2\pi^2 + \frac{1}{\sqrt{2}})\alpha_n(t), n \geq 1, \end{cases} \tag{5.3.6}$$

(6) The operators $\tilde{A}, \tilde{b}$ and are obtained from the operators $\mathscr{A}, b$ and C in the following way: $\tilde{\mathscr{A}} = \mathscr{F}\mathscr{A}\mathscr{F}^{-1}, \tilde{b} = \mathscr{F}b$ and $\tilde{C} = C\mathscr{F}^{-1}$.

where the notation $\{\alpha_n\}_{n\in\mathbb{N}} := \mathscr{F}\overline{\xi}$ has been used. The solution of this optimal control problem is given by the state feedback $u^*(t) = -\frac{1}{\rho_2}\tilde{b}^T\tilde{Q}(\alpha_0 \quad \alpha_1 \quad \ldots)^T$ where the positive self-adjoint operator $\tilde{Q} \in \mathscr{L}(l^2(\mathbb{N}))$ is the solution of the following operator Riccati equation

$$\tilde{\mathscr{A}}^T\tilde{Q} + \tilde{Q}\mathscr{A} + \rho_1\tilde{c}^T\tilde{c} - \frac{1}{\rho_2}\tilde{Q}\tilde{b}\tilde{b}^T\tilde{Q} = 0, \tag{5.3.7}$$

where $\tilde{c} = \tilde{b}^T$. By considering the expressions of the infinite matrices $\tilde{A}, \tilde{b}$ and $\tilde{c}$ and by denoting the infinite matrix $\tilde{Q}$ by $(q_{ij})_{i,j\geq 0}$, (5.3.7) may be expressed as

$$\begin{pmatrix} \lambda_0 q_{00} & \lambda_0 q_{01} & \lambda_0 q_{02} & \cdots \\ \lambda_1 q_{10} & \lambda_1 q_{11} & \lambda_1 q_{12} & \cdots \\ \lambda_2 q_{20} & \lambda_2 q_{21} & \lambda_2 q_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \begin{pmatrix} \lambda_0 q_{00} & \lambda_1 q_{01} & \lambda_2 q_{02} & \cdots \\ \lambda_0 q_{10} & \lambda_1 q_{11} & \lambda_2 q_{12} & \cdots \\ \lambda_0 q_{20} & \lambda_1 q_{21} & \lambda_2 q_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \begin{pmatrix} \rho_1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$-\frac{1}{\rho_2}\begin{pmatrix} q_{00}^2 & q_{00}q_{01} & q_{00}q_{02} & \cdots \\ q_{10}q_{00} & q_{10}q_{01} & q_{10}q_{02} & \cdots \\ q_{20}q_{00} & q_{20}q_{01} & q_{20}q_{02} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{5.3.8}$$

First observe that the unknown q_{00} is subject to the scalar equation

$$2\lambda_0 q_{00} + \rho_1 - \frac{1}{\rho_2}q_{00}^2 = 0, \tag{5.3.9}$$

whose solutions are[(7)] $q_{00}^+ = \frac{2\lambda_0 + \sqrt{4\lambda_0^2 + \frac{4\rho_1}{\rho_2}}}{2}\rho_2$ and $q_{00}^- = \frac{2\lambda_0 - \sqrt{4\lambda_0^2 + \frac{4\rho_1}{\rho_2}}}{2}\rho_2$. In what follows we shall keep the positive solution owing to the positivity of $\tilde{Q}$. By looking at the element at the position $(1,2)$ in the matrix equation (5.3.8) yields

$$(\lambda_0 + \lambda_1)q_{01} - \frac{1}{\rho_2}q_{00}q_{01} = 0.$$

As the term $\lambda_0 + \lambda_1 - \frac{1}{\rho_2}q_{00} < 0$, the only solution is $q_{01} = 0$. Iterating the process by looking at the element at position $(1, i+1), i \geq 2$ in (5.3.8), one gets the equation

$$(\lambda_0 + \lambda_i)q_{0i} - \frac{1}{\rho_2}q_{00}q_{0i} = 0,$$

which yields the solution $q_{0i} = 0$ for any $i \geq 1$. As the operator matrix Q is self-adjoint, it is symmetric, which implies that $q_{i0} = 0, i \geq 1$. By incorporating this in (5.3.8) one finds that q_{ij} is 0 for any $i, j \geq 0$ except for the element q_{00}. Consequently, the solution $\tilde{Q} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ of (5.3.8) satisfies $\tilde{Q}\{\alpha_n\}_{n\in\mathbb{N}} = (q_{00}^+\alpha_0 \quad 0 \quad 0 \quad \ldots)^T$ for any sequence $\{\alpha_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. This implies that the optimal feedback $u^*(t)$ which is the solution to the optimal control problem (5.3.6) is given by

$$u^*(t) = -\frac{1}{\rho}\tilde{b}^T\tilde{Q}\{\alpha_n(t)\}_{n\in\mathbb{N}} = -\frac{1}{\rho_2}q_{00}^+\alpha_0(t) =: \tilde{k}\{\alpha_n\}_{n\in\mathbb{N}},$$

(7) The exponents "+" and "-" refer to the positive and the negative solutions, respectively.

where $\tilde{k} : l^2(\mathbb{N}) \to \mathbb{R}$. Note that according to the isomorphism $\mathscr{F}$, the solution of the operator Riccati equation (5.2.7) is given by the positive self-adjoint operator $Q \in \mathscr{L}(X)$ defined by $Q = \mathscr{F}^{-1}\tilde{Q}\mathscr{F}$. Now take any $\overline{\xi} \in X$. It holds that

$$\begin{aligned} Q\overline{\xi} = \mathscr{F}^{-1}\tilde{Q}\mathscr{F}\overline{\xi} &= \mathscr{F}^{-1}\tilde{Q}\{\langle \overline{\xi}, \phi_n \rangle_X\}_{n\in\mathbb{N}} \\ &= \mathscr{F}^{-1}(q_{00}^+ \langle \overline{\xi}, \phi_0 \rangle_X \quad 0 \quad 0 \quad \ldots)^T \\ &= q_{00}^+ \langle \overline{\xi}, \phi_0 \rangle_X \phi_0 = q_{00}^+ 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz. \end{aligned}$$

It is easy to see that the operator Q is self-adjoint. Moreover, observe that

$$\langle Q\overline{\xi}, \overline{\xi} \rangle_X = q_{00}^+ \langle 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz, \overline{\xi} \rangle_X = q_{00}^+ \left(\int_0^1 \overline{\xi}(z) dz \right)^2 \geq 0,$$

which means that the operator Q is positive. Furthermore, let us consider $\overline{\xi} \in D(A)$, see (4.4.3). It is easy to see that $Q\overline{\xi} \in D(A^*) = D(A)$. Besides it holds that

$$\begin{aligned} &\mathscr{A}^* Q\overline{\xi} + Q\mathscr{A}\overline{\xi} + \rho_1 C^* C\overline{\xi} - \frac{1}{\rho_2} Qbb^* Q\overline{\xi} \\ &= \frac{d^2 Q\overline{\xi}}{dz^2} + Q\frac{d^2\overline{\xi}}{dz^2} + \sqrt{2}Q\overline{\xi} + \rho_1 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz - \frac{1}{\rho_2} Qbb^* q_{00}^+ 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz \\ &= \frac{d^2}{dz^2} \left(q_{00}^+ 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz \right) + q_{00}^+ 1_{[0,1]} \int_0^1 \frac{d^2\overline{\xi}}{dz^2} dz + \sqrt{2}Q\overline{\xi} \\ &\qquad + \rho_1 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz - \frac{1}{\rho_2} q_{00}^+ Q \left(1_{[0,1]} \int_0^1 \overline{\xi}(z) dz \right) \\ &= (\sqrt{2} q_{00}^+ + \rho_1 - \frac{1}{\rho_2} (q_{00}^+)^2) 1_{[0,1]} \int_0^1 \overline{\xi}(z) dz = 0, \end{aligned}$$

where the definition of $D(A)$ and relation (5.3.9) have been used. The corresponding state feedback $K : X \to \mathbb{R}$ takes the form $K = -\frac{1}{\rho_2} b^* Q$, i.e. $K\overline{\xi} = -\frac{1}{\rho_2} q_{00}^+ \int_0^1 \overline{\xi}(z) dz$, with $\overline{\xi} \in D(A)$, which may also be found as $K = \tilde{k}\mathscr{F}$.

It remains to show that once connected to the dynamics (5.3.4) the state feedback K stabilizes the latter on Y. Notice that the eigenvalues of the operator $\mathscr{A}$, namely $\{\lambda_n\}_{n\in\mathbb{N}} = \{-n^2\pi^2 + \frac{1}{\sqrt{2}}\}_{n\in\mathbb{N}}$, satisfy the following inequality

$$\sup_{m\in\mathbb{N}} \sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{1}{|\lambda_n - \lambda_m|^2} \leq \frac{1}{6\pi^2},$$

see Abouzaïd et al. (2021) for instance. Consequently, the operator $\mathscr{A} + bK$ is a self-adjoint Riesz-spectral operator whose eigenvalues are given by [8]

$$\{\lambda_n^o\}_{n\in\mathbb{N}} = \{-\sqrt{\lambda_0^2 + \frac{\rho_1}{\rho_2}}\} \cup \{-n^2\pi^2 + \frac{1}{\sqrt{2}}\}_{n\in\mathbb{N}_0}.$$

[8] In order to avoid eigenvalues with larger multiplicities than 1, the condition $\frac{\rho_1}{\rho_2} \neq n^4\pi^4 - \sqrt{2}n^2\pi^2$ has to be imposed for any $n \geq 1$.

The corresponding eigenfunctions are the same as the one of the operator $\mathscr{A}$, i.e. $\{\phi_n^o\}_{n\in\mathbb{N}} = \{1_{[0,1]}(z)\}\cup\{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}_0}$ and form an orthonormal basis of X. The series in (5.2.9) becomes in our context

$$\sum_{n=0}^{\infty}\frac{\|\phi_n^o\|_X\|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|} = \sqrt{2}\sum_{n=0}^{\infty}\frac{1}{|\lambda_n|} = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\frac{\rho_1}{\rho_2}}} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-\frac{1}{\sqrt{2}}},$$

which is a convergent series. Consequently, the state feedback K stabilizes exponentially the linear dynamics (5.3.4) on the space Y according to Proposition 5.2.1.

We are now able to state the main result of this section, which makes the link between the stability of the linearized system (5.3.4) with the tracking of the reference profile $r(z)$ for the nonlinear system (5.3.1).

Theorem 5.3.1 *The optimal control law $u(t) = \tilde{u}(t) - \sqrt{2}$ with $\tilde{u}(t)$ being the state feedback $\tilde{u}(t) = -\frac{1}{\rho_2}q_{00}^{+}\int_0^1 \xi(z,t)dz$, stabilizes locally and exponentially the nonlinear system (5.3.1) around the reference profile $r(z) = 1_{[0,1]}(z)$.*

Proof. See above in this Section and Theorem 4.3.1. □

Despite the control law $u(t)$ is locally stabilizing for (5.3.1) and optimal for the corresponding linearized system, it is not necessarily optimal for the nonlinear system (5.3.1), for the same cost function. Similar arguments as those presented in Ikeda and Šiljak (1990) and Aksikas (2005) could be used to study the inverse optimality of $u(t)$ for the nonlinear system (5.3.1), i.e., to see which cost should be minimized by $u(t)$ for the nonlinear system (5.3.1). This question will not be tackled in this thesis.

5.3.2 Discussion on the global stability of the closed-loop system

Here we shall see that the optimal state feedback computed as the solution of (5.3.5) stabilizes even globally the nonlinear system (5.3.2) with the chosen parameters. However, another reference profile, which is different from $r(z) = 1_{[0,1]}(z)$, and specific parameters will be chosen to illustrate that this is not always the case.

5.3.2.1 Global exponential stability with the reference profile $r(z) = 1_{[0,1]}(z)$

Observe that, since the operator $\frac{d^2}{dz^2} - \frac{1}{\rho_2}q_{00}^{+}1_{[0,1]}\int_0^1$ is a Riesz-spectral operator with eigenvalues $\{-\frac{1}{\rho_2}q_{00}^{+}\}\cup\{-n^2\pi^2\}_{n\in\mathbb{N}_0} =: \{\kappa_n\}_{n\in\mathbb{N}}$, the semigroup generated by this operator, denoted $(\mathbb{T}(t))_{t\geq 0}$, satisfies

$$\mathbb{T}(t)\xi_0 = \sum_{n=0}^{\infty} e^{\kappa_n t}\langle\xi_0,\phi_n\rangle_X\phi_n,$$

for any initial condition $\xi_0 \in X$. Its norm satisfies the estimate

$$\|\mathbb{T}(t)\xi_0\|_X^2 = \sum_{n=0}^{\infty} e^{2\kappa_n t}\langle\xi_0,\phi_0\rangle_X^2 \leq e^{2\sup_{n\in\mathbb{N}}\{\kappa_n\}t}\sum_{n=0}^{\infty}\langle\xi_0,\phi_0\rangle_X^2 = e^{2\sup_{n\in\mathbb{N}}\{\kappa_n\}t}\|\xi_0\|_X^2.$$

Owing to this fact, one may express the mild solution of the PDE (5.3.2) with initial condition ξ_0 as

$$\xi(t) = \mathbb{T}(t)\xi_0 + \int_0^t \mathbb{T}(t-s)[\sqrt{(\xi(s)+1)^2+1} - \sqrt{2}]ds. \tag{5.3.10}$$

Taking the norm of both sides of (5.3.10) and using the notation[(9)] $\sup_{n\in\mathbb{N}}\{\kappa_n\} := \kappa^* < -1$ imply that

$$\begin{aligned}\|\xi(t)\|_X &\leq e^{\kappa^* t}\|\xi_0\|_X + \int_0^t e^{\kappa^*(t-s)}\|\sqrt{(\xi(s)+1)^2+1} - \sqrt{2}\|_X ds\\ &\leq e^{\kappa^* t}\|\xi_0\|_X + \int_0^t e^{\kappa^*(t-s)}\|\xi(s)\|_X ds,\end{aligned}$$

where the Lipschitz continuity of the nonlinear operator $\tilde{f}\cdot = \sqrt{(\cdot+1)^2+1} - \sqrt{2}$ has been used. Applying Gronwall's inequality to the function $e^{-\kappa^* t}\|\xi(t)\|_X$ yields

$$e^{-\kappa^* t}\|\xi(t)\|_X \leq e^t\|\xi_0\|_X,$$

or equivalently

$$\|\xi(t)\|_X \leq e^{(\kappa^*+1)t}\|\xi_0\|_X,$$

which shows the global exponential stability of the closed-loop nonlinear system (5.3.2).

This fact could lead us to think that the proposed method based on the extended Fréchet differentiability conditions is not useful since it can be shown in another easier way that the linear feedback stabilizes globally the nonlinear dynamics. However, this is not true and will be discussed in the next section.

5.3.2.2 Changes in the parameters

As another reference profile, let us consider $r(z) = \frac{1}{4}1_{[0,1]}(z), z \in [0,1]$. We shall see that the conclusion highlighted in the previous section does not hold necessarily.

Indeed, for this new reference, the corresponding input steady-state value u^e, which is the solution of (5.3.1) at the equilibrium, is given by $u^e = -\sqrt{\frac{17}{16}}$. Consequently the nonlinear "around equilibrium" system is written as

$$\left\{\begin{array}{l}\frac{\partial\xi}{\partial t} = \frac{\partial^2\xi}{\partial z^2} + \sqrt{(\xi+\frac{1}{4})^2+1} - 1_{[0,1]}(z)\sqrt{\frac{17}{16}} + b(z)\tilde{u}(t),\\ \frac{\partial\xi}{\partial z}(0,t) = 0 = \frac{\partial\xi}{\partial z}(1,t),\end{array}\right. \tag{5.3.11}$$

where the variable $\xi(z,t) = x(z,t) - r(z)$. Note that the nonlinear operator $\tilde{f}$ defined by $\sqrt{(\xi+\frac{1}{4})^2+1} - \sqrt{\frac{17}{16}}$ is still uniformly Lipschitz continuous with a Lipschitz

(9) The relation $\kappa^* < -1$ holds true since the eigenvalue $-\frac{1}{\rho_2}q_{00}^+ = -\lambda_0 - \sqrt{\lambda_0^2 + \frac{\rho_1}{\rho_2}} < -2\lambda_0 = -\sqrt{2}$ and all others eigenvalues $\{-n^2\pi^2\}_{n\in\mathbb{N}_0}$ are less than -1.

constant given by $l_{\tilde{f}} := \sup_{\xi\in\mathbb{R}} |\tilde{f}'(\xi)| = 1$. The corresponding linearized equations around $\xi = 0$ are given by

$$\begin{cases} \frac{\partial \overline{\xi}}{\partial t} = \frac{\partial^2 \xi}{\partial z^2} + \frac{1}{\sqrt{17}}\overline{\xi} + b\tilde{u}(t), \\ \frac{\partial \overline{\xi}}{\partial z}(0,t) = 0 = \frac{\partial \overline{\xi}}{\partial z}(1,t). \end{cases} \tag{5.3.12}$$

As it has been done for the previous reference profile, it can be shown by using similar arguments that the optimal feedback operator K that minimizes the cost functional

$$J(\xi_0, \tilde{u}) = \int_0^\infty (\rho_1 \langle c, \overline{\xi}(t)\rangle_X^2 + \rho_2 \tilde{u}^2(t))dt$$

is given by the bounded linear operator $K : X \to \mathbb{R}$ expressed as

$$K\overline{\xi} = -\frac{1}{\rho_2}\hat{q}_{00}^+ \int_0^1 \overline{\xi}(z)dz, \tag{5.3.13}$$

where $\hat{q}_{00}^+$ is the positive solution of the following scalar Riccati equation

$$\frac{1}{\rho_2}(\hat{q}_{00}^+)^2 - \frac{2}{\sqrt{17}}\hat{q}_{00}^+ + \rho_1 = 0,$$

which is given by $\hat{q}_{00}^+ = \left(\frac{1}{\sqrt{17}} + \sqrt{\frac{1}{17} + \frac{\rho_1}{\rho_2}}\right)\rho_2$. Consequently the closed-loop nonlinear system is expressed as

$$\begin{cases} \frac{\partial \xi}{\partial t} = \frac{\partial^2 \xi}{\partial z^2} + \sqrt{(\xi + \frac{1}{4})^2 + 1} - 1_{[0,1]}\sqrt{\frac{17}{16}} - \left(\frac{1}{\sqrt{17}} + \sqrt{\frac{1}{17} + \frac{\rho_1}{\rho_2}}\right) 1_{[0,1]} \int_0^1 \xi dz, \\ \frac{\partial \xi}{\partial z}(0,t) = 0 = \frac{\partial \xi}{\partial z}(1,t), \end{cases} \tag{5.3.14}$$

and its solution is known to be locally exponentially stable according to Theorem 4.3.1. As it was pointed out before, the operator

$$\frac{d^2}{dz^2} - \left(\frac{1}{\sqrt{17}} + \sqrt{\frac{1}{17} + \frac{\rho_1}{\rho_2}}\right) 1_{[0,1]} \int_0^1 =: \mathbb{A}, \tag{5.3.15}$$

defined on $D(\mathbb{A}) = D(A)$, is a Riesz-spectral operator whose spectrum is composed of only the eigenvalues $\{\kappa_n\}_{n\in\mathbb{N}} = \{-\frac{1}{\sqrt{17}} - \sqrt{\frac{1}{17} + \frac{\rho_1}{\rho_2}}\} \cup \{-\pi^2 n^2\}_{n\in\mathbb{N}_0}$. The semigroup that is generated by this operator is denoted by $(\mathbb{T}(t))_{t\geq 0}$. Let us choose for instance the parameters $\rho_1 = 3$ and $\rho_2 = 17$. In this way the largest eigenvalue of the operator $\mathbb{A}$ is given by $\kappa^* = \frac{-3}{\sqrt{17}} \simeq -0.7276$. Observe now that the mild solution of (5.3.14) is given by

$$\xi(t) = \mathbb{T}(t)\xi_0 + \int_0^t \mathbb{T}(t-s)[\sqrt{(\xi(s) + \frac{1}{4})^2 + 1} - \sqrt{\frac{17}{16}}]ds,$$

which implies that the estimate one may perform on the norm of ξ is $\|\xi(t)\|_X \leq e^{(\kappa^*+1)t}\|\xi_0\|_X$. As the quantity $\kappa^* + 1$ is positive, this does not give any information about global exponential stability.

5.3.3 Numerical simulations

Here we aim at illustrating the exponential convergence of the state trajectory x of the nonlinear system (5.3.1) to the constant reference profile $r(z)$. The influence of the parameters ρ_1 and ρ_2 on the state x and the output u will also be depicted.

5.3.3.1 Fixed values of the weight parameters

As parameters for the numerical simulations, we choose the reference profile $r(z) = \frac{1}{4}1_{[0,1]}(z)$. The method that is used to compute the state trajectory consists in discretizing the space variable z into n equal pieces, $n = 40$. The diffusion operator with Neumann boundary conditions has been discretized by means of finite differences, resulting in a matrix $D_n \in \mathbb{R}^{n\times n}$. The set of resulting n ordinary differential equations, expressed as

$$\dot{\xi}_n(t) = D_n\xi_n(t) + \tilde{f}(\xi_n(t)) + b_n\tilde{u}(t), \tag{5.3.16}$$

with $\xi_n(t) = (\xi(0,t), \xi(h,t), \dots, \xi(1,t))^T \in \mathbb{R}^n$ and $b_n = (1, \dots, 1)^T$ has then been linearized, which yields the linear finite-dimensional system

$$\dot{\overline{\xi}}_n(t) = D_n\overline{\xi}_n(t) + \frac{1}{\sqrt{17}}\overline{\xi}_n(t) + b_n\tilde{u}(t).$$

Based on this linearization, the optimal control input $\tilde{u}(t)$, solution of (5.3.5), has been computed. This has been performed by using the routine `lqr` of Matlab©, for which it has been observed that it produces the same optimal control as the one found analytically in (5.3.13). Then the resulting set of n nonlinear ordinary differential equations (5.3.16) is integrated via the routine `ode23s` of Matlab©. As initial condition for the x variable, we have chosen the function $x_0(z) = 4z^3 - 6z^2 + 1$. It is easy to show that this function satisfies the boundary conditions $\frac{dx_0}{dz}(0) = 0 = \frac{dx_0}{dz}(1)$ and is sufficiently regular such that it lies in $D(A)$.

As parameters for the penalties on the state and on the input, we fixed $\rho_1 = 3$ and $\rho_2 = 17$.

The open-loop state trajectory $x(z,t)$ of (5.3.1) with $u(t) \equiv 0$ is shown in Figure 5.1. The closed-loop state trajectory $x(z,t)$ solution of (5.3.1) with $u(t) = K(x(t) - r) - \sqrt{r^2+1}$ where K is given in (5.3.13) is represented in Figure 5.2. Therein it can be observed that the reference profile $r(z) = \frac{1}{4}1_{[0,1]}(z)$ is reached.

Moreover, the $X-$norm of the difference between the state x and the reference profile r is depicted in Figure 5.3. The exponential decay of that quantity may be seen. Besides, the corresponding control input $u(t) = \tilde{u}(t) + u^e$ is represented in Figure 5.4 where the steady-state value of the control u^e is given by $-\sqrt{r^2+1}$.

5.3.3.2 Variations in the weight parameters

Now we shall illustrate the influence of the weighting parameters ρ_1 and ρ_2. Therefore, we fixed the value of ρ_2 to 1 and we considered three values for the parameter ρ_1, namely $1, 6$ and 72. The three corresponding state trajectories are depicted in Figure 5.5. In addition, the $X-$norm of the error between x and r and the optimal control

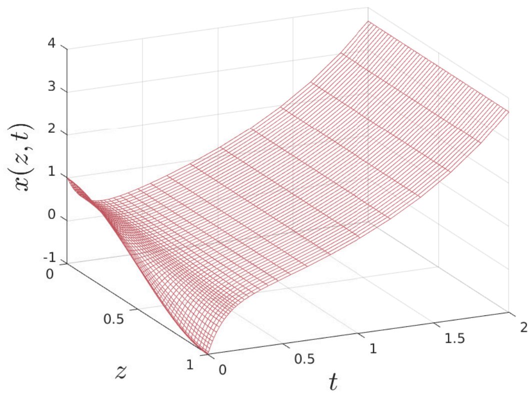

Figure 5.1 – State trajectory x of the open-loop system (5.3.1) with $u(t) \equiv 0$.

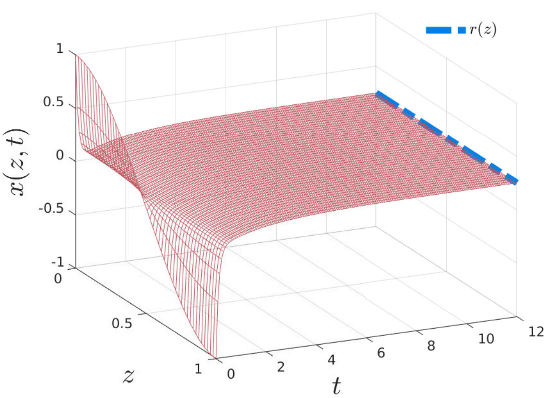

Figure 5.2 – State trajectory x of the closed-loop system (5.3.1) with the optimal control law $u(t) = K(x(t) - r) - \sqrt{r^2 + 1}$, where K is defined in (5.3.13).

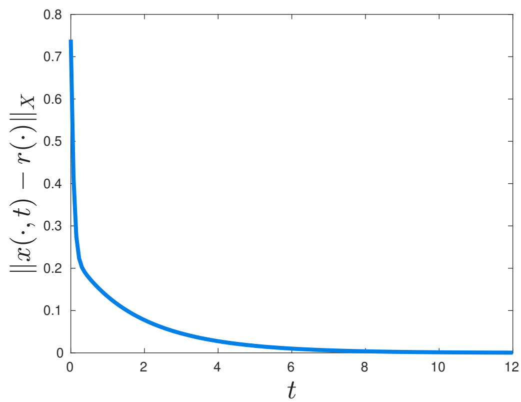

Figure 5.3 – $X-$norm of the deviation variable $x(z,t)-r(z)$ as a function of t.

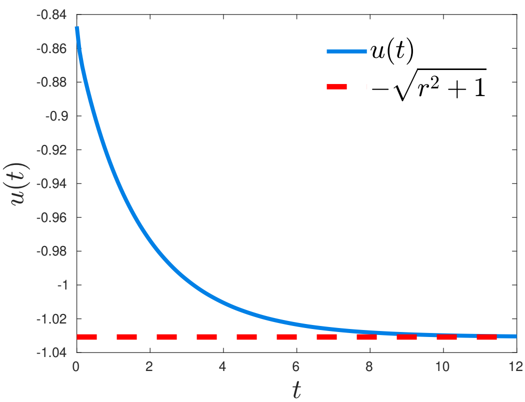

Figure 5.4 – Optimal control law $u(t) = K(x(t)-r) - \sqrt{r^2+1}$, where K is defined in (5.3.13).

inputs are given in Figures 5.6 and 5.7, respectively. It can be noticed that the larger the parameter ρ_1 is, the faster the input and the error go to 0. This can be explained by the fact that the quantity $-\frac{1}{\sqrt{17}} - \sqrt{\frac{1}{17} + \frac{\rho_1}{\rho_2}}$ is an eigenvalue of the operator $\mathbb{A}$ given in (5.3.15). For the 3 values of ρ_1, this eigenvalue is given by $-1.2715, -2.7040$ and -8.7313, successively. In particular, for these parameters, this eigenvalue is the largest of the operator $\mathbb{A}$ since the others are given by $\{-n^2\pi^2\}_{n\in\mathbb{N}_0}$.

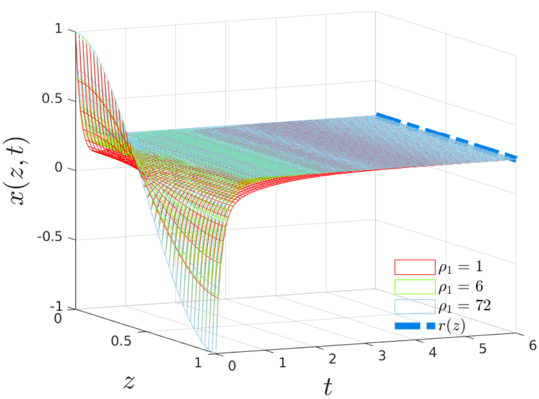

Figure 5.5 – State trajectory x of the closed-loop system (5.3.1) with $u(t) = K(x(t) - r) - \sqrt{r^2+1}$, where K is defined in (5.3.13). The trajectory is depicted for any parameters $\rho_1 = 1, \rho_1 = 6, \rho_1 = 72$, each time with $\rho_2 = 1$.

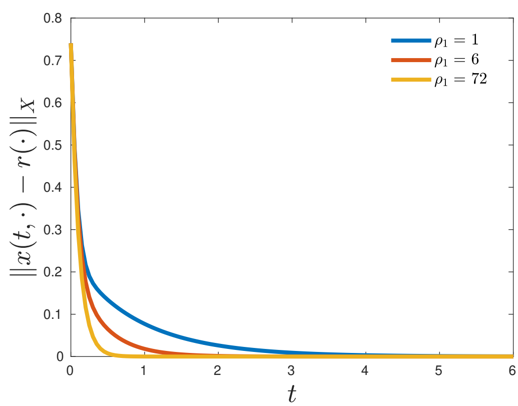

Figure 5.6 – $X-$norm of the deviation variable $x(z,t) - r(z)$ as a function of t for any parameters $\rho_1 = 1, \rho_1 = 6, \rho_1 = 72$, each time with $\rho_2 = 1$.

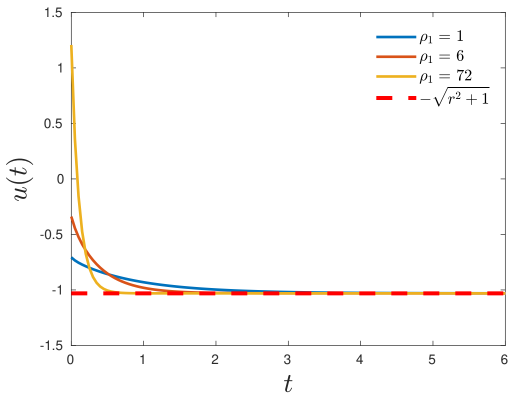

Figure 5.7 – Optimal control law $u(t) = K(x(t) - r) - \sqrt{r^2 + 1}$, where K is defined in (5.3.13), for the parameters $\rho_1 = 1, \rho_1 = 6, \rho_1 = 72$, each time with $\rho_2 = 1$.

Chapter 6

Control of a structurally known plug-flow reactor

Contents

This chapter is more application oriented since it is mostly dedicated to the regulation of the temperature inside a plug-flow reactor where no diffusion phenomenon is considered. As presented in Chapter 1, the control variable that is under consideration is the heat exchanger temperature. This chapter aims at constructing a controller which stabilizes globally and exponentially the dynamics of a plug-flow reactor while regulating an output function at some constant reference. This is the main reason why it is different from Chapters 4 and 5 and can be viewed as an extension of the methodology developed therein, where only local stability and stabilization are considered.

A generalization of the proportional-integral (PI) adaptive control will be developed in order to tackle the nonlinear aspects of the model that is under consideration. In particular, the idea is to develop an integral adaptive control law perturbed by a nonlinear term that enables to write the time derivative of some Lyapunov function as the time derivative of a Lyapunov function corresponding to a linear system. The control objective is on one hand the convergence of a scalar output which corresponds to an averaged temperature in the reactor towards a reference temperature. On the other hand, it is required that the system trajectories (trajectories around the equilibrium) are exponentially stable. The proposed control law will be shown to fulfill the control objective despite the known disturbances applied to the system. The condition for the feasability of the control objective is expressed as an inequality on the integral gain in relation with the system parameters, more specifically what could be called a small-gain condition.

Numerical simulations will be performed to show the effectiveness of the proposed method while it shows also the feasability of the control heat exchanger temperature despite the fact that no constraints are imposed on the latter.

The field of control which is of interest in this chapter is closely related to adaptive control, for which a brief review of the literature is given in the next section.

6.1 Introduction on adaptive control

Adaptive control of dynamical systems, both finite or infinite-dimensional, is a kind of control which adapts to the system parameters or to uncertainties. It has been widely studied in the literature in the last decades. Such techniques are deeply considered in Krstic et al. (1995a) for nonlinear finite-dimensional systems. We refer to Logemann and Townley (1997) to get an overview of adaptive control for infinite-dimensional systems, where high-gain, low-gain and switching controllers are considered.

6.1.1 Proportional Integral Derivative control

A very often used and popular field of adaptive control is related to proportional–integral (PI), proportional–derivative (PD) or proportional–integral–derivative (PID) control. A large class of processes in the industry are monitored by means of such control techniques.

A good reference for multivariable PI control of linear infinite-dimensional systems is Pohjolainen (1982) wherein a small gain condition needs to hold in order to get a robust stabilizing PI controller that tracks constant reference signals. Moreover it is assumed that the operator dynamics generates a holomorphic semigroup on the considered Banach state space. The question of PI regulation has also been studied in Logemann and Zwart (1992) for linear multivariable infinite-dimensional minimum-phase systems with a property of "relative-degree" one. The authors showed that under certain high-gain conditions, the PI controller yields a stable closed-loop system while it tracks asymptotically constant reference signals. The controller also ensures robustness properties such as perturbations induced by nonlinearities in the feedback loop. Few years later, the authors of Xu and hamadi Jerbi (1995) generalized the ideas of Pohjolainen (1982) to single-input, single-output linear systems whose operator dynamics does not necessarily generates a holomorphic semigroup, covering a larger class of linear infinite-dimensional systems, for instance systems governed by hyperbolic partial differential equations (PDEs). The case of infinite-dimensional linear regular systems subject to an input nonlinearity is studied in Logemann and

Adam (2001). Extension of the results of Xu and hamadi Jerbi (1995) for multi-input multi-output systems has been studied more recently in Boulite et al. (2009).

Attention has also been paid in the literature to PI control of nonlinear systems. In Martins et al. (2014) a PI controller has been designed for a nonlinear distributed parameter system derived from the hyperbolic Saint-Venant PDEs. The stability of the closed-loop system is studied by means of operator techniques. Later in Guiver et al. (2017) low–gain integral control is studied for nonlinear finite-dimensional systems, where the nonlinearity enters in the system via the input. Note that the Saint-Venant equations have also been studied in Trinh et al. (2017) and Trinh et al. (2015) for which PI control is considered on a linearized version of the model. The input and output variables are supposed to be on the boundary of the considered domain.

Adaptive control via PI, PD or PID is also established in Song (2018).

6.1.2 Adaptive control for chemical processes

Adaptive control methods are also often used in chemical process engineering, and in particular with attention to the control of chemical/bio reactors. The on-line estimation and adaptive control of such systems are deeply studied in the book Bastin and Dochain (1990) for instance.

6.1.2.1 Extremum Seeking Control (ESC)

A field of adaptive control which is developing a lot and often used for chemical processes is known as extremum-seeking-control (ESC). Roughly speaking it consists in optimizing an objective function depending on the state of a dynamical system while stabilizing the corresponding system and estimating unknown parameters. For instance ESC has been applied to an infinite-dimensional system consisting in the dynamics of a plug-flow tubular reactor with unknown reaction kinetics in Hudon et al. (2005). Therein, the on-line optimization of a product concentration component for the Williams-Otto reaction is studied. Few years later, a similar controller has been developed in Hudon et al. (2008) for the Van der Vusse chemical reaction.

An overview of ESC applied to chemical processes is given in Dochain et al. (2011).

6.1.2.2 PI control for chemical processes

Chemical processes have also attracted PI control or output feedback regulation such as in Jadot et al. (1999) where a robust saturated output feedback is designed on a stirred-tank reactor. Under some mild assumption, a temperature control with classical PI compensation is used in Alvarez-Ramirez and Puebla (2001) to stabilize the dynamics of a chemical reactor. For that kind of devices, PI cascade control has been shown to regulate the outlet concentration in Urrea et al. (2008). A general way of controlling infinite-dimensional systems described by hyperbolic PDEs via a PI controller is described and developed in Aguilar-Garnica et al. (2011), where particular applications in process engineering are studied. The authors study first the regulation of the temperature in a heat exchanger and then regulate the outlet concentration in a nonisothermal plug-flow tubular reactor. In Nájera et al. (2016), feed and temperature measurments ensure the design of a robust output feedback controller for a highly exothermic gas-phase packed-bed tubular reactor. The regulation of the temperature in an axial dispersion plug-flow tubular reactor by means of an adaptive controller based on partial temperature measurments has been described and developed in Beniich et al. (2017). More specifically the authors steer the temperature in the reactor in a ball of some prescribed and arbitrary small radius around an a priori temperature reference. Plug-flow tubular reactors have also been considered in Zárate-Navarro et al. (2019) where a PI controller with variable coefficients has been designed and where dissipation conditions together with the second law of thermodynamics have been used.

The question of integral control for a plug-flow tubular reactor is considered in the next three sections. It generalizes the classical PI control action and incorporates an additional term that aims at managing the nonlinear nature of the model.

6.2 Problem description

Let us recall that the time evolution of the temperature and the concentration in a nonisothermal plug-flow tubular reactor are governed by the following set of partial differential equations (PDEs)[(1)], see (1.2.1) where $\lambda_{ea} = 0$ and $D_{ma} = 0$:

$$\begin{cases} \partial_\tau T(\zeta,\tau) = -v\partial_\zeta T(\zeta,\tau) - \frac{\Delta H}{\rho C_p} k_0 C(\zeta,\tau) e^{\frac{-E}{RT(\zeta,\tau)}} \\ \qquad\qquad + \frac{4h}{\rho C_p d}(1_{[0,L]}(\zeta) T_w(\tau) - T(\zeta,\tau)) + \frac{v}{\varepsilon_T} 1_{[0,\varepsilon_T]}(\zeta) w_T(\tau), \\ \partial_\tau C(\zeta,\tau) = -v\partial_\zeta C(\zeta,\tau) - k_0 C(\zeta,\tau) e^{\frac{-E}{RT(\zeta,\tau)}} + \frac{v}{\varepsilon_C} 1_{[0,\varepsilon_C]}(\zeta) w_C(\tau), \end{cases} \tag{6.2.1}$$

where $T(\zeta,\tau)$ and $C(\zeta,\tau)$ represent the temperature and the concentration at time $\tau \in [0,+\infty)$ and position $\zeta \in [0,L]$, respectively. The parameters $\varepsilon_T \in [0,L]$ and $\varepsilon_C \in [0,L]$ represent the widths of the windows on which w_T and w_C act, respectively. The perturbations $w_T[K]$ and $w_C[\frac{mol}{l}]$ are assumed to be constant and model uncertainties on the temperature and concentration at the inlet of the reactor, respectively. The meaning and the units of the parameters in (6.2.1) are summarized in Table 1.1.

For the meaning of the variables T_w and for the physical constraints on the variables T and C, we refer to Chapter 1, Section 1.2.

As the parameters λ_{ea} and D_{ma} are set to 0 here, the boundary conditions (1.2.3) becomes

$$T(0,\tau) = T_{in}, C(0,\tau) = C_{in}, \tag{6.2.2}$$

for $\tau \in [0,+\infty)$. In order to work with a dimensionless model, we perform the changes of variables (2.3.2) to get the following set of PDEs

$$\begin{cases} \partial_t \theta_1(z,t) = -\partial_z \theta_1(z,t) + \delta f(\theta_1,\theta_2) + \beta(1_{[0,1]}(z)\theta_w(t) - \theta_1(z,t)) \\ \qquad\qquad + \frac{L}{\varepsilon_T} 1_{[0,\varepsilon_T/L]}(z)(1+d_T), \\ \partial_t \theta_2(z,t) = -\partial_z \theta_2(z,t) + f(\theta_1,\theta_2) - \frac{L}{\varepsilon_C} 1_{[0,\varepsilon_C/L]}(z)(1-d_C), \end{cases} \tag{6.2.3}$$

[(1)]The notations ∂_τ and ∂_ζ stand for the operators $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\zeta}$, respectively.

where the nonlinear function $f(x,y) := \alpha(1-y)e^{\frac{\mu x}{1+x}}$ for $x \geq -1$ and $y \in [0,1]$. The parameters δ, μ, α and β are given by (2.3.4) while the constants d_T and d_C are given by $d_T = (w_T - T_{in})/T_{in}$ and $d_C = (C_{in} - w_C)/C_{in}$, respectively. We shall adopt the notations $\eta_T := \frac{\varepsilon_T}{L}$ and $\eta_C := \frac{\varepsilon_C}{L}$ in what follows. The boundary conditions (6.2.2) are now expressed as

$$\theta_1(0,t) = 0, \theta_2(0,t) = 0. \tag{6.2.4}$$

For the model (6.2.3) with (6.2.4) we assume to have access to pointwise measurments of the dimensionless temperature. The place where the measure is collected is at the outlet of the reactor, i.e., at $z = 1$. Therefore let us define the output function as $y(t) := \theta_1(t,1)$. The PDEs (6.2.3) together with the boundary conditions (6.2.4) and the output y may be written as the abstract controlled and observed Cauchy problem

$$\begin{cases} \dot{\theta}(t) = A\theta(t) + F(\theta(t)) + Bu(t) + \Gamma w(t), \\ y(t) = C\theta(t), \\ \theta(0) = \theta_0, \end{cases} \tag{6.2.5}$$

where the associated state space is $X := L^2([0,1];\mathbb{R}) \times L^2([0,1];\mathbb{R})$. The unbounded linear operator A is defined by $A\theta = \begin{pmatrix} -d_z\theta_1 - \beta\theta_1 & 0 \\ 0 & -d_z\theta_2 \end{pmatrix}$ for $\theta := \begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix}$ in $D(A)$ expressed as

$$D(A) := \{\theta \in Y, \theta_1(0) = 0, \theta_2(0) = 0\},$$

with $Y := H^1([0,1];\mathbb{R}) \times H^1([0,1];\mathbb{R})$. The nonlinear operator $F : D(F) \subset X \to X$ is given by $\begin{pmatrix} \delta f(\theta_1,\theta_2) & f(\theta_1,\theta_2) \end{pmatrix}^T$ for θ in the closed and convex subset $D(F)$ defined by $\{\theta \in X, -1 \leq \theta_1(z), 0 \leq \theta_2(z) \leq 1\}$. The control operator $B : \mathbb{R} \to X$ distributes the scalar input u along the reactor through the characteristic function $1_{[0,1]}(z)$. It is thus expressed as $Bu = \begin{pmatrix} \beta 1_{[0,1]}(z)u & 0 \end{pmatrix}^T$. The control input is the heat exchanger temperature $\theta_w(t)$. According to the definition of the output function, the unbounded output operator $C : X \to \mathbb{R}$ is such that $C\theta = \theta_1(1)$. The perturbations in the dynamics (6.2.3) are modeled via the operator Γ, which is given as $\Gamma : \mathbb{R}^2 \to X, \Gamma w = \begin{pmatrix} \frac{1}{\eta_T}1_{[0,\eta_T]}(z)w_1 & -\frac{1}{\eta_C}1_{[0,\eta_C]}(z)w_2 \end{pmatrix}^T$, with $w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^T$, where the components $w_1 := 1 + d_T$ and $w_2 := 1 - d_C$.

Note that the homogeneous part of (6.2.5), characterized by the operator $A + F$ is known to possess a unique mild solution, see Laabissi et al. (2001); Winkin et al. (2000) among others and Chapter 2, Section 2.3. In terms of semigroups, it means that $A + F$ generates a semigroup of nonlinear operators on $D(A) \cap D(F)$. Connecting inputs and perturbations yields the following expression for the mild solution of (6.2.5)

$$\theta(t) = T(t)\theta_0 + \int_0^t T(t-s)F(\theta(s))ds + \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)\Gamma w(s)ds,$$

where $(T(t))_{t\geq 0}$ denotes the linear semigroup generated by the operator A.

We end this section by characterizing the equilibria of (6.2.3). An equilibrium triplet $(\theta_1^e, \theta_2^e, u^e)$ of (6.2.3) is the solution of the following set of nonlinear differential equations (ODEs)[2]

$$\begin{cases} d_z\theta_1^e(z) = -\beta\theta_1^e(z) + \delta f(\theta_1^e,\theta_2^e) + \beta 1_{[0,1]}(z)u^e + \frac{1}{\eta_T}1_{[0,\eta_T]}(z)(1+d_T), \\ d_z\theta_2^e(z) = f(\theta_1^e,\theta_2^e) - \frac{1}{\eta_C}1_{[0,\eta_C]}(z)(1-d_C), \\ \theta_1^e(0) = 0, \theta_2^e(0) = 0. \end{cases} \tag{6.2.6}$$

Note that once the equilibrium u^e is fixed, the set of ODEs (6.2.6) possesses a unique solution since only initial conditions are taken into account. The notation $u^e := \theta_w^e$ has been used.

6.3 Control description

In this section we focus on the construction of an adaptive controller whose objective is to steer the output of the system (6.2.3) to a predetermined value while it has to stabilize the state trajectory of (6.2.3) in some sense.

In order to be able to prove asymptotic regulation of the output of (6.2.3) we shall approximate the output operator C by an operator which is bounded on the state space we are working in, X. Let us consider $\rho(A) \ni \lambda > 0$, sufficiently large. We introduce the operator $C_\lambda^1 : D(A) \to \mathbb{R}$ as follows

$$C_\lambda^1\theta = \lambda\int_0^1 e^{-\lambda(1-z)}\theta_1(z)dz =: C_\lambda\theta_1, \tag{6.3.1}$$

where the operator C_λ works only on the first component of θ. It is easy to see that C_λ is bounded since according to the Cauchy-Schwarz inequality we have

$$\begin{aligned} |C_\lambda\theta_1| \leq \lambda\|e^{-\lambda(1-\cdot)}\|_{L^2}\|\theta_1\|_{L^2} &= \lambda\left(\int_0^1 e^{-2\lambda(1-z)}dz\right)^{\frac{1}{2}}\|\theta_1\|_{L^2} \\ &= \sqrt{\frac{\lambda(1-e^{-2\lambda})}{2}}\|\theta_1\|_{L^2}. \end{aligned}$$

Moreover, performing an integration by parts on (6.3.1) yields that

$$\begin{aligned} C_\lambda\theta_1 &= \left[\theta_1(z)e^{-\lambda(1-z)}\right]_0^1 - \int_0^1 e^{-\lambda(1-z)}\frac{d\theta_1}{dz}dz \\ &= \theta_1(1) - e^{-\lambda}\theta_1(0) - \int_0^1 e^{-\lambda(1-z)}\frac{d\theta_1}{dz}dz. \end{aligned} \tag{6.3.2}$$

The last term of the previous equation can be bounded as

$$\left|\int_0^1 e^{-\lambda(1-z)}\frac{d\theta_1}{dz}dz\right| \leq \left(\int_0^1 e^{-2\lambda(1-z)}dz\right)^{\frac{1}{2}}\|\frac{d\theta_1}{dz}\|_{L^2} = \sqrt{\frac{1-e^{-2\lambda}}{2\lambda}}\|\frac{d\theta_1}{dz}\|_{L^2},$$

[2]The notation d_z stands for the operator $\frac{d}{dz}$.

which is well defined since functions on the domain of A are also in Y. Consequently taking the limit for λ going to ∞ both sides of (6.3.2) entails that

$$\lim_{\lambda\to\infty} C_\lambda \theta_1 = \theta_1(1),$$

which means that the limit for λ going to ∞ of (6.3.1) coincides with the exact output operator applied on the domain of A. This concept is known as the Yosida approximation of the output operator C, see Weiss (1994) and references therein. Let us now introduce the following adaptive controller

$$\begin{cases} u(t) = k_I \mathfrak{z}(t) + \tilde{u}(t), \\ \dot{\mathfrak{z}}(t) = C_\lambda \theta_1 - y_r - \eta(\mathfrak{z}(t) - \mathfrak{z}^e), \end{cases} \tag{6.3.3}$$

where $\tilde{u}(t)$ is an additional control variable tending to 0 when t approaches ∞ and that has to be assigned during the design. The constant $\mathfrak{z}^e$ is the value of $\mathfrak{z}$ at steady-state. In other words, $\mathfrak{z}^e$ is such that the condition $C_\lambda \theta_1^e = y_r$ is met when replacing u^e by $k_I\mathfrak{z}^e$ in (6.2.6). The equilibrium dimensionless temperature and concentration are denoted by θ_1^e and θ_2^e, respectively, and are supposed to be known. The scalar y_r is the asymptotic value of the approximated output $C_\lambda \theta_1$ which is fixed a priori and the coefficients $\eta(\geq 0)$ and k_I are real numbers that will be assigned during the design procedure. With this setting the control objective can be formulated mathematically as follows:

- The control $u(t)$ introduced in (6.3.3) regulates the output $C_\lambda \theta_1$ asymptotically to y_r, i.e.
$$\lim_{t\to\infty} C_\lambda \theta_1 = y_r;$$
- Moreover, $u(t)$ is such that the $X\times\mathbb{R}-$deviation between the state trajectory of the system composed by the variables $(\theta_1, \theta_2, \mathfrak{z})$ from the corresponding equilibrium $(\theta_1^e, \theta_2^e, \mathfrak{z}^e)$ tends exponentially fast to 0 when t goes to ∞, that is
$$\left\| \begin{smallmatrix} \theta_1(\cdot,t)-\theta_1^e(\cdot) \\ \theta_2(\cdot,t)-\theta_2(\cdot) \\ \mathfrak{z}(t)-\mathfrak{z}^e \end{smallmatrix} \right\|_{X\times\mathbb{R}} \leq \mathscr{M} e^{-\gamma t} \left\| \begin{smallmatrix} \theta_1(\cdot,0)-\theta_1^e(\cdot) \\ \theta_2(\cdot,0)-\theta_2^e(\cdot) \\ \mathfrak{z}(0)-\mathfrak{z}^e \end{smallmatrix} \right\|_{X\times\mathbb{R}},$$
for positive constants $\mathscr{M}$ and γ and any initial conditions $\left(\begin{smallmatrix} \theta_1(0,\cdot) & \theta_2(0,\cdot) & \mathfrak{z}(0) \end{smallmatrix} \right)^T \in D(A)\times\mathbb{R}$.

Note that since the space X is a Hilbert space, the product space $X\times\mathbb{R}$, noted $\mathscr{X}$, remains Hilbert in which the norm is induced by the following inner product

$$\left\langle \begin{pmatrix} \mathfrak{f}_1 \\ \mathfrak{f}_2 \\ \mathfrak{z}_1 \end{pmatrix}, \begin{pmatrix} \mathfrak{l}_1 \\ \mathfrak{l}_2 \\ \mathfrak{z}_2 \end{pmatrix} \right\rangle_{\mathscr{X}} := \langle \mathfrak{f}_1, \mathfrak{l}_1\rangle_{L^2} + \langle \mathfrak{f}_2, \mathfrak{l}_2\rangle_{L^2} + \mathfrak{z}_1\mathfrak{z}_2, \tag{6.3.4}$$

for $\mathfrak{f}_i, \mathfrak{l}_i \in L^2([0,1];\mathbb{R}), i = 1,2$, and $\mathfrak{z}_i \in \mathbb{R}, i = 1,2$.

6.4 Regulation of the output and exponential stability of the closed-loop system

We start this section by introducing the variables $\tilde{\theta}_1 := \theta_1 - \theta_1^e, \tilde{\theta}_2 := \theta_2 - \theta_2^e$ and $\tilde{\mathfrak{z}} = \mathfrak{z} - \mathfrak{z}^e$. In this new variables the first component of (6.2.3) together with (6.3.3) transforms as:

$$\begin{aligned}
\partial_t \tilde{\theta}_1 = \partial_t \theta_1 &= -\partial_z \tilde{\theta}_1 - d_z \theta_1^e + \delta f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) + \beta (1_{[0,1]} k_I (\tilde{\mathfrak{z}} + \mathfrak{z}^e) + \tilde{u}) \\
&\quad - \beta \tilde{\theta}_1 - \beta \theta_1^e + \frac{1}{\eta_T} 1_{[0,\eta_T]} (1 + d_T) \\
&= -\partial_z \tilde{\theta}_1 - \beta \tilde{\theta}_1 + \delta f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) + \beta k_I 1_{[0,1]} \tilde{\mathfrak{z}} + \beta 1_{[0,1]} \tilde{u} \\
&\quad + \left(-d_z \theta_1^e - \beta \theta_1^e + \beta k_I \mathfrak{z}^e 1_{[0,1]} + \frac{1}{\eta_T} 1_{[0,\eta_T]} (1 + d_T) \right) \\
&\overset{(6.2.6)}{=} -\partial_z \tilde{\theta}_1 - \beta \tilde{\theta}_1 + \delta \left(f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) - f(\theta_1^e, \theta_2^e) \right) + \beta k_I 1_{[0,1]} \tilde{\mathfrak{z}} + \beta 1_{[0,1]} \tilde{u}.
\end{aligned} \tag{6.4.1}$$

The dimensionless concentration around the equilibrium is written in the following way

$$\begin{aligned}
\partial_t \tilde{\theta}_2 = \partial_t \theta_2 &= -\partial_z \tilde{\theta}_2 - d_z \theta_2^e + f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) \\
&\overset{(6.2.6)}{=} -\partial_z \tilde{\theta}_2 + f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) - f(\theta_1^e, \theta_2^e).
\end{aligned} \tag{6.4.2}$$

The differential equation associated to the new variable $\tilde{\mathfrak{z}}$ is given by

$$\dot{\tilde{\mathfrak{z}}} = \dot{\mathfrak{z}} = C_\lambda \tilde{\theta}_1 + \underbrace{C_\lambda \theta_1^e - y_r}_{=0} - \eta (\tilde{\mathfrak{z}} + \mathfrak{z}^e - \mathfrak{z}^e) = C_\lambda \tilde{\theta}_1 - \eta \tilde{\mathfrak{z}}. \tag{6.4.3}$$

In order to prove exponential stability of the system composed of (6.4.1), (6.4.2) and (6.4.3) in the $\mathscr{X}-$norm, let us introduce the following Lyapunov functional candidate

$$\begin{aligned}
&V : \mathbb{R}^+ \to \mathbb{R}^+ \\
&\quad t \mapsto \frac{1}{2} \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1^2 dz + \frac{1}{2} \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2^2 dz + \frac{1}{2} (1 - e^{-\lambda}) \tilde{\mathfrak{z}}^2,
\end{aligned} \tag{6.4.4}$$

where λ is the same constant that has been chosen for the definition of the operator C_λ. Note that the function V is equivalent to the square of the $\mathscr{X}-$norm of $\left(\tilde{\theta}_1 \ \tilde{\theta}_2 \ \tilde{\mathfrak{z}} \right)^T$ in the sense that there exist positive constants m and M such that

$$m \left\| \begin{matrix} \tilde{\theta}_1(\cdot,t) \\ \tilde{\theta}_2(\cdot,t) \\ \tilde{\mathfrak{z}}(t) \end{matrix} \right\|_{\mathscr{X}}^2 \leq V(t) \leq M \left\| \begin{matrix} \tilde{\theta}_1(\cdot,t) \\ \tilde{\theta}_2(\cdot,t) \\ \tilde{\mathfrak{z}}(t) \end{matrix} \right\|_{\mathscr{X}}^2. \tag{6.4.5}$$

Observe that the positive function $\lambda e^{-\lambda(1-z)} \in [\lambda e^{-\lambda}, \lambda]$ for a.e. $z \in [0,1]$. This is sufficient to prove that (6.4.5) holds true with m and M given by $\frac{1}{2} \min\{\lambda e^{-\lambda}, 1 - e^{-\lambda}\}$ and $\frac{1}{2} \max\{\lambda, 1 - e^{-\lambda}\}$, respectively.

Before going into the characterization of the exponential stability of (6.2.3) with (6.3.3) the following lemma makes the link between the Lyapunov function V and a weighted version of the latter.

Lemma 6.4.1 *Let us consider a matrix $N \in \mathbb{R}^{3\times 3}$ such that $N = N^T$. Then the inequality*

$$\frac{1}{2}\int_0^1 (\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,) N\lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\mathfrak{z}} \end{pmatrix} dz \le [\max \sigma(N)] V(t)$$

holds, where V is defined in (6.4.4).

Proof. Observe that V may be written as

$$V(t) = \frac{1}{2}\int_0^1 (\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,) I_{3\times 3}\lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\mathfrak{z}} \end{pmatrix} dz.$$

Now we consider a weighted version of V, denoted by V^N, given as

$$V^N(t) = \frac{1}{2}\int_0^1 (\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,) N\lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\mathfrak{z}} \end{pmatrix} dz.$$

Since N is symmetric it is orthogonally diagonalizable, that is, $N = K^T DK$ where K is an orthogonal matrix whose columns are the eigenvectors of N and D is a diagonal matrix containing the eigenvalues of N on its diagonal. The function V^N may be written in another way as

$$V^N(t) = \frac{1}{2}\int_0^1 (\,\tilde{\theta}_1^K\ \tilde{\theta}_2^K\ \tilde{\mathfrak{z}}^K\,) D\lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1^K \\ \tilde{\theta}_2^K \\ \tilde{\mathfrak{z}}^K \end{pmatrix} dz,$$

where the change of coordinates $(\,\tilde{\theta}_1^K\ \tilde{\theta}_2^K\ \tilde{\mathfrak{z}}^K\,)^T = K(\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,)^T$ has been introduced. Since the diagonal matrix D contains the eigenvalues of N, the following estimation holds for $V^N(t)$:

$$\begin{aligned} V^N(t) &\le [\max \sigma(N)] \frac{1}{2}\int_0^1 (\,\tilde{\theta}_1^K\ \tilde{\theta}_2^K\ \tilde{\mathfrak{z}}^K\,) \lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1^K \\ \tilde{\theta}_2^K \\ \tilde{\mathfrak{z}}^K \end{pmatrix} dz \\ &= [\max \sigma(N)] \frac{1}{2}\int_0^1 (\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,) K^T K\lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\mathfrak{z}} \end{pmatrix} dz. \end{aligned}$$

By using the orthogonality of the matrix K, we conclude that

$$V^N(t) \le [\max \sigma(N)] V(t).$$

□

The following proposition gives necessary conditions ensuring that the closed-loop system (6.2.3) together with (6.3.3) is exponentially stable for a particular nonlinear feedback $\tilde{u}(t) := g(\tilde{\theta}_1, \tilde{\theta}_2)$.

Proposition 6.4.1 *System (6.2.3) in closed-loop with (6.3.3) where the additional control input $\tilde{u}(t)$ is defined for $t \geq 0$ by*

$$\tilde{u}(t) = \frac{1}{\beta \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 dz} \left[- \left| \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 (d_z \tilde{\theta}_1) dz \right| \right. \tag{6.4.6}$$

$$- \left| \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (d_z \tilde{\theta}_2) dz \right|$$

$$- \int_0^1 \lambda e^{-\lambda(1-z)} (\delta \tilde{\theta}_1 + \tilde{\theta}_2)(f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) - f(\theta_1^e, \theta_2^e)) dz$$

$$\left. - \kappa (\tilde{\theta}_1^2(0) + \tilde{\theta}_1^2(1)) - k_C \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2^2 dz \right], \tag{6.4.7}$$

with $k_C > 0$ and $\kappa > 0$, is exponentially stable provided that the coefficient k_I satisfies $4\beta\eta - (\beta k_I + 1 - e^{-\lambda})^2 > 0$.

Proof. Let us fix the parameters $\lambda > 0, k_C > 0$ and $\kappa > 0$ and consider the Lyapunov functional candidate (6.4.4). Taking its time derivative along the state trajectories of (6.4.1)–(6.4.3) implies that

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 (\partial_t \tilde{\theta}_1) dz + \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (\partial_t \tilde{\theta}_2) dz + (1 - e^{-\lambda}) \tilde{\mathfrak{z}} \dot{\tilde{\mathfrak{z}}} \\
&= - \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 (d_z \tilde{\theta}_1) dz - \beta \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1^2 dz \\
&+ \int_0^1 \lambda e^{-\lambda(1-z)} \delta \tilde{\theta}_1 (f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) - f(\theta_1^e, \theta_2^e)) dz \\
&+ \beta k_I \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 \tilde{\mathfrak{z}} dz + \tilde{u} \beta \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 dz - \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (d_z \tilde{\theta}_2) dz \\
&+ \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (f(\tilde{\theta}_1 + \theta_1^e, \tilde{\theta}_2 + \theta_2^e) - f(\theta_1^e, \theta_2^e)) dz \\
&+ (1 - e^{-\lambda}) \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 \tilde{\mathfrak{z}} dz - (1 - e^{-\lambda}) \eta \tilde{\mathfrak{z}}^2.
\end{aligned}$$

Injecting the expression of $\tilde{u}$ (6.4.7) in the time derivative of V yields that

$$\begin{aligned}
\dot{V}(t) = & \underbrace{- \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 (d_z \tilde{\theta}_1) dz - \left| \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 (d_z \tilde{\theta}_1) dz \right|}_{\leq 0} \underbrace{- \kappa (\tilde{\theta}_1^2(0) + \tilde{\theta}_1^2(1))}_{\leq 0} \\
& \underbrace{- \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (d_z \tilde{\theta}_2) dz - \left| \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2 (d_z \tilde{\theta}_2) dz \right|}_{\leq 0} - \beta \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1^2 dz \\
& - k_C \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_2^2 dz + (\beta k_I + 1 - e^{-\lambda}) \int_0^1 \lambda e^{-\lambda(1-z)} \tilde{\theta}_1 \tilde{\mathfrak{z}} dz - (1 - e^{-\lambda}) \eta \tilde{\mathfrak{z}}^2
\end{aligned}$$

$$\leq \int_0^1 \begin{pmatrix} \tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\mathfrak{z}} \end{pmatrix} \begin{pmatrix} -\beta & 0 & \frac{\beta k_I+1-e^{-\lambda}}{2} \\ 0 & -k_C & 0 \\ \frac{\beta k_I+1-e^{-\lambda}}{2} & 0 & -\eta \end{pmatrix} \lambda e^{-\lambda(1-z)} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\mathfrak{z}} \end{pmatrix} dz.$$

Applying Lemma 6.4.1 to the previous expression with

$$N = \begin{pmatrix} -\beta & 0 & \frac{\beta k_I+1-e^{-\lambda}}{2} \\ 0 & -k_C & 0 \\ \frac{\beta k_I+1-e^{-\lambda}}{2} & 0 & -\eta \end{pmatrix}$$

implies that

$$\dot{V}(t) \leq 2 \max \sigma(N) V(t),$$

where the elements of $\sigma(N)$ are given by

$$\begin{aligned}
\sigma_1 &= -k_C, \\
\sigma_2 &= \frac{-(\eta+\beta) + \sqrt{(\eta+\beta)^2 - (4\beta\eta - (\beta k_I + 1 - e^{-\lambda})^2)}}{2}, \\
\sigma_3 &= \frac{-(\eta+\beta) - \sqrt{(\eta+\beta)^2 - (4\beta\eta - (\beta k_I + 1 - e^{-\lambda})^2)}}{2}.
\end{aligned}$$

The assumption $4\beta\eta - (\beta k_I + 1 - e^{-\lambda})^2 > 0$ ensures that $\max_{i=1,2,3} \sigma_i < 0$. Consequently,

$$\dot{V}(t) \leq -\Omega V(t),$$

with $\Omega := -2\max_{i=1,2,3} \sigma_i > 0$, which proves exponential stability. □

Remark 6.4.1 *In order to properly define the additional control input $\tilde{u}(t)$ in Proposition 6.4.1, one should ensure that the denominator is not* 0 *at any time. This is not perfomed in this thesis but a partial answer concerning the asymptotic behavior of $\tilde{u}(t)$ is given in Proposition 6.4.2 herebelow.*

Due to the exponential convergence of the function $V(t)$ to 0 when t goes to ∞, i.e.

$$V(t) \leq e^{-\Omega t} V(0), \tag{6.4.8}$$

and the relation (6.4.5), we may conclude that the $\mathscr{X}-$norm of the state vector

$$\begin{pmatrix} \tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\mathfrak{z}} \end{pmatrix}$$

converges also to 0 exponentially fast as t goes to ∞. More particularly we have that

$$\left\| \begin{matrix} \tilde{\theta}_1(\cdot,t) \\ \tilde{\theta}_2(\cdot,t) \\ \tilde{\mathfrak{z}}(t) \end{matrix} \right\|_{\mathscr{X}} \leq \sqrt{\frac{M}{m}} e^{-\frac{\Omega}{2}t} \left\| \begin{matrix} \tilde{\theta}_1(\cdot,0) \\ \tilde{\theta}_2(\cdot,0) \\ \tilde{\mathfrak{z}}(0) \end{matrix} \right\|_{\mathscr{X}},$$

where the positive constants m and M have been introduced in (6.4.5). In addition let us prove that exponential stability of $\begin{pmatrix} \tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\mathfrak{z}} \end{pmatrix}$ implies that the approximate output converges towards the reference, y_r. Therefore, remark that

$$|C_\lambda \theta_1 - y_r| = |C_\lambda \theta_1 - C_\lambda \theta_1^e + C_\lambda \theta_1^e - y_r| = |C_\lambda \tilde{\theta}_1| \leq \sqrt{\frac{\lambda(1-e^{-2\lambda})}{2}}\|\tilde{\theta}_1\|_{L^2},$$

which converges exponentially fast to 0 as t tends to ∞ due to Proposition 6.4.1.

The last point of that section is dedicated to the convergence of the additional control input, $\tilde{u}(t)$.

Proposition 6.4.2 *The additional control law $\tilde{u}(t)$, whose expression is given in (6.4.7), converges to* 0 *as t tends to ∞ whenever the limit exists.*

Proof. Note that as the $\mathscr{X}-$norm of $\begin{pmatrix} \tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\mathfrak{z}} \end{pmatrix}$ tends to 0 exponentially fast as t tends to ∞ so does $\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_1 dz$. Observe that $\tilde{u}(t)$ may be written as

$$\begin{aligned}\tilde{u}(t) =& \frac{-\left|\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_1(d_z\tilde{\theta}_1)dz\right| - \left|\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_2(d_z\tilde{\theta}_2)dz\right|}{k(t)} \\ &- \frac{\kappa(\tilde{\theta}_1^2(1)+\tilde{\theta}_1^2(0))+g(t)}{k(t)},\end{aligned}$$

where the functions g and k are defined for positive t by

$$\begin{aligned}g(t) :=& -\int_0^1 \lambda e^{-\lambda(1-z)}(\delta\tilde{\theta}_1+\tilde{\theta}_2)(f(\tilde{\theta}_1+\theta_1^e,\tilde{\theta}_2+\theta_2^e)-f(\theta_1^e,\theta_2^e))dz \\ &- k_C\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_2^2 dz, \\ k(t) :=& \beta\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_1 dz,\end{aligned}$$

respectively. Applying the Cauchy-Schwarz and the Young inequalities combined with the Lipschitz continuity of f to the first term of the function g is sufficient to show that $g(t)$ converges to 0 whenever t goes to ∞. This together with the assumption that the limit for t going to ∞ of $\tilde{u}(t)$ exists implies that

$$\begin{aligned}\lim_{t\to\infty}\Big[&-\left|\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_1(d_z\tilde{\theta}_1)dz\right| - \left|\int_0^1 \lambda e^{-\lambda(1-z)}\tilde{\theta}_2(d_z\tilde{\theta}_2)dz\right| \\ &-\kappa(\tilde{\theta}_1^2(1)+\tilde{\theta}_1^2(0))\Big] = 0. \end{aligned} \tag{6.4.9}$$

The non positivity of each term of the previous expression entails that

$$\lim_{t\to\infty}\tilde{\theta}_1(1,t) = 0, \lim_{t\to\infty}\tilde{\theta}_1(0,t) = 0.$$

Let us take (6.4.1) and define the function $\mathfrak{g}(t) := \int_0^1 \tilde{\theta}_1 dz$. According to (6.4.1), the time derivative of $\mathfrak{g}$ is subject to the following differential equation

$$\dot{\mathfrak{g}}(t) = -\tilde{\theta}_1(1,t) + \tilde{\theta}_1(0,t) - \beta\mathfrak{g}(t)$$

Notation	Value	Notation	Value
L	$1m$	C_{in}	$0.02\frac{\mathrm{mol}}{l}$
v	$0.025\frac{m}{s}$	T_{in}	$340K$
E	$11250\frac{kJ}{kg}$	$\frac{4h}{\rho C_p d}$	$0.2\frac{1}{s}$
k_0	$10^6\frac{1}{s}$	$\frac{\Delta H}{\rho C_p}$	$4250\frac{m^3K}{kg}$
R	$1.986\frac{kJ}{kgK}$		

Table 6.1 – Simulation parameters.

$$
\begin{aligned}
&+\delta\int_0^1 \left[f(\tilde{\theta}_1(z,t)+\theta_1^e(z),\tilde{\theta}_2(z,t)+\theta_2^e(z))-f(\theta_1^e(z),\theta_2^e(z))\right]dz \\
&+\beta k_I\tilde{\mathfrak{z}}(t)+\beta\tilde{u}(t).
\end{aligned} \tag{6.4.10}
$$

Taking the limit for t going to ∞ both sides of (6.4.10) yields that

$$\lim_{t\to\infty}\tilde{u}(t)=\frac{1}{\beta}\lim_{t\to\infty}\dot{\mathfrak{g}}(t).$$

Since the function $\mathfrak{g}$ converges to a real number (which in particular is 0) and the limit of its derivative exists, it can only be 0. Consequently, $\lim_{t\to\infty}\tilde{u}(t)=0$. $\square$

6.5 Numerical simulations

The results presented in Section 6.4 are illustrated in this section by means of numerical simulations. The values of the parameters used in the model are given in Table 6.1, see Aksikas (2005) for instance. We took the two following functions as initial conditions for T and C:

$$
\begin{aligned}
T(\zeta,0)&=T_{in}+0.08T_{in}\left(-\left(\frac{\zeta}{L}\right)^3+\left(\frac{\zeta}{L}\right)^2+\frac{\zeta}{L}\right),\\
C(\zeta,0)&=C_{in}-0.7C_{in}\left(-\left(\frac{\zeta}{L}\right)^3+\left(\frac{\zeta}{L}\right)^2+\frac{\zeta}{L}\right).
\end{aligned}
$$

They can be viewed as perturbations around the inlet temperature and concentration, respectively. Note that the values presented in Table 6.1 entails that

$$\mu=16.6607,\alpha=2.3248,\delta=-0.25,\beta=8.$$

For the parameters of the controller, we choosed

$$k_I=-0.125,k_C=6,\eta=6,\kappa=0.01,\lambda=75,$$

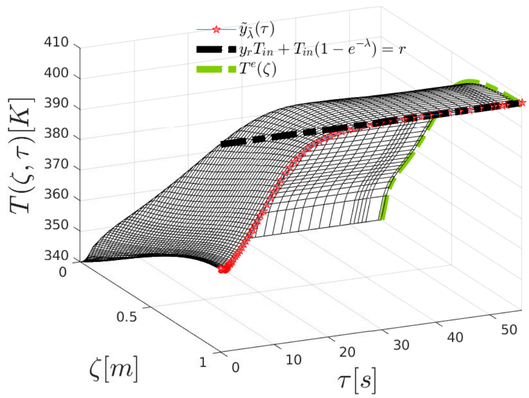

Figure 6.1 – Temperature $T(\zeta,\tau)[K]$ as a function of space $\zeta[m]$ and time $\tau[s]$.

in such a way that the condition $4\beta\eta - (\beta k_I + 1 - e^{-\lambda})^2 > 0$ is satisfied. The reference value that has to be tracked by the dimensionless output, y_r, is 0.2. Note that the equilibria, θ_1^e, θ_2^e and $\mathfrak{z}^e$ have been computed first. Then, the system (6.2.3) in closed-loop with (6.3.3) where $\tilde{u}$ is given in (6.4.7), has been discretized in the space coordinate via finite differences and then integrated via the Matlab routine `ode15s`. The number of discretization points has been fixed to 200 for the $z-$coordinate while it has been chosen by the algorithm for the $t-$coordinate (variable step numerical integration method). The final dimensionless time that has been chosen is 1.4, which corresponds to $56s$. Note that the simulations have been performed without taking disturbances into account, i.e. by considering $w_T = 0, w_C = 0$. The temperature $T(\tau,\zeta)$ and the concentration $C(\tau,\zeta)$ are depicted in Figures 6.1 and 6.2, respectively, wherein the corresponding equilibria, $T^e(\zeta)$ and $C^e(\zeta)$, are overlaid. Moreover, the value of the reference is highlighted in the figure corresponding to the temperature, see Figure 6.1.

Figure 6.3 aims at showing the approximated output (which approximates the temperature at $\zeta = L(=1)$) together with the temperature trajectory at the outlet of the reactor, i.e. for $\zeta = L$, which corresponds to the real output/measurment on the system (6.2.1). Let us highlight the link between the approximation of the dimensionless and dimensional output trajectories. It holds that

$$
\begin{aligned}
y_\lambda(t) := C_\lambda \theta_1(\cdot,t) &= \lambda \int_0^1 e^{-\lambda(1-z)} \theta_1(t,z) dz \\
&= \lambda \int_0^1 e^{-\lambda(1-z)} \frac{T(zL, tL/v) - T_{in}}{T_{in}} dz \\
&\overset{\zeta = Lz}{=} \frac{\lambda}{L} \int_0^L e^{-\frac{\lambda}{L}(L-\zeta)} \frac{T(\zeta, tL/v) - T_{in}}{T_{in}} d\zeta
\end{aligned}
$$

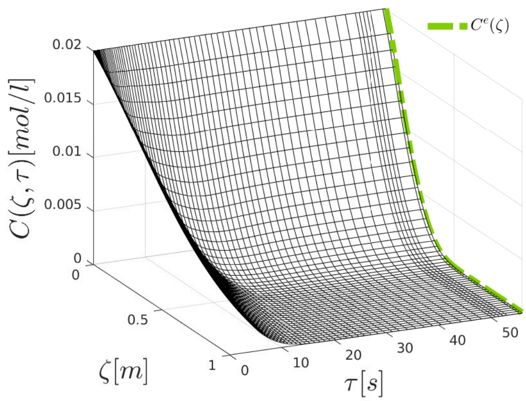

Figure 6.2 – Concentration $C(\zeta,\tau)[mol/l]$ as a function of space $\zeta[m]$ and time $\tau[s]$.

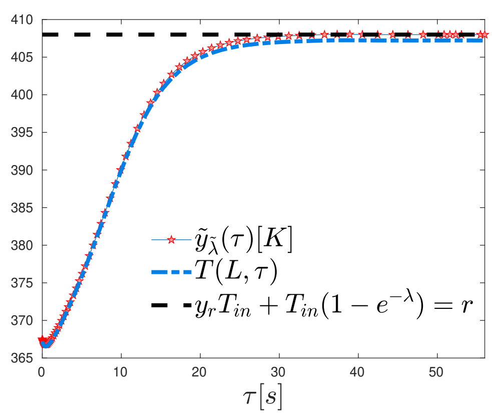

Figure 6.3 – Approximated output trajectory, $\tilde{y}_{\tilde{\lambda}}(\tau) := [C_\lambda \theta_1]T_{in} + (1-e^{-\lambda})T_{in}$ together with the temperature at $\zeta = L, T(\tau, L)$. The reference to be tracked is denoted by $r = y_r T_{in} + (1-e^{-\lambda})T_{in}$.

$$\begin{aligned}
&\overset{\tau=tL/v,\tilde{\lambda}=\frac{\lambda}{L}}{=} \frac{1}{T_{in}}\tilde{\lambda}\int_0^L e^{-\tilde{\lambda}(L-\zeta)}T(\zeta,\tau)d\zeta - \tilde{\lambda}\int_0^L e^{-\tilde{\lambda}(L-\zeta)}d\zeta \\
&= \frac{1}{T_{in}}\tilde{\lambda}\int_0^L e^{-\tilde{\lambda}(L-\zeta)}T(\zeta,\tau)d\zeta - (1-e^{-\lambda}) =: \frac{\tilde{y}_{\tilde{\lambda}}(\tau)-(1-e^{-\lambda})T_{in}}{T_{in}},
\end{aligned}$$

where $\tilde{y}_{\lambda}(\tau)$ represents an approximation of the temperature for $\zeta = L$. This means that regulating the approximated dimensionless temperature at the outlet, $y_{\lambda}(t)$, is equivalent in regulating the scaled approximated and dimensional outlet temperature, $\frac{\tilde{y}_{\tilde{\lambda}}(\tau)-T_{in}}{T_{in}}$, with an additional correction term which comes from the approximation of the output, $e^{-\lambda}$. Consequently, the approximated dimensional output, $\tilde{y}_{\tilde{\lambda}}(\tau)$ may be expressed as

$$\tilde{y}_{\tilde{\lambda}}(\tau) = y_{\lambda}(t)T_{in} + (1-e^{-\lambda})T_{in}, \tag{6.5.1}$$

with $\tau = tL/v$. Note that the limit for τ going to ∞ of (6.5.1) is $y_rT_{in} + (1-e^{-\lambda})T_{in}$, where y_r is a reference value that has been chosen. From a practical point of view, the value that has to be tracked asymptotically by the dimensional approximated output, r, is chosen at the beginning of the procedure. Then, the corresponding reference value, y_r, is computed as

$$y_r = \frac{r-(1-e^{-\lambda})T_{in}}{T_{in}}.$$

That value of y_r combined with the second part of Proposition 6.4.1 ensures that the reference r is tracked asymptotically by $\tilde{y}_{\tilde{\lambda}}(\tau)$. These facts are summarized in Figure 6.3. It can be observed that the approximation of $T(L,\tau)$, denoted by $\tilde{y}_{\tilde{\lambda}}(\tau)$ is quite accurate since its maximal deviation from $T(\tau,L)$ is about $0.7811K$. The dimensionless value $y_r = 0.2$ corresponds to an asymptotic dimensional reference of $408K$. Note that the function $\tilde{y}_{\tilde{\lambda}}(\tau)$ is also represented in Figure 6.1 on the surface corresponding to the temperature.

In Figure 6.4 the input trajectory $T_w(\tau)$ is depicted. According to the change of variable $u = \frac{T_w-T_{in}}{T_{in}}$, the relation $u(t) = k_I\mathfrak{z}(t) + \tilde{u}(t)$ (6.4) and Proposition 6.4.2, the asymptotic value of T_w is $k_I\mathfrak{z}^eT_{in} + T_{in}$.

It can be observed that the control action lies within the values $368.6861K$ and $410.7719K$, which correspond to $95.5360°C$ and $137.6219°C$, respectively. These bounds are quite realistic from a physical point of view despite that no constraints have been put on the control action. A perspective would be the introduction of control constraints in order to ensure that the input is realistic at any time. This would lead to rethink the proof of the exponential stability.

The Lyapunov functional V has been represented in Figure 6.5. The estimate $e^{-\Omega t}V(0)$ is also highlighted in order to show that (6.4.8) is satisfied. Note that $\Omega = 12$.

Figure 6.6 is dedicated to the $\mathscr{X}-$norm of the state trajectory $(\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,)^T$. It means that the control input $k_I\mathfrak{z}(t) + \tilde{u}(t)$ steers the $\mathscr{X}-$norm of $(\,\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}}\,)^T$ to 0 exponentially fast as it is stated in Proposition 6.4.1. More specifically it steers also the $X-$norm of $(\,\tilde{\theta}_1\ \tilde{\theta}_2\,)^T$ to 0. Interpreting these quantities with the original dimensions entails that the heat exchanger control variable $T_w(\tau) = T_{in} + T_{in}(k_I\mathfrak{z}(\tau v/L) + \tilde{u}(\tau v/L))$ drives the

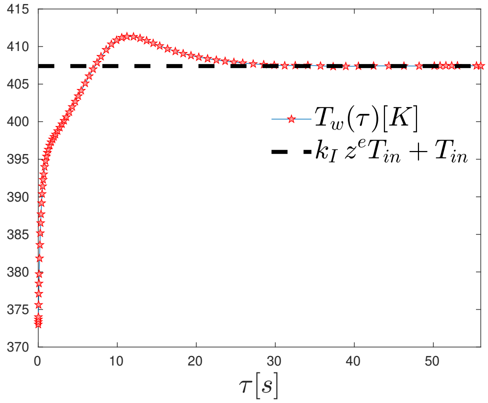

Figure 6.4 – Input trajectory, $T_w(\tau)$ together with its asymptotic value, $k_I \mathfrak{z}^e T_{in} + T_{in}$.

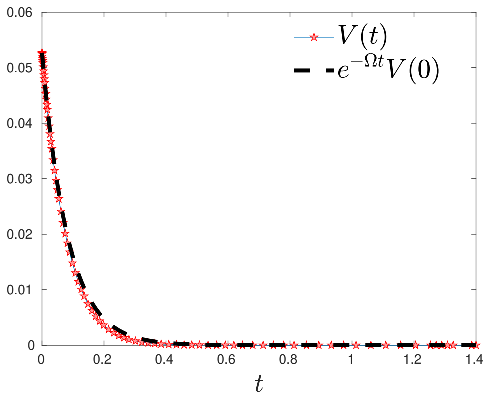

Figure 6.5 – Lyapunov functional $V(t)$ defined in (6.4.4) together with the function $e^{-\Omega t}V(0)$.

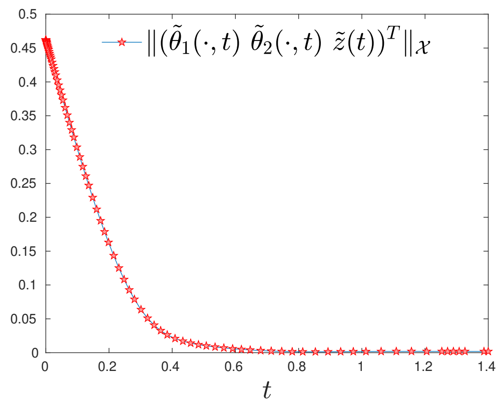

Figure 6.6 – $\mathscr{X}-$norm of the state trajectory $(\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\mathfrak{z}})^T$.

$L^2([0,L];\mathbb{R})\times L^2([0,L];\mathbb{R})-$norm of

$$\left(\tfrac{1}{T_{in}}(T(\zeta,\tau)-T^e(\zeta))\ \tfrac{1}{C_{in}}(C^e(\zeta)-C(\zeta,\tau))\right)^T$$

to 0 when τ goes to ∞. The link between the $X-$ and the $L^2([0,L];\mathbb{R})\times L^2([0,L];\mathbb{R})-$ norms can be made by noticing that for any $f\in L^2([0,L];\mathbb{R})$ and $\tilde{f}\in L^2([0,1];\mathbb{R})$ defined by $\tilde{f}(z)=f(Lz), z\in[0,1]$, it holds

$$\begin{aligned}\|f\|^2_{L^2(0,L)} &= \int_0^L f^2(\zeta)d\zeta = \int_0^L f^2\left(L\frac{\zeta}{L}\right)d\zeta \\ &= \int_0^L \tilde{f}^2\left(\frac{\zeta}{L}\right)d\zeta = \int_0^1 \tilde{f}^2(\lambda)Ld\lambda = L\|\tilde{f}\|^2_{L^2(0,1)}.\end{aligned}$$

Note that due to condition $4\beta\eta-(\beta k_I+1-e^{-\lambda})>0$, there exists a continuum of admissible values for k_I which ensures exponential stability together with output regulation. This interval of admissible values is $[-1.8571, 1.6071]$ for the set of parameters chosen in this section.

Let us now introduce setpoint changes during the time integration. We define the following times for which the reference r changes:

$$\overrightarrow{T}=\{52s, 104s, 156s, 208s\}=:\{T_i\}_{i=1}^4.$$

At these time instants, the value of the reference have been fixed as

$$\overrightarrow{r}=\{408K, 425K, 399.5K, 414.8K\}=:\{r_i\}_{i=1}^4. \tag{6.5.2}$$

Consequently, the reference may be defined as follows

$$r(\tau)=r_1 1_{[0,T_1]}(\tau)+\sum_{i=2}^4 r_i 1_{[T_{i-1},T_i]}(\tau). \tag{6.5.3}$$

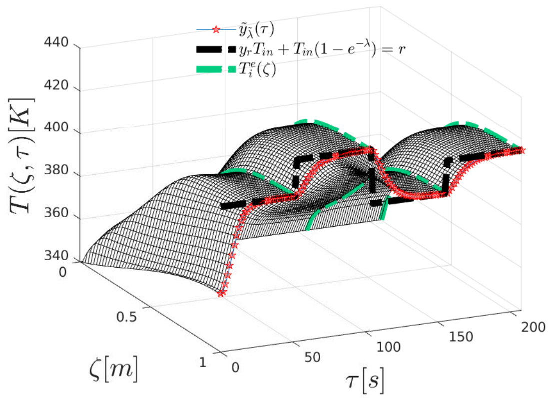

Figure 6.7 – Temperature $T(\zeta,\tau)[K]$ as a function of space $\zeta[m]$ and time $\tau[s]$.

In Figures 6.7 and 6.8 the surfaces corresponding to the temperature and the concentration are depicted, respectively. It can be seen that the outlet temperature tracks each reference on their corresponding time intervals. The equilibria are also shown in these figures.

The output of the system, i.e. the approximated outlet temperature, is represented in Figure 6.9 while the associated control action, $T_w(\tau)$, is presented in Figure 6.10.

We end this chapter by presenting the $\mathscr{X}-$norm of the state vector $\begin{pmatrix}\tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\mathfrak{z}}\end{pmatrix}$ as a function of time, see Figure 6.11. One observes that the equilibrium corresponding to each reference value of the vector (6.5.2) is reached exponentially fast on each subinterval $[0,T_1 v/L]$ and $[T_{i-1}v/L, T_i v/L], i=2,3,4$. The dimensionless values $\{y_{r_i}\}_{i=1}^4$ associated to $\{r_i\}_{i=1}^4$ are computed as

$$y_{r_i} = \frac{r_i - (1-e^{-\lambda})T_{in}}{T_{in}}.$$

We conclude by insisting on the fact that the approach that has been considered here assumes a complete knowledge of the equilibria of (6.2.1), i.e. the knowledge of the solution of (6.2.6). Even more, it is supposed that the entire dynamics are known, even the perturbations, which is often a bit strong to assume, particularly when working with chemical reactions. Indeed, due to slowness of the reaction, the reaction rates are complicated to estimate and a priori knowledge of the dynamics is a strong assumption. Moreover, since perturbations are especially entering the system in a probabilistic way, these are not known a priori.

Therefore, state estimators and in particular Kalman filters are often a good manner to overcome these difficulties and fit better the reality. However, this can be really challenging to construct when working with nonlinear infinite-dimensional systems.

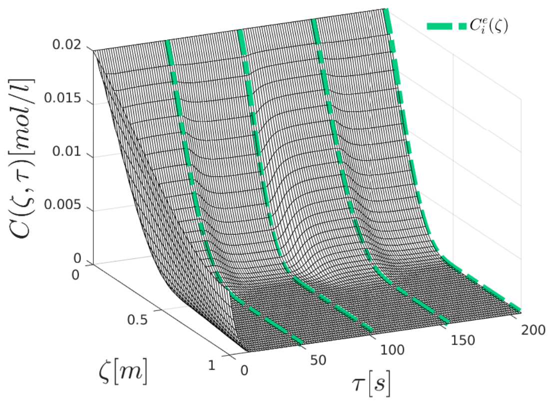

Figure 6.8 – Concentration $C(\zeta,\tau)[K]$ as a function of space $\zeta[m]$ and time $\tau[s]$.

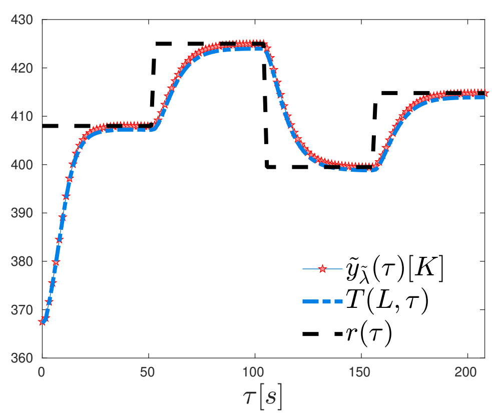

Figure 6.9 – Approximated output trajectory, $\tilde{y}_{\tilde{\lambda}}(\tau) := [C_\lambda \theta_1]T_{in} + (1 - e^{-\lambda})T_{in}$ together with the temperature at $\zeta = L, T(L,\tau)$. The reference to be tracked is denoted by $r(\tau)$ and is defined in (6.5.3).

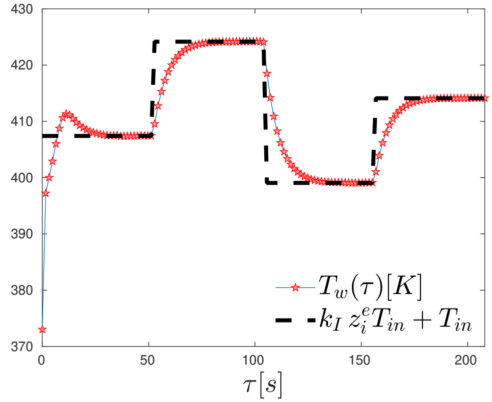

Figure 6.10 – Input trajectory, $T_w(\tau)$ together with each asymptotic values, $k_I \mathfrak{z}_i^e T_{in} + T_{in}$.

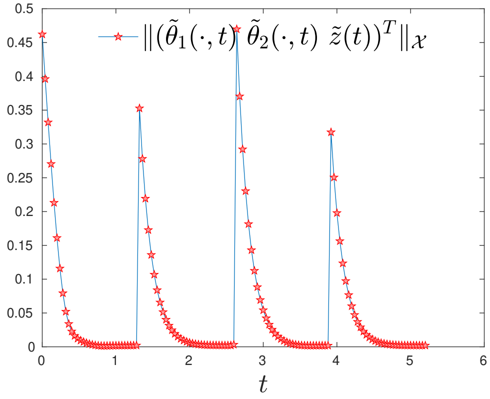

Figure 6.11 – $\mathscr{X}$ –norm of the state trajectory $(\tilde{\theta}_1\, \tilde{\theta}_2\, \tilde{\mathfrak{z}})^T$.

This is induced by the unboundedness of the output operator which renders the operator dynamics nonlinear and unbounded. The literature is quite not well developed now for such topics and even the well-posedness is a complicated and involved question for such types of dynamical systems. Recently this question has been adressed in Schwenninger (2020) where the notion of nonlinear boundary control system has been defined. Results on Luenberger-type observers for semilinear systems can also be found in Meurer (2013). For more general aspects of state estimators we refer to Dochain (2003) or Mohd Ali et al. (2015) among others.

Chapter 7

Funnel control for distributed-parameter systems: from the linear case to the nonlinear case

Contents

This part of the thesis is a continuation of the study of the question raised at the end of Chapter 6 concerning the knowledge of the equilibria and of the dynamics of a system when performing regulation of some output trajectory. The chapter is concerned with the development of an adaptive control law that aims at forcing the scalar output of semilinear infinite-dimensional systems to track some reference signal that satisfies some regularity assumptions. The control method that will be developed is known as funnel control. It will bring our attention for systems that can be written in an abstract way as nonlinear infinite-dimensional systems, whose description will be given here below. In particular, global Lipschitz continuity of the nonlinear part as well as bounded-input-state bounded-output (BISBO) property of the system will be required in order to ensure feasability of the proposed control method.

7.1 General considerations on funnel control

Based on an abstract differential relation between the input and the output of a dynamical system, a funnel controller is an adaptive model-free output feedback control whose objective aims at letting evolve the output error tracking in a prescribed funnel for which the boundaries are bounded away from 0. This does not guarantee asymptotic tracking of the reference but control of the transient behavior of the tracking error. This control method has attracted a lot of attention in the last few years.

It has been thoroughly developed in Ilchmann et al. (2002) for systems with relative degree one (see the definition below). Since then, a lot of attention has been given to the identification of classes of systems for which funnel control is feasible. In particular, in Ilchmann et al. (2002) they have shown that the funnel control approach is feasible for linear finite-dimensional systems, infinite-dimensional linear regular systems, nonlinear finite-dimensional systems, nonlinear delay systems and systems with hysteresis. Details for nonlinear systems with relative degree one are also available in Ilchmann et al. (2005). A few years later, funnel control has been extended and widely developed for MIMO nonlinear systems with known strict relative degree in Berger et al. (2018). The new funnel controller they introduced involves the $r-1$ derivatives of the tracking error, where r stands for the relative degree of the system. Note that a very recent survey on funnel control for different types of systems can be found in Berger et al. (2021c). Funnel control has also been considered in several fields of aplications due to its quite simple architecture. It has been shown appropriate for the regulation of the reaction temperature in chemical reactors, see Ilchmann and Trenn (2004). More recently, the position of a moving water tank has been controlled via a funnel controller in Berger et al. (2022). A linearized version of the Saint-Venant Exner infinite-dimensional dynamics has been used as internal dynamics. This shows that funnel control becomes more and more attractive for systems driven by infinite-dimensional internal dynamics. In that way, this topic has been considered in Berger et al. (2020) wherein it is proved that some class of infinite-dimensional linear systems fits the required assumptions for funnel control to be feasible. Moreover, they proved that linear-infinite dimensional systems that can be written in Byrnes-Isidori form, see Ilchmann et al. (2016), are encompassed in that new class. Meanwhile, funnel control for a class of boundary control (BC) systems has been developed in Puche et al. (2021). There, thanks to appropriate tools of nonlinear functional analysis, the authors show that a class of hyperbolic port-Hamiltonian systems controlled and observed at the boundary is contained in the class of BC systems they consider. More recently, funnel control has also been applied to a nonlinear infinite-dimensional reaction-diffusion equation coupled with the nonlinear Fitzhugh-Nagumo model, which represent together defibrillation processes of the human heart, see Berger et al. (2021a). As another recent attraction for funel control, Berger (2021) developed a funnel controller for the Fokker-Planck equation corresponding to a multi-dimensional Ornstein–Uhlenbeck process on an unbounded spatial domain. This constitues another application of funnel control to an infinite-dimensional model. Note also that funnel control has been lately coupled to model-predictive-control (Funnel MPC) for nonlinear systems with relative degree one, see Berger et al. (2021b).

In what follows, we shall introduce here a quite general class of nonlinear infinite-dimensional systems to which funnel control can be applied. Based on the Byrnes-Isidori form for linear systems, some change of variables that aims at extracting the output dynamics of the system, which is assumed to be finite-dimensional, is performed. Based on this transformation, funnel control is shown to be feasible provided that the remaining part of the dynamics satisfies a BISBO stability assumption. Moreover, a way of getting this BISBO stability condition is presented.

7.2 Mathematical description of funnel control

Here we introduce the differential relation we consider to make the link between the input and the output of a (not necessarily known) dynamical system together with the control objective that will be of interest in the sequel of the chapter.

Let us start by recalling the notion of relative degree for a linear time invariant finite-dimensional system (LTI system).

Definition 7.2.1 *Consider real matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, with n, m and p are natural numbers. The system Σ whose internal dynamics are given by*

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), x_0 \in \mathbb{R}^n, \end{array}\right. \tag{7.2.1}$$

where the inputs and the outputs are given by the functions $u(t)$ and $y(t)$, respectively. Σ is said to have relative degree $r \in \mathbb{N}$ if $CA^jB = 0$ for $j = 1, \cdots, r-2$ and $CA^{r-1}B \neq 0$.

This notion can be generalized for nonlinear finite-dimensional systems and also for linear infinite-dimensional systems, see e.g. Isidori (1995) or Ilchmann et al. (2016) among others. Intuitively speaking, the relative degree of a system is the minimal number of times the output has to be differentiated in order to see the input appearing explicitely.

7.2.1 General framework

The scalar differential equation making the connection between the input and the output of a dynamical system whose internal dynamics are not necessarily known (model-free) are assumed to be given as

$$\left\{\begin{array}{l} \dot{y}(t) = N(d(t), T(y)(t)) + \Gamma(d(t), T(y)(t))u(t), \\ y(0) = y_0, \end{array}\right. \tag{7.2.2}$$

where the following conditions are assumed to hold.

Assumption 7.2.1 *The disturbance $d \in L^\infty(\mathbb{R}^+,\mathbb{R})$, the nonlinear function N is in $C(\mathbb{R}^2,\mathbb{R})$ and the gain function $\Gamma \in C(\mathbb{R}^2,\mathbb{R})$ is positive in the sense that*

$$\Gamma(d,\rho) > 0$$

for all $(d,\rho) \in \mathbb{R}^2$.

Assumption 7.2.2 *The map $T : C(\mathbb{R}^+,\mathbb{R}) \to L^\infty_{loc}(\mathbb{R}^+,\mathbb{R})$ is a (possibly nonlinear) operator which satisfies the following conditions:*

1. *Bounded trajectories are mapped into bounded trajectories (BIBO[a] property), i.e. for all $k_1 > 0$, there exists $k_2 > 0$ such that for all $y \in C(\mathbb{R}^+,\mathbb{R})$,*
$$\sup_{t\in\mathbb{R}^+} |y(t)| \le k_1 \Rightarrow \sup_{t\in\mathbb{R}^+} |T(y)(t)| \le k_2. \tag{7.2.3}$$
2. *The operator T is causal, i.e. for any $t \in \mathbb{R}^+$ and any $y,\hat{y} \in C(\mathbb{R}^+,\mathbb{R})$*
$$y_{|[0,t)} = \hat{y}_{|[0,t)} \Rightarrow T(y)_{|[0,t)} = T(\hat{y})_{|[0,t)}, \tag{7.2.4}$$
where $f_{|I}$ denotes the restriction of the function f to the interval I.
3. *T is locally Lipschitz in the sense that for all $t \in \mathbb{R}^+$ and all $y \in C([0,t],\mathbb{R})$ there exist positive constants τ, δ and ρ such that for any $y_1,y_2 \in C(\mathbb{R}^+,\mathbb{R})$ with $y_{i_{|[0,t]}} = y, i = 1,2$ and $|y_i(s) - y(t)| < \delta$ for all $s \in [t,t+\tau]$ and $i = 1,2$ it holds that*
$$\|(T(y_1) - T(y_2))_{|[t,t+\tau]}\|_\infty \le \rho \|(y_1 - y_2)_{|[t,t+\tau]}\|_\infty, \tag{7.2.5}$$
where $\|f_{|[t,t+\tau]}\|_\infty := \sup_{s\in[t,t+\tau]} |f(s)|$.

[a] Bounded-Input Bounded-Output

The class of systems governed by (7.2.2) with Assumptions 7.2.1–7.2.2 is presented in (Berger et al., 2020, Section 1) for systems with (possible) memory and relative degree $r \in \mathbb{N}$. Here we consider systems with no memory and relative degree one. This class is quite general and encompasses systems with infinite-dimensional internal dynamics as shown in Berger et al. (2020) and Ilchmann et al. (2016) for instance. However it is still not clear which classes of distributed-parameter systems (DPS) may be written as the input-output equation (7.2.2) as it is explicitly mentioned in Berger et al. (2020). To this question we shall give a partial answer hereafter.

7.2.2 Control objective

The control objective for a system whose output differential equation admits the representation (7.2.2) consists in developing an output error feedback $u(t) = \mathscr{G}(t, e(t))$ with $e(t) = y(t) - y_{\text{ref}}(t)$ for a reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, such that, when connected to (7.2.2), it results in a closed-loop system for which the error $e(t)$ evolves in a prescribed performance funnel

$$\mathscr{F}_\phi := \left\{(t,e) \in \mathbb{R}^+ \times \mathbb{R},\, \phi(t)|e(t)| < 1\right\}, \tag{7.2.6}$$

where the function ϕ is assumed to belong to

$$\begin{aligned}\Phi := \Big\{\phi \in C(\mathbb{R}^+,\mathbb{R}),\, \phi, \dot{\phi} \in L^\infty(\mathbb{R}^+,\mathbb{R}),& \\ \phi(t) > 0, \forall t \in \mathbb{R}^+ \text{ and } \liminf_{t\to\infty} \phi(t) > 0\Big\}.&\end{aligned} \tag{7.2.7}$$

This control objective is also considered in (Berger et al., 2020, Section 1) for systems with arbitrary relative degree $r \in \mathbb{N}$. As described in Berger et al. (2020), Ilchmann et al. (2016) and Berger et al. (2018), a controller that achieves the output tracking performance described above is expressed as

$$u(t) = \frac{-e(t)}{1 - \phi^2(t)e^2(t)}, \tag{7.2.8}$$

with $\phi \in \Phi$ and $\phi(0)|e(0)| < 1$. The controller (7.2.8) is called a funnel controller and can be viewed as the output error feedback $u(t) = -k(t)e(t)$ with a time-varying (adaptive) gain $k(t) = \frac{1}{1-\phi^2(t)e^2(t)}$. The following theorem, coming from Berger et al. (2018) with $r = 1$, characterizes the effectiveness of the controller (7.2.8) in terms of existence and uniqueness of solutions of the closed-loop systems and also in terms of output tracking performance.

Theorem 7.2.1 *Consider a system (7.2.2) with Assumptions 7.2.1–7.2.2. Let $y_{ref} \in W^{1,\infty}(\mathbb{R}^+,\mathbb{R}), \phi \in \Phi$ and $y_0 \in \mathbb{R}$ such that the condition $\phi(0)|e(0)| < 1$ holds. Then the funnel controller (7.2.8) applied to (7.2.2) results in a closed-loop system whose solution $y : [0,\omega) \to \mathbb{R}, \omega \in (0,\infty]$, has the following properties:*

1. *The solution is global, i.e. $\omega = \infty$;*
2. *The input $u : \mathbb{R}^+ \to \mathbb{R}$, the gain function $k : \mathbb{R}^+ \to \mathbb{R}$ and the output $y : \mathbb{R}^+ \to \mathbb{R}$ are bounded;*
3. *The tracking error $e : \mathbb{R}^+ \to \mathbb{R}$ evolves in the funnel $\mathscr{F}_\phi$ and is bounded away from the funnel boundaries in the sense that there exists $\varepsilon > 0$ such that, for all $t \geq 0, |e(t)| \leq \frac{1}{\phi(t)} - \varepsilon$.*

This theorem will be a paramount tool in the next section, in order to prove output tracking control of the scalar output of some class of nonlinear DPS.

7.3 Systems driven by nonlinear infinite-dimensional dynamics

The main novelty here relies on the fact that we shall consider a class of nonlinear infinite-dimensional systems that admits an input-output differential description of the form (7.2.2). This consitutes also the difference with respect to e.g. Berger et al. (2020) wherein a class of operators T is introduced, which comes from systems modeled by linear infinite-dimensional internal dynamics. Our contribution enlarges the class of systems for which funnel control is feasible since, to the best of our knowledge, our class of nonlinear infinite-dimensional systems is shown to be appropriate for funnel control for the first time here. However, examples in which nonlinear infinite-dimensional systems are considered have already been studied for funnel control in Berger et al. (2021a). Our work also extends the Byrnes Isidori form studied in Ilchmann et al. (2016) to nonlinear infinite-dimensional systems that satisfy some relatively standard assumptions.

7.3.1 Abstract description and related assumptions

Here we introduce a class of nonlinear infinite-dimensional systems to which funnel control will be considered. The ad hoc assumptions that enable funnel control are also presented.

Let H be a real (separable) Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$. The nonlinear systems that we consider are governed by the following controlled and perturbed abstract ordinary differential equation

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + b(u(t) + d(t)), \\ y(t) = \langle x(t), c \rangle_H, \\ x(0) = x_0 \in H, \end{cases} \tag{7.3.1}$$

for which we make the following three assumptions.

Assumption 7.3.1 *The (unbounded) linear operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous (C_0) semigroup of bounded linear operators on H.*

Assumption 7.3.2 *The nonlinear operator $f : H \to H$ is uniformly Lipschitz continuous on H.*

Assumption 7.3.3 *The vectors b and c are in $D(A)$ and $D(A^*)$, respectively, where $D(A^*)$ is the domain of the adjoint of the operator A. Moreover, it is assumed without loss of generality that $\langle b, c \rangle_H > 0$. The disturbance d is bounded, i.e. $d \in L^\infty(\mathbb{R}^+, \mathbb{R})$.*

Note that the scalar functions u and y stand for the input and the output, respectively. The way the disturbance d enters the system may be interpreted as uncertainties on the input function for instance. According to (Curtain and Zwart, 2020, Theorem 11.1.5), Assumptions 7.3.1 and 7.3.2 implies that the homogenenous[(1)] part of (7.3.1) has a unique mild solution on $[0,\infty)$ which can even be a classical one provided that $x_0 \in D(A)$. We emphasize the fact that, despite that Assumption 7.3.3 may be seen as restrictive, if the shape functions b and c do not lie in $D(A)$ and $D(A^*)$, it is always possible to approximate them, as accurately as desired, by functions of $D(A)$ and $D(A^*)$ since they are both dense subspaces of H. However, in the case where b and c should be approximated, the controller (7.2.8) could be affected even by small variations and produce undesirable effects. This should be taken into account in the design procedure.

7.3.2 Byrnes-Isidori decomposition

It is shown in this subsection how the abstract differential equation (7.3.1) can be transformed into (7.2.2) by using a change of variables as it is made for linear systems in Ilchmann et al. (2016).

Let us therefore consider the following proposition that enables us to decompose the state space H into the direct sum of two linear subspaces, see e.g. Ilchmann et al. (2016) and Byrnes et al. (1998).

Proposition 7.3.1 *Let Assumption 7.3.3 holds. Then the state space H can be decomposed as*

$$H := span\{c\} \oplus \{b\}^\perp =: \mathscr{C} \oplus \mathscr{I}, \tag{7.3.2}$$

where $\{b\}^\perp := \{f \in H, \langle f,b\rangle_H = 0\}$.

Proof. We start by showing that the intersection between $\mathscr{C}$ and $\mathscr{I}$ is reduced to the origin of H. Assume by contradiction that there exists $f \neq 0$ such that $f \in \mathscr{C}$ and $f \in \mathscr{I}$. Since $f \in \mathscr{C}$, there exists $\lambda \in \mathbb{R}_0$ such that $f = \lambda c$. As f is in $\mathscr{I}$ too, f is orthogonal to b, i.e. $\langle f,b\rangle_H = 0$. According to $f = \lambda c$, there holds $\langle c,b\rangle_H = 0$, which contradicts Assumption 7.3.3. Now we show that the direct sum $\mathscr{C} \oplus \mathscr{I}$ is contained in H. This follows directly by noting that $c \in H$ and by the definition of $\{b\}^\perp$. Let us end the proof by demonstrating the inclusion $H \subseteq \mathscr{C} \oplus \mathscr{I}$. Take any $f \in H$. Obviously f may be written as

$$f = \left(\frac{\langle f,b\rangle_H}{\langle c,b\rangle_H} c\right) + \left(f - \frac{\langle f,b\rangle_H}{\langle c,b\rangle_H} c\right), \tag{7.3.3}$$

where it is easy to see that $\frac{\langle f,b\rangle_H}{\langle c,b\rangle_H} c \in \mathscr{C}$ and $f - \frac{\langle f,b\rangle_H}{\langle c,b\rangle_H} c \in \mathscr{I}$. $\square$

According to (7.3.3) let us introcude the operator cP and $I - cP =: P_{\mathscr{I}}$ where the projection operator $P : H \to \mathbb{R}$ is defined as

$$Pf = \frac{\langle f,b\rangle}{\langle c,b\rangle},$$

[(1)]By "homogenenous" we mean uncontrolled and unperturbed.

for any $f \in H$. Then the decomposition (7.3.3) may be written as $f = cPf + P_{\mathscr{I}} f$. One shall now consider the following proposition that characterizes the subspaces $\mathscr{C}$ and $\mathscr{I}$.

Proposition 7.3.2 *The subspaces $\mathscr{C}$ and $\mathscr{I}$ satisfy $\mathscr{C} = cPH$ and $\mathscr{I} = P_{\mathscr{I}}H$, respectively.*

Proof. We start by showing that $\mathscr{C} = cPH$. First let us take $f \in \mathscr{C}$. Then there exists $\lambda \in \mathbb{R}$ such that $f = \lambda c$. Now observe that there exists $g \in H$ such that $f = cPg$. This g is given by λc. Conversely, let us consider $g \in cPH$. This entails that there exists $h \in H$ such that $g = cPh$. Then it is easy to see that $g \in \mathscr{C}$ since the constant λ such that $g = \lambda c$ is given by $\lambda = Ph$. Now we focus on the equality $\mathscr{I} = P_{\mathscr{I}}H$. Consider $f \in \mathscr{I}$. The question of existence of a function $g \in H$ such that $f = P_{\mathscr{I}} g$ is solved by taking $g = f$ because the relation $\langle f, b\rangle_H = 0$ holds true. The inclusion $P_{\mathscr{I}}H \subseteq \mathscr{I}$ is proved as follows. Take any $f \in P_{\mathscr{I}}H$. Then there exists $h \in H$ such that $f = P_{\mathscr{I}}h$, that is, $f = h - \frac{\langle h,b\rangle_H}{\langle c,b\rangle_H} c$. This is straightforward to see that this f is orthogonal to b, i.e. $f \in \mathscr{I}$. □

According to Propositions 7.3.1 and 7.3.2 any element $f \in H$ may be decomposed uniquely in $\mathscr{C} \oplus \mathscr{I}$ as

$$f = cPf + P_{\mathscr{I}} f.$$

Moreover, observe that for any function $f \in H$ there holds $\langle P_{\mathscr{I}} f, b\rangle_H = 0$. Note that the operator $P_{\mathscr{I}}$ is not an orthogonal projector on $\mathscr{I}$. Such a projector will be denoted by $P^{\perp} : H \to \mathscr{I}$ and defined as $P^{\perp} f = f - b\frac{\langle f,b\rangle_H}{\langle b,b\rangle_H}$ for any $f \in H$. Now we shall introduce some operators that will be of importance in order to transform the system (7.3.1). Let us consider the operator $U : H \to \mathbb{R} \times \mathscr{I}$ defined as

$$Uf = \begin{pmatrix} Pf \\ P_{\mathscr{I}} f \end{pmatrix}. \tag{7.3.4}$$

This operator is boundedly invertible, see Ilchmann et al. (2016), with inverse $U^{-1} : \mathbb{R} \times \mathscr{I} \to H$ expressed as

$$U^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha c + \eta. \tag{7.3.5}$$

The adjoint of U^{-1}, denoted by $U^{-*} : H \to \mathbb{R} \times \mathscr{I}$, reads as follows

$$U^{-*} f = \begin{pmatrix} \langle f, c\rangle_H \\ P^{\perp} f \end{pmatrix}. \tag{7.3.6}$$

Consequently, the operator $U^* : \mathbb{R} \times \mathscr{I} \to H$ is given by

$$U^* \begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha \frac{b}{\langle c, b\rangle_H} + \eta - \langle c, \eta\rangle_H \frac{b}{\langle c, b\rangle_H}. \tag{7.3.7}$$

Like the operators U and U^{-1}, the operators U^* and U^{-*} are linear and bounded. This allows us to perform the change of variables for system (7.3.1) defined by $x =$

$U^*\xi, \xi \in \mathbb{R} \times \mathscr{I}$. By the invertibility of U^*, one gets

$$\xi = U^{-*}x = \begin{pmatrix} \langle x, c\rangle_H \\ P^{\perp}x \end{pmatrix}. \tag{7.3.8}$$

By setting $\eta := P^{\perp}x$, one has that $\xi = \binom{y}{\eta}$. Now we take a look at the dynamics of the variable ξ. In the new variables (7.3.8), the system (7.3.1) reads as

$$\tilde{\Sigma}: \begin{cases} \dot{\xi}(t) = U^{-*}AU^*\xi(t) + U^{-*}f(U^*\xi(t)) + U^{-*}bu(t) + U^{-*}bd(t), \\ y(t) = (1 \quad 0)\xi(t), \\ \xi(0) = U^{-*}x_0 =: \xi_0. \end{cases} \tag{7.3.9}$$

Note that the systems Σ and $\tilde{\Sigma}$ are equivalent according to the state transformation induced by U^{-*}. By using the definitions of U^{-*} and U^*, see (7.3.6) and (7.3.7), the linear operator $U^{-*}AU^*$ can be rewritten as

$$\begin{aligned} U^{-*}AU^*\xi = \begin{pmatrix} \langle AU^*\xi, c\rangle_H \\ P^{\perp}AU^*\xi \end{pmatrix} &= \begin{pmatrix} \langle \xi, UA^*c\rangle_{\mathbb{R}\times\mathscr{I}} \\ P^{\perp}AU^*\xi \end{pmatrix} \\ &= \begin{pmatrix} y\frac{\langle A^*c,b\rangle_H}{\langle c,b\rangle_H} + \langle \eta, P_{\mathscr{I}}A^*c\rangle_H \\ y\frac{P^{\perp}Ab}{\langle c,b\rangle_H} + P^{\perp}A\eta - \frac{\langle c,\eta\rangle_H}{\langle c,b\rangle_H}P^{\perp}Ab \end{pmatrix} \\ &=: \begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}\xi, \end{aligned}$$

where the operators P_0, S, R and Q are defined as follows

$$\begin{aligned} &P_0 : \mathbb{R} \to \mathbb{R}, P_0 y = y\frac{\langle A^*c, b\rangle_H}{\langle c, b\rangle_H}, \\ &S : \mathscr{I} \to \mathbb{R}, S\eta = \langle \eta, P_{\mathscr{I}}A^*c\rangle_H, \\ &R : \mathbb{R} \to \mathscr{I}, Ry = y\frac{P^{\perp}Ab}{\langle c, b\rangle_H}, \\ &Q : D(Q) \subset \mathscr{I} \to \mathscr{I}, Q\eta = P^{\perp}A\eta - \frac{\langle c, \eta\rangle_H}{\langle c, b\rangle_H}P^{\perp}Ab, \end{aligned} \tag{7.3.10}$$

where $D(Q) = D(A) \cap \mathscr{I}$. According to Ilchmann et al. (2016), the operator Q is the infinitesimal generator of a C_0-semigroup[(2)] $(T_Q(t))_{t\geq 0}$ on $\mathscr{I}$. Moreover, since the operators P_0, S and R are bounded, the operator $\left(\begin{smallmatrix} P_0 & S \\ R & Q \end{smallmatrix}\right)$ is still the generator of a C_0-semigroup, see e.g. Curtain and Zwart (2020); Pazy (1983); Engel and Nagel (2006). By using (7.3.6) and (7.3.7), we can write the nonlinear part of (7.3.9) as

$$U^{-*}f(U^*\xi(t)) = \begin{pmatrix} \langle f(U^*\xi(t)), c\rangle_H \\ f(U^*\xi(t)) - \frac{\langle f(U^*\xi(t)),b\rangle_H}{\langle b,b\rangle_H}b \end{pmatrix}$$

[(2)] Without loss of generality there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_Q(t)\| \leq Me^{\omega t}$.

$$=: \begin{pmatrix} \langle \tilde{f}(y(t),\eta(t)),c\rangle_H \\ \tilde{f}(y(t),\eta(t)) - \frac{\langle \tilde{f}(y(t),\eta(t)),b\rangle_H}{\langle b,b\rangle_H} b \end{pmatrix},$$

where the nonlinear operator $\tilde{f}:\mathbb{R}\times\mathscr{I}\to H$ is defined as $\tilde{f}(y,\eta)=f(y\frac{b}{\langle c,b\rangle_H}+\eta-\langle c,\eta\rangle_H\frac{b}{\langle c,b\rangle_H})$. By using Assumption 7.3.2 and the fact that U^{-*} and U^* are linear bounded operators, the nonlinear operator $U^{-*}f(U^*\cdot)$ is still uniformly Lipschitz continuous from $\mathbb{R}\times\mathscr{I}$ into $\mathbb{R}\times\mathscr{I}$. This entails that the homogeneous part of (7.3.9) possesses a unique mild solution on $[0,\infty)$. Taking any initial condition in $\mathbb{R}\times D(Q)$ implies that this solution is classical. Now observe that the term $U^{-*}b$ is expressed as $\left(\langle b,c\rangle_H\ 0\right)^T$. From these observations it follows that the dynamics of y and η may be written as

$$\dot{y}(t)=P_0y(t)+S\eta(t)+\langle \tilde{f}(y(t),\eta(t)),c\rangle_H+\gamma u(t)+\gamma d(t) \tag{7.3.11}$$

and

$$\dot{\eta}(t)=Ry(t)+Q\eta(t)+\tilde{f}(y(t),\eta(t))-\frac{\langle \tilde{f}(y(t),\eta(t)),b\rangle_H}{\langle b,b\rangle_H}b, \tag{7.3.12}$$

with initial conditions $y(0)=y_0$ and $\eta(0)=\eta_0$, respectively, where $\gamma:=\langle b,c\rangle_H$.

7.3.3 Feasability of funnel control

The main theorem of this chapter is stated and proved here. That is, by considering some quite easily checkable assumptions, it is proved that the transformed equations (7.3.11)–(7.3.12) satisfy Assumptions 7.2.1–7.2.2, which implies that funnel control is feasible for (7.3.1) according to Theorem 7.2.1.

Therefore, observe that (7.3.11) admits the representation (7.2.2), where

1. The gain function $\Gamma:\mathbb{R}^2\to\mathbb{R}$ is defined as $\Gamma(d,\rho)=\gamma>0$;
2. The well-defined nonlinear operator $T:C(\mathbb{R}^+,\mathbb{R})\to L^\infty_{\mathrm{loc}}(\mathbb{R}^+,\mathbb{R})$ has the form

$$T(y)(t)=P_0y(t)+S\eta(t)+\langle \tilde{f}(y(t),\eta(t)),c\rangle_H, \tag{7.3.13}$$

where $\eta(t)$ is the mild solution of (7.3.12);

3. The function $N:\mathbb{R}^2\to\mathbb{R}$ reads as $N(d,\rho)=\gamma d+\rho$.

This shows that Assumption 7.2.1 on system (7.2.2) is satisfied. It remains to show that the nonlinear operator T given by (7.3.13) possesses the three properties of Assumption 7.2.2. This constitutes the main result of this section. Before going into the details of this result, we shall denote by $\Sigma_{y\to\eta}$ the system which can be viewed as a system with input y and output η and whose dynamics are described by (7.3.12). We make the following assumption on that system.

Assumption 7.3.4 *The system $\Sigma_{y\to\eta}$ whose dynamics are governed by (7.3.12) is BISBO stable in the following sense: for all $k > 0$ and all $\hat{k} > 0$, there exists $\tilde{k} > 0$ such that for all $y \in C(\mathbb{R}^+,\mathbb{R})$ and all $\eta_0 \in \mathscr{I}$,*

$$\sup_{t\in\mathbb{R}^+} |y(t)| \leq k \text{ and } \|\eta_0\| \leq \hat{k} \Rightarrow \sup_{t\in\mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}. \tag{7.3.14}$$

Theorem 7.3.3 *The operator T defined in (7.3.13), which arises from the nonlinear system (7.3.1) via the change of variables (7.3.8), satisfies Assumption 7.2.2.*

Proof. The proof is divided into three steps, according to the three items of Assumption 7.2.2.
Step 1: In order to show that T maps bounded trajectories into bounded ones, let us fix $k_1 > 0, \hat{k} > 0$ and $y \in C(\mathbb{R}^+,\mathbb{R}), \eta_0 \in \mathscr{I}$ such that $\sup_{t\in\mathbb{R}^+} |y(t)| \leq k_1$ and $\|\eta_0\| \leq \hat{k}$. There exists a positive $\tilde{k}$ such that, for this y, the mild solution of (7.3.12) with initial condition $\eta_0 \in \mathscr{I}$ satisfies $\sup_{t\in\mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}$, according to Assumption 7.3.4. Thanks to the expression (7.3.13) of T, the boundedness of the operator S and the Cauchy-Schwarz inequality, one may write that

$$|T(y)(t)| \leq |P_0||y(t)| + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})}\|\eta(t)\| + \|\tilde{f}(y(t),\eta(t))\|_H\|c\|_H.$$

Assumption 7.3.2 allows us to write

$$\begin{aligned}|T(y)(t)| &\leq |P_0||y(t)| + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})}\|\eta(t)\| \\ &\quad + \|\tilde{f}(y(t),\eta(t)) - \tilde{f}(0,0)\|_H\|c\|_H + \|\tilde{f}(0,0)\|_H\|c\|_H \\ &\leq |P_0||y(t)| + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})}\|\eta(t)\| + (l_1|y(t)| + l_2\|\eta(t)\|)\|c\|_H + \sigma\|c\|_H,\end{aligned} \tag{7.3.15}$$

where $l_1 > 0$ and $l_2 > 0$ denote the Lipschitz constants of the operator $\tilde{f}$ associated with y and η, respectively, and where the positive constant σ is such that[(3)] $\|f(0)\|_H \leq \sigma$. Consequently,

$$\sup_{t\in\mathbb{R}^+} |T(y)(t)| \leq |P_0|k_1 + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})}\tilde{k} + \tilde{\sigma}\|c\|_H =: k_2,$$

where $\tilde{\sigma} = \sigma + l_1 k_1 + l_2\tilde{k}$, which proves that T satisfies the BIBO condition required in Assumption 7.2.2.
Step 2: The causality can be easily established by noting that, for a fixed $y \in C(\mathbb{R}^+,\mathbb{R})$ the corresponding mild solution of (7.3.12) is unique. This entails that for $y,\hat{y} \in C(\mathbb{R}^+,\mathbb{R})$ such that $y_{|[0,t)} = \hat{y}_{|[0,t)}$, the corresponding mild solutions of (7.3.12), denoted by η and $\hat{\eta}$, respectively, satisfy $\eta_{|[0,t)} = \hat{\eta}_{|[0,t)}$. In view of the expression (7.3.13) of T, it follows that (7.2.4) holds.

[(3)]This is valid since the nonlinear operator f maps the whole space H into itself, meaning that any point in H has a finite image by f in the $H-$norm.

Step 3: For the local Lipschitz continuity, let us consider $t \geq 0$ and $y \in C([0,t], \mathbb{R})$. Now let us take $y_1, y_2 \in C(\mathbb{R}^+, \mathbb{R})$ such that y_i coincides with y up to time t for $i = 1, 2$. The mild solutions of (7.3.12) with input y_i and starting at time t are given by

$$\eta_i(\tilde{t}) = T_Q(\tilde{t} - t)\eta_{i,t} + \int_t^{\tilde{t}} T_Q(\tilde{t} - s) R y_i(s) ds + \int_t^{\tilde{t}} T_Q(\tilde{t} - s) P^{\perp} \tilde{f}(y_i(s), \eta_i(s)) ds$$

for any $\tilde{t} \in [t, t + \tau]$ with τ being an arbitrary positive constant. Note that the functions $\eta_{1,t}$ and $\eta_{2,t}$ correspond to $\eta_1(t)$ and $\eta_2(t)$, respectively. Since, by assumption, $y_1(t) = y(t) = y_2(t)$ and $\eta_1(0) = \eta_0 = \eta_2(0)$ and by the uniqueness of the mild solution of (7.3.12), the relation $\eta_{1,t} = \eta_{2,t}$ holds true. Consequently,

$$\begin{aligned} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq & \int_t^{\tilde{t}} \|T_Q(\tilde{t} - s) R (y_1(s) - y_2(s))\| ds \\ & + \int_t^{\tilde{t}} \|T_Q(\tilde{t} - s) P^{\perp} (\tilde{f}(y_1(s), \eta_1(s)) - \tilde{f}(y_2(s), \eta_2(s)))\| ds. \end{aligned}$$

Assumption 7.3.2 together with the boundedness of the operator R implies that

$$\begin{aligned} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq & (\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})} + 2l_1) \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} |y_1(s) - y_2(s)| ds \\ & + 2l_2 \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} \|\eta_1(s) - \eta_2(s)\| ds, \end{aligned}$$

where l_1 and l_2 are the positive constants introduced in (7.3.15). We shall use the notation $\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})} + 2l_1 =: \mathfrak{g}$ in what follows. Applying Gronwall's lemma to the function $e^{-|\omega|\tilde{t}} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\|$ yields the inequality

$$\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq \mathfrak{g} M e^{|\omega|\tilde{t}} e^{2Ml_2(\tilde{t}-t)} \int_t^{\tilde{t}} e^{-|\omega|s} |y_1(s) - y_2(s)| ds.$$

Taking the supremum over all $\tilde{t}$ in $[t, t+\tau]$ on both sides yields the estimate

$$\sup_{\tilde{t} \in [t,t+\tau]} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq \mathfrak{g} M e^{(|\omega| + 2Ml_2)\tau} \tau \sup_{\tilde{t} \in [t,t+\tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|. \tag{7.3.16}$$

The notation $\mathfrak{f}_\tau := \mathfrak{g} M e^{(|\omega| + 2Ml_2)\tau} \tau$ will be used for the sake of simplicity. According to the definition (7.3.13) of the nonlinear operator T, it holds that

$$\begin{aligned} & |T(y_1)(\tilde{t}) - T(y_2)(\tilde{t})| \leq |P_0| |y_1(\tilde{t}) - y_2(\tilde{t})| \\ & \quad + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| + l_1 \|c\|_H |y_1(\tilde{t}) - y_2(\tilde{t})| + l_2 \|c\|_H \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\|, \end{aligned}$$

which, combined with (7.3.16), leads to

$$\sup_{\tilde{t} \in [t,t+\tau]} |T(y_1)(\tilde{t}) - T(y_2)(\tilde{t})| \leq \rho \sup_{\tilde{t} \in [t,t+\tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|,$$

where $\rho := |P_0| + \|S\|_{\mathscr{L}(\mathscr{I},\mathbb{R})} \mathfrak{f}_\tau + l_1 \|c\|_H + l_2 \|c\|_H \mathfrak{f}_\tau$. □

This means that funnel control is feasible for a nonlinear infinite-dimensional system of the form (7.3.1) which satisfies Assumptions 7.3.1, 7.3.2, 7.3.3 and 7.3.4.

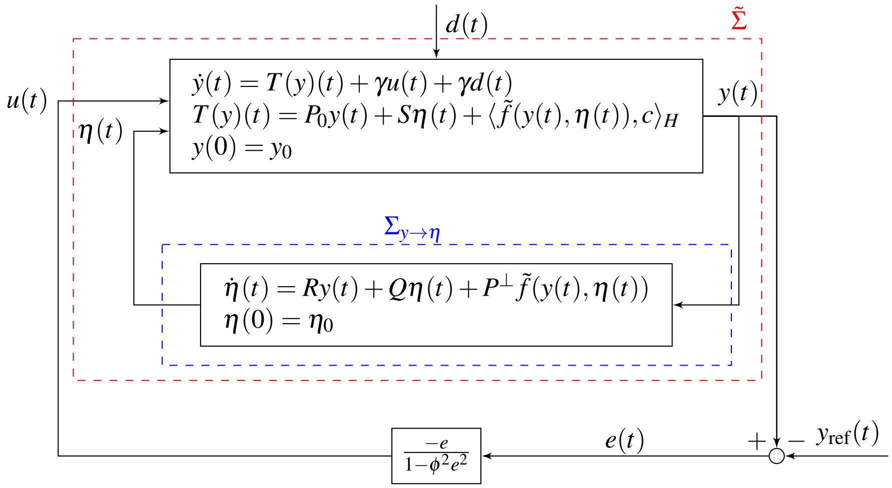

Figure 7.1 – Interconnection of $\tilde{\Sigma}$ and the funnel controller (7.2.8).

Moreover the closed-loop system which consists of the interconnection of (7.3.1), described by (7.3.11)–(7.3.12) (system $\tilde{\Sigma}$), with the funnel controller (7.2.8) has the properties described in Theorem 7.2.1. This system is depicted in Figure 7.1.

We state hereafter a useful criterion for checking Assumption 7.3.4.

Proposition 7.3.4 *Assuming that the semigroup $(T_Q(t))_{t\geq 0}$ is exponentially stable on $\mathscr{I}$ and that the nonlinear operator f satisfies $\|f(x)\|_H \leq \hat{\sigma}$, for some constant $\hat{\sigma} > 0$ independent of x, for any $x \in H$, is sufficient to ensure that Assumption 7.3.4 is satisfied.*

Proof. Let us fix $k_1 > 0, \hat{k} > 0$ and $y \in C(\mathbb{R}^+, \mathbb{R}), \eta_0 \in \mathscr{I}$ such that $\sup_{t\in\mathbb{R}^+} |y(t)| \leq k_1$ and $\|\eta_0\| \leq \hat{k}$. As the semigroup $(T_Q(t))_{t\geq 0}$ is exponentially stable, the inequality $\|T_Q(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}$ holds for some $\tilde{M} \geq 1$ and $\tilde{\omega} > 0$. Moreover the mild solution of (7.3.12) with initial condition $\eta_0 \in \mathscr{I}$ and with the function y fixed above is given by

$$\eta(t) = T_Q(t)\eta_0 + \int_0^t T_Q(t-s)Ry(s)ds + \int_0^t T_Q(t-s)P^{\perp}\tilde{f}(y(s),\eta(s))ds.$$

By taking the $H-$norm (restricted to $\mathscr{I}$) on both sides and by using the exponential stability of $(T_Q(t))_{t\geq 0}$, one gets that

$$\|\eta(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}\|\eta_0\| + \int_0^t \tilde{M}e^{-\tilde{\omega}(t-s)}\|Ry(s)\|ds + \int_0^t \tilde{M}e^{-\tilde{\omega}(t-s)}\|P^{\perp}\tilde{f}(y(s),\eta(s))\|ds.$$

The boundedness of the operator R combined with the definition of $P^{\perp}$ and the assumption on the operator f entails that

$$\|\eta(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}\|\eta_0\| + \tilde{M}e^{-\tilde{\omega}t}\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})}\int_0^t e^{\tilde{\omega}s}|y(s)|ds + 2\hat{\sigma}\tilde{M}e^{-\tilde{\omega}t}\int_0^t e^{\tilde{\omega}s}ds.$$

It follows, by using the estimate $\sup_{t\in\mathbb{R}^+}|y(t)| \leq k_1$, that

$$\|\eta(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}\left(\|\eta_0\| + (\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})}k_1 + 2\hat{\sigma})\int_0^t e^{\tilde{\omega}s}ds\right).$$

Hence

$$\|\eta(t)\| \leq \tilde{M}\left(e^{-\tilde{\omega}t}\|\eta_0\| + \frac{\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})}k_1 + 2\hat{\sigma}}{\tilde{\omega}}(1 - e^{-\tilde{\omega}t})\right) \leq \kappa, \tag{7.3.17}$$

where $\kappa := \tilde{M}\left(\hat{k} + \frac{\|R\|_{\mathscr{L}(\mathbb{R},\mathscr{I})}k_1 + 2\hat{\sigma}}{\tilde{\omega}}\right)$ does not depend on t. □

7.4 Applications: from wave to hyperbolic systems

Here we consider different nonlinear infinite-dimensional systems to which the funnel control (7.2.8) is applied. The first control problem we are dealing with is the regulation of the average temperature in a nonisothermal plug-flow tubular reactor whose dynamics are given by the unperturbed part of (6.2.3). Then two different versions of the damped sine-Gordon equation will be considered, one in which Dirichlet boundary conditions are taken into account and the other involving mixed boundary conditions, Dirichlet at one boundary and Neumann at the other. After writing any system in an abstract way, one shall verify that Assumptions 7.3.1, 7.3.2 and 7.3.3 are satisfied. Moreover, Proposition 7.3.4 will be used to characterize the BISBO stability required in Assumption 7.3.4.

7.4.1 A nonlinear plug-flow tubular reactor model

Let us consider the unperturbed[(4)] and dimensionless dynamics of a PFTR described by (6.2.3), i.e.

$$\left\{\begin{array}{l} \partial_t\theta_1 = -\partial_z\theta_1 + \delta f(\theta_1,\theta_2) + \beta(1_{[0,1]}(z)\theta_w(t) - \theta_1) \\ \partial_t\theta_2 = -\partial_z\theta_2 + f(\theta_1,\theta_2) \\ \theta(0,t) = 0, \theta_2(0,t) = 0, \end{array}\right. \tag{7.4.1}$$

where the dimensionless temporal and spatial variables satisfy $t \geq 0$ and $z \in [0,1]$. Note that the same notations as those considered in Chapter 6 for ∂_t and ∂_z are considered here. As it is explained in Chapter 6 the state components θ_1 and θ_2 stand for the dimensionless temperature and reactant concentration, respectively. What is different

[(4)]By unperturbed we mean that $1 + d_T = 0$ and $1 - d_C = 0$.

from (6.2.3) here is that the nonlinear operator is not only defined on the closed and convex subset $D = \{\theta := (\theta_1 \quad \theta_2)^T \in X, -1 \leq \theta_1, 0 \leq \theta_2 \leq 1\}$ but it is extended on the whole space $X := L^2([0,1];\mathbb{R}) \times L^2([0,1];\mathbb{R})$ in the following way

$$f(\theta_1,\theta_2) = \begin{cases} 0 \text{ if } \theta_1 < -1 \\ \alpha e^{\frac{\mu\theta_1}{1+\theta_1}} \text{ if } \theta_1 \geq -1 \text{ and } \theta_2 < 0 \\ \alpha(1-\theta_2)e^{\frac{\mu\theta_1}{1+\theta_1}} \text{ if } \theta_1 \geq -1 \text{ and } 0 \leq \theta_2 \leq 1 \\ 0 \text{ otherwise.} \end{cases} \tag{7.4.2}$$

Note that this definition of f implies that the latter is uniformly Lipschitz continuous as a pointwise function defined from $\mathbb{R}^2$ into $\mathbb{R}$. Hence it possesses that property viewed as an operator from X into X. Moreover, it satisfies $|f(x,y)| \leq \alpha e^{\mu}$ for any $(x \quad y)^T \in \mathbb{R}^2$. The positive constants α, β, μ and δ depend on the model parameters as in (2.3.4). The scalar control variable, denoted by $\theta_w(t)$, is due to a heat exchanger that acts as a distributed control input along the reactor through the characteristic function $1_{[0,1]}(z)$.

The control objective that is taken into account here consists in the tracking of the following output function which corresponds to the mean value of the temperature along the reactor:

$$y(t) = \int_0^1 \theta_1(z,t)dz. \tag{7.4.3}$$

In order to reach this goal, we shall develop a funnel controller producing an input $\theta_w(t)$ which will take the form (7.2.8) for some reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+,\mathbb{R})$. First observe that (7.4.1) admits the abstract representation

$$\begin{cases} \dot{x}(t) = Ax(t) + F(x(t)) + Bu(t) \\ x(0) = x_0 \in X \\ y(t) = \langle c, x(t)\rangle_X, \end{cases} \tag{7.4.4}$$

where the state space is chosen as being X equipped with the inner product

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_X := \langle x_1, w_1\rangle_{L^2} + \langle x_2, w_2\rangle_{L^2}, \tag{7.4.5}$$

for $(x_1 \quad x_2)^T, (w_1 \quad w_2)^T \in X$. Note that $\langle x_1, w_1\rangle_{L^2}$ stands for the standard inner product on $L^2([0,1];\mathbb{R})$ defined by

$$\langle x_1, w_1\rangle_{L^2} = \int_0^1 x_1(z)w_1(z)dz. \tag{7.4.6}$$

The unbounded linear operator A is given by $A = \begin{pmatrix} -d_z - \beta I & 0 \\ 0 & -d_z \end{pmatrix}$ on the dense linear subspace

$$D(A) = \left\{(x_1 \quad x_2)^T \in H^1([0,1];\mathbb{R}) \times H^1([0,1];\mathbb{R}), x_1(0) = 0 = x_2(0)\right\}.$$

From that definition of $(A, D(A))$ and according to Example 2.1.3, the operator A is the infinitesimal generator of a contraction (and even exponentially stable) C_0-semigroup on X. Hence Assumption 7.3.1 is satisfied. The nonlinear operator $F: X \to X$ is given by $F(x_1,x_2) = (\delta f(x_1,x_2) \quad f(x_1,x_2))^T$. Due to the definition of f given in (7.4.2), the nonlinear operator F is uniformly Lipschitz continuous on X. Moreover, it satisfies $\|F(x_1,x_2)\|_X \leq \alpha e^{\mu}\sqrt{\delta^2+1}$ for any $(x_1 \quad x_2)^T \in X$ as a same kind of property holds for the pointwise component $f: \mathbb{R}^2 \to \mathbb{R}$. This implies that Assumption 7.3.2 is fulfilled. The control operator $B: \mathbb{R} \to X$ is given by $Bu = (\beta 1_{[0,1]}(z) \quad 0)^T u$ while the observation operator $C: X \to \mathbb{R}$ takes the form

$$C(x_1 \quad x_2)^T = \left\langle \begin{pmatrix} 1_{[0,1]} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle_X . \tag{7.4.7}$$

Hence the functions b and c satisfy $b(z) = \beta c(z) = (\beta 1_{[0,1]}(z) \quad 0)^T$. Note that the domain $D(A^*)$ of the adjoint operator of A is given by

$$\left\{ (x_1 \quad x_2)^T \in H^1([0,1];\mathbb{R}) \times H^1([0,1];\mathbb{R}), x_1(1) = 0 = x_2(1) \right\}.$$

It is obvious that b and c do not lie in $D(A)$ and $D(A^*)$, respectively. In order to overcome this difficulty, we approximate the function $1_{[0,1]}(z)$ by a linear combination of elements of the orthonormal basis $\left\{ \sqrt{2}\sin(n\pi z) \right\}_{n\in\mathbb{N}_0}$ of $L^2([0,1];\mathbb{R})$. For a fixed $N \in \mathbb{N}_0$, the $N-$th order approximation of $1_{[0,1]}(z)$, denoted by $1_N(z)$, is given by

$$1_N(z) = \sum_{n=1}^{N} \langle 1_{[0,1]}(\cdot), \sqrt{2}\sin(n\pi\cdot)\rangle_{L^2(0,1)} \sqrt{2}\sin(n\pi z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{4}{n\pi}\sin(n\pi z). \tag{7.4.8}$$

As this approximation lies in $H^1([0,1];\mathbb{R})$ and vanishes both for $z = 0$ and $z = 1$, the approximations of $b(z)$ and $c(z)$, denoted by $b_N(z)$ and $c_N(z)$ and whose expressions are given by $(\beta 1_N(z) \quad 0)^T$ and $(1_N(z) \quad 0)^T$ are in $D(A)$ and $D(A^*)$, respectively. Now observe that the inner product between b_N and c_N is given by

$$\langle b_N, c_N\rangle_X = \beta\langle 1_N, 1_N\rangle_{L^2(0,1)} = \beta \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{8}{n^2\pi^2} > 0,$$

which implies that Assumption 7.3.3 is satisfied. Now it remains to show that the system $\Sigma_{y\to\eta}$ is BISBO stable in the sense of Assumption 7.3.4. We shall use the characterization of Proposition 7.3.4 to reach this aim. To this end, let us first introduce the decomposition of the state space X as in (7.3.2), which is given here by:

$$\begin{aligned} X &= \operatorname{span}\{c_N\} \oplus \{b_N\}^{\perp} = \operatorname{span}\{c_N\} \oplus \{c_N\}^{\perp} \\ &= \operatorname{span}\left\{ \left(\begin{smallmatrix} 1_N \\ 0 \end{smallmatrix}\right) \right\} \oplus \left\{ \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in X, \left\langle \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1_N \\ 0 \end{smallmatrix}\right) \right\rangle_X = 0 \right\} = \mathscr{C} \oplus \mathscr{I}, \end{aligned}$$

where $\mathscr{I}$ may also be written as

$$\mathscr{I} = \left\{ x_1 \in L^2([0,1];\mathbb{R}), \langle x_1, 1_N\rangle_{L^2} = 0 \right\} \times L^2([0,1];\mathbb{R}). \tag{7.4.9}$$

The operator $Q : D(Q) = D(A) \cap \mathscr{I} \to \mathscr{I}$ whose definition is given in (7.3.10) is expressed as

$$Q\eta = P^{\perp}A\eta - \frac{\langle c_N, \eta\rangle_X}{\langle c_N, b_N\rangle_X} P^{\perp}Ab_N,$$

for $\eta = (\eta_1 \quad \eta_2)^T \in D(Q)$. Expanding the application of Q to η gives rise to

$$\begin{aligned} Q\eta &= A\eta - \frac{\langle A\eta, b_N\rangle_X}{\langle b_N, b_N\rangle_X} b_N \\ &= \begin{pmatrix} -d_z\eta_1 - \beta\eta_1 \\ -d_z\eta_2 \end{pmatrix} + \frac{\langle d_z\eta_1, 1_N\rangle_{L^2(0,1)}}{\langle 1_N, 1_N\rangle_{L^2(0,1)}} \begin{pmatrix} 1_N \\ 0 \end{pmatrix} =: \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{aligned}$$

where the relation $\langle \eta_1, 1_N\rangle_{L^2} = 0$ has been used. The operator $Q_1 : D(Q_1) \subset \{1_N\}^{\perp} \to \{1_N\}^{\perp}$ is defined as

$$Q_1\eta_1 = -d_z\eta_1 - \beta\eta_1 + \frac{\langle d_z\eta_1, 1_N\rangle_{L^2}}{\langle 1_N, 1_N\rangle_{L^2}} 1_N,$$

for $\eta_1 \in D(Q_1)$ given by

$$D(Q_1) = \left\{x_1 \in H^1([0,1];\mathbb{R}), x_1(0) = 0\right\} \cap \{1_N\}^{\perp},$$

whereas $Q_2\eta_2 = -d_z\eta_2$ on

$$D(Q_2) = \left\{x_2 \in H^1([0,1];\mathbb{R}), x_2(0) = 0\right\}.$$

According to Lyapunov's Theorem, see e.g. (Curtain and Zwart, 2020, Theorem 4.1.3), the semigroup generated by the operator Q is exponentially stable if and only if there exists a positive self-adjoint operator $P \in \mathscr{L}(\mathscr{I})$ such that

$$\langle Q\eta, P\eta\rangle_{\mathscr{I}} + \langle P\eta, Q\eta\rangle_{\mathscr{I}} \leq -\langle \eta, \eta\rangle_{\mathscr{I}},$$

for all $\eta \in D(Q)$. Note that, for any two functions in $\mathscr{I}$, their inner product on $\mathscr{I}$ is the same as the one defined in (7.4.5). Let us propose the following form for the operator P

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Let us define P_1 as $P_1\eta_1 = \frac{1}{2\beta}\eta_1$ and P_2 as

$$(P_2\eta_2)(z) = (1-z)\eta_2(z), z \in [0,1].$$

Now observe that, for $\eta_1 \in D(Q_1)$, there holds

$$\begin{aligned} &\langle Q_1\eta_1, P_1\eta_1\rangle + \langle P_1\eta_1, Q_1\eta_1\rangle = 2\langle Q_1\eta_1, P_1\eta_1\rangle \\ &= -\frac{1}{\beta}\langle d_z\eta_1, \eta_1\rangle - \langle \eta_1, \eta_1\rangle + \frac{1}{\beta}\frac{\langle d_z\eta_1, 1_N\rangle_{L^2}}{\langle 1_N, 1_N\rangle_{L^2}}\langle 1_N, \eta_1\rangle \end{aligned}$$

$$= -\frac{1}{2\beta}\eta_1^2(1) - \langle \eta_1, \eta_1 \rangle \leq -\langle \eta_1, \eta_1 \rangle,$$

where the fact that η_1 and 1_N are orthogonal has been used. Moreover, for any $\eta_2 \in D(Q_2)$, one has that

$$\begin{aligned}&\langle Q_2\eta_2, P_2\eta_2\rangle + \langle P_2\eta_2, Q_2\eta_2\rangle \\ &= -2\int_0^1 (1-z)\eta_2(d_z\eta_2)dz = \left[-(1-z)\eta_2^2(z)\right]_0^1 - \langle \eta_2, \eta_2\rangle = -\langle \eta_2, \eta_2\rangle.\end{aligned}$$

Thus the semigroup generated by the operator Q is exponentially stable. This fact combined with the properties of the nonlinear operator F and Proposition 7.3.4 ensures that Assumption 7.3.4 is satisfied. Hence funnel control is feasible for the system (7.4.1) in which the indicator function $1_{[0,1]}$ is approximated by 1_N, which means that considering a heat exchanger temperature expressed as

$$\theta_w(t) = \frac{-e(t)}{1-\phi^2(t)e^2(t)}, \tag{7.4.10}$$

where $e(t) = y(t) - y_{\text{ref}}(t), y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R}), \phi \in \Phi$, and y is given by (7.4.3), yields a closed-loop system which possesses the properties stated in Theorem 7.2.1.

As an illustration of the results, we shall report numerical simulations hereafter. The parameters related to the model (7.4.1) have been chosen as follows:

$$\delta = 0.25, \alpha = 2.3248, \mu = 16.6607, \beta = 8,$$

see Aksikas (2005). The number of basis functions N in the approximation of the function b has been set at $N = 100$. As initial conditions for θ_1 and θ_2 we consider the following functions

$$\theta_1(z,0) = 0.02(-z^3 + z^2 + z), \theta_2(z,0) = 0.7(-z^3 + z^2 + z).$$

The reference signal that has to be tracked by the output (7.4.3) is

$$y_{\text{ref}}(t) = \frac{1}{20} + \frac{1}{20}\arctan(t).$$

The funnel boundary is chosen as $\frac{1}{\phi(t)} = e^{-2t} + 0.0025$. Obviously, there holds $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R}))$ while $\phi \in \Phi$ defined in (7.2.7).

The output trajectory (7.4.3) and the reference signal $y_{\text{ref}}(t)$ are depicted in Figure 7.2 and the funnel control, $\theta_w(t)$, given by (7.4.10), is shown in Figure 7.3. The tracking error is depicted in Figure 7.4. The state trajectories corresponding to $\theta_1(t,z)$ and $\theta_2(t,z)$ are represented in Figures 7.5 and 7.6, respectively.

Remark 7.4.1 *The method that has been used to compute numerically the state trajectories is based on a finite-dimensional approximation of the operators A and F by means of finite differences. The spatial coordinate has been discretized into n equal pieces, $n = 50$. Based on this, a finite-dimensional approximation of the operator A has been obtained, let us denote it by $\hat{A}_n \in \mathbb{R}^{2n\times 2n}$. Let us also denote by $X_n \in \mathbb{R}^{2n}$ the approximation of the state vector X. Its components are* *given by*

$$X_n^i(t) = \theta_1((i-1)h,t), X_n^{i+n}(t) = \theta_2((i-1)h,t), i = 1,\dots,n,$$

where h *stands for the discretization step* ($h = \frac{1}{n-1}$)*. The nonlinear term,* F*, has been discretized in order to obtain a vector,* $\hat{F}_n$*, whose action on the state* X_n *is given by*

$$F_n^i(X_n) = \delta f(X_n^i, X_n^{i+n}), F_n^{i+n}(X_n) = f(X_n^i, X_n^{i+n}), i = 1,\dots,n.$$

The control operator b_N *has been discretized too, yielding a control vector,* $\hat{b}_{N,n} \in \mathbb{R}^{2n}$*, whose components are expressed as*

$$\hat{b}_{N,n}^i = \beta 1_N((i-1)h), \hat{b}_{N,i+n} = 0, i = 1,\dots,n.$$

The corresponding output function, (7.4.7), has also been computed numerically via the routine `trapz` *of Matlab©. It gives a numerical approximation of the integral of a function via the trapezium method. Let us denote by* $y_n(t)$ *the approximation of the output function. Then, the linear finite-dimensional approximation of (7.4.4),* $\dot{X}_n(t) = \hat{A}_n X_n(t) + \hat{F}_n(X_n(t)) + \hat{b}_{N,n} u_n(t)$*, has been numerically integrated via the routine* `ode15s` *of Matlab©, where the approximated funnel controller is given by*

$$u_n(t) = \frac{-e_n(t)}{1-\phi^2(t)e_n^2(t)}, e_n(t) = y_n(t) - y_{ref}(t),$$

which constitutes an approximation of the funnel controller (7.4.10).

7.4.2 Different models of damped sine-Gordon equations

Two sine-Gordon PDEs are considered here, involving the same PDEs but with different boundary conditions. For any of these two models we consider uniform distributed actuation by a scalar input function designed as a funnel control that aims at regulating some average quantity which can be viewed as the average of a linear combination of the state components.

7.4.2.1 Mixed-boundary conditions

The model that will be of interest here is governed by the following dynamics

$$\begin{cases} \partial_{tt}^2 x = \partial_{zz}^2 x - \alpha \partial_t x + \nu \sin(x) + b(z)u(t) \\ x(0,t) = 0, \partial_z x(1,t) = 0, \end{cases} \tag{7.4.11}$$

where the space variable $z \in [0,1]$ and $t \in \mathbb{R}^+$ denotes the time variable. The parameters ν and α are such that $\nu \in \mathbb{R}_0$ and $\alpha > \pi$. The homogeneous part of this nonlinear PDE ($u \equiv 0$) has already been introduced in (2.2.4), for which well-posedness has

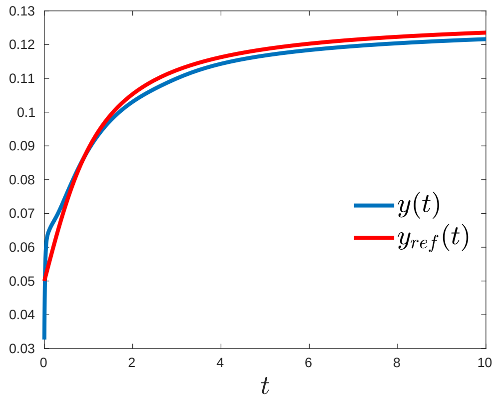

Figure 7.2 – Output trajectory (7.4.3) with the reference signal $y_{\mathrm{ref}}(t)$.

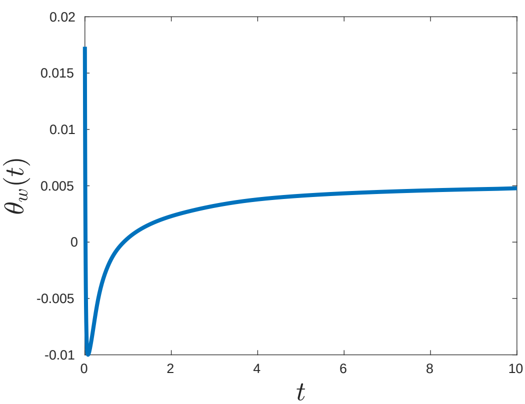

Figure 7.3 – Input trajectory.

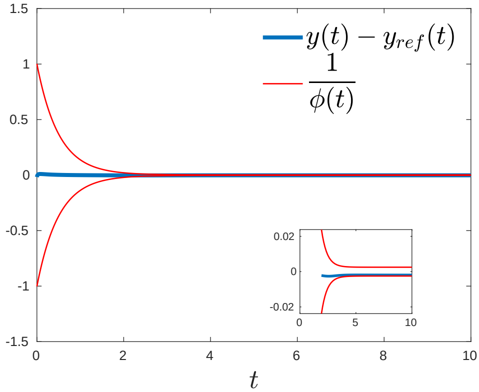

Figure 7.4 – Output error trajectory $y(t) - y_{\mathrm{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

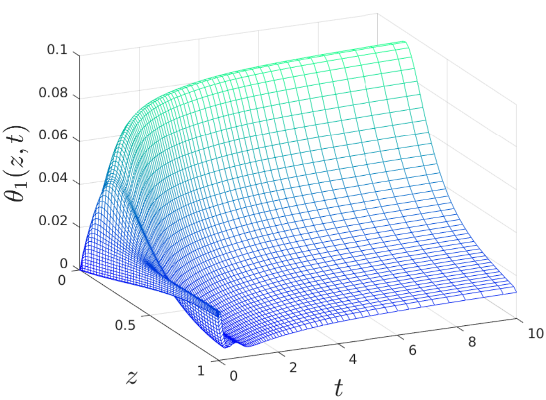

Figure 7.5 – Closed-loop state trajectory $\theta_1(t,z)$.

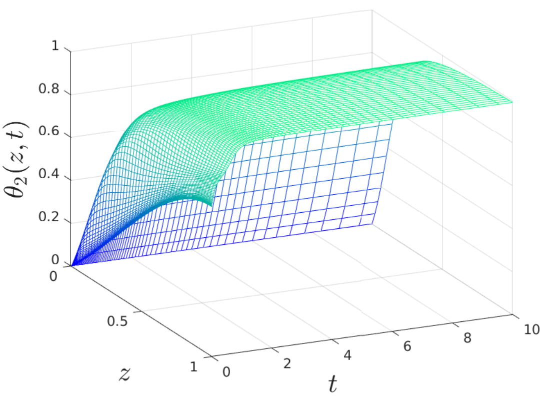

Figure 7.6 – Closed-loop state trajectory $\theta_2(t,z)$.

been studied in terms of existence and uniqueness of a mild solution (even a classical solution) on the positive real line $\mathbb{R}^+$. The latter encompasses many phenomena in physics as the dynamics of a Josephson junction driven by a current source, see e.g. Temam (1997); Cuevas-Maraver et al. (2014), as well as the dynamics of mechanical transmission lines, see Cirillo et al. (1981) among others. The stability of the homogeneous dynamics of (7.4.11) has been investigated for Dirichlet and Neumann boundary conditions in Dickey (1976) and Callegari and Reiss (1973). For control problems related to (7.4.11), we refer to Dolgopolik et al. (2016) and Efimov et al. (2019) for instance, where boundary energy control and robust input-to-state stability are developed.

Let us consider the operator $A_0 = -\frac{d^2}{dz^2}$ on the domain

$$D(A_0) = \left\{ x \in H^2([0,1];\mathbb{R}), x(0) = 0 = \frac{dx}{dz}(1) \right\}.$$

As the operator A_0 is self-adjoint and coercive[(5)], it admits a unique nonnegative square-root, see e.g. (Curtain and Zwart, 2020, Lemma A.3.82), which satisfies

$$D(A_0^{\frac{1}{2}}) = \left\{ x \in H^1([0,1];\mathbb{R}), x(0) = 0 \right\}.$$

This allows us to consider the Hilbert state space $Z = D(A_0^{\frac{1}{2}}) \times X$ equipped with the inner product[(6)]

$$\left\langle \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_Z := \langle d_z \zeta_1, d_z w_1 \rangle_X + \langle \zeta_2, w_2 \rangle_X, \tag{7.4.12}$$

[(5)]By using Poincaré's inequality it can be seen that, for any $x \in D(A_0)$, the relation $\langle A_0 x, x \rangle_{L^2} \geq \frac{\pi^2}{4} \|x\|_{L^2}^2$ holds.

[(6)]The equality $\langle d_z \zeta_1, d_z w_1 \rangle_X = \langle A_0^{\frac{1}{2}} \zeta_1, A_0^{\frac{1}{2}} w_1 \rangle_X$ holds.

where $\zeta_1, w_1 \in D(A_0^{\frac{1}{2}})$ and $\zeta_2, w_2 \in X$ with $X := L^2([0,1];\mathbb{R})$, see (2.1.37). The inner product on X is the same as the one defined in (7.4.6). Let us consider the state variable $\zeta = \begin{pmatrix} x \\ \partial_t x \end{pmatrix} =: \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$. Hence, as is has already been shown in Chapter 2, Section 2.2, the PDE (7.4.11) may be written as (7.4.4), where the operator A is given by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix} \tag{7.4.13}$$

on $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$. According to (Curtain and Zwart, 2020, Example 2.3.5), the operator A is the generator of a contraction C_0-semigroup on Z. Note that the adjoint operator of A, denoted by A^*, is expressed as $A^* = \begin{pmatrix} 0 & -I \\ A_0 & -\alpha I \end{pmatrix}$ on $D(A^*) = D(A)$. The nonlinear operator $F: Z \to Z$ is expressed as $F(\zeta_1, \zeta_2) = \begin{pmatrix} 0 \\ \nu \sin(\zeta_1) \end{pmatrix}$. The latter is uniformly Lipschitz continuous and satisfies $\|F(\zeta_1, \zeta_2)\|_Z \leq |\nu|$ for any $(\zeta_1 \quad \zeta_2)^T \in Z$. Consequently, Assumptions 7.3.1 and 7.3.2 are satisfied.

Here we consider that the operator $B: \mathbb{R} \to Z$ is defined for $u \in \mathbb{R}$ by $Bu = b(z)u = \begin{pmatrix} 0 \\ b_N(z) \end{pmatrix} u$ where $b_N(z) = 1_N(z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{4}{n\pi} \sin(n\pi z)$ is the function defined in (7.4.8). Obviously $b \in D(A)$. For the function c, we choose the expression

$$c(z) = \begin{pmatrix} \frac{2}{\pi^2}(\alpha + \sqrt{\alpha^2 - \pi^2}) \sin(\frac{\pi}{2} z) \\ \sin(\frac{\pi}{2} z) \end{pmatrix},$$

so that the output trajectory corresponding to (7.4.11) is given by

$$\begin{aligned} y(t) &= \langle c, \zeta(t) \rangle_Z \\ &= \frac{\alpha + \sqrt{\alpha^2 - \pi^2}}{\pi} \int_0^1 \cos\left(\frac{\pi}{2} z\right) \partial_z x(t,z) dz + \int_0^1 \sin\left(\frac{\pi}{2} z\right) \partial_t x(t,z) dz. \end{aligned} \tag{7.4.14}$$

As it is shown later, this choice of c is made in order to obtain a particular form of the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$ and it does not necessarily has a meaningful physical interpretation in terms of measurements on the system (7.4.11). It can be easily seen that the function $c \in D(A^*)$. A straightforward computation shows that

$$\langle b, c \rangle_Z = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{16}{\pi^2 (4n^2 - 1)} > 0,$$

which entails that Assumption 7.3.3 is satisfied. Before showing that funnel control is feasible in our context, let us introduce the decomposition of the state space

$$Z = \operatorname{span}\{c\} \oplus \{b\}^{\perp} = \mathscr{C} \oplus \mathscr{I},$$

with

$$\mathscr{I} = D(A_0^{\frac{1}{2}}) \times \{x \in X, \langle x, b_N \rangle_{L^2} = 0\}. \tag{7.4.15}$$

According to (7.3.8) the system (7.4.11) admits the representation (7.3.11)–(7.3.12), in which we shall focus on the linear part, i.e. the operator $\left(\begin{smallmatrix} P_0 & S \\ R & Q \end{smallmatrix}\right)$. In order to show that funnel control is feasible for (7.4.11), one should use the criterion of BISBO stability stated in Proposition 7.3.4. Therefore it remains to show that the C_0-semigroup generated by the operator Q, see (7.3.10), is exponentially stable. First observe that the operator P_0 takes the form $P_0 y = p_0 y$, where

$$\begin{aligned} p_0 &= \frac{\langle A^* c, b\rangle_Z}{\langle b, c\rangle_Z} \\ &= \frac{\langle \frac{1}{2}(\alpha + \sqrt{\alpha^2 - \pi^2}) \sin\left(\frac{\pi}{2}z\right) - \alpha \sin\left(\frac{\pi}{2}z\right), b_N(z)\rangle_X}{\langle \sin\left(\frac{\pi}{2}z\right), b_N(z)\rangle_X} = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - \pi^2} < 0, \end{aligned}$$

which means that the semigroup generated by P_0, which is given by

$$\left(e^{(-\frac{\alpha}{2}+\frac{1}{2}\sqrt{\alpha^2-\pi^2})t}\right)_{t\geq 0},$$

is exponentially stable. Secondly, let us compute the operator $S : \mathscr{I} \to \mathbb{R}, S\eta = \langle \eta, P_{\mathscr{I}} A^* c\rangle_Z$. Observe that

$$\begin{aligned} P_{\mathscr{I}} A^* c &= A^* c - P_0 c \\ &= \begin{pmatrix} -1 \\ -\frac{\alpha}{2}+\frac{1}{2}\sqrt{\alpha^2-\pi^2} \end{pmatrix} \sin\left(\frac{\pi}{2}z\right) \\ &\quad + \left(\frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2-\pi^2}\right) \begin{pmatrix} \frac{2}{\pi^2}(\alpha+\sqrt{\alpha^2-\pi^2}) \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{2}z\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Consequently, the operator $\left(\begin{smallmatrix} P_0 & S \\ R & Q \end{smallmatrix}\right)$ is a triangular operator of the form $\left(\begin{smallmatrix} P_0 & 0 \\ R & Q \end{smallmatrix}\right)$. As it is similar to the operator A, the corresponding semigroups are also similar, i.e. denoting by $(S(t))_{t\geq 0}$ and by $(\tilde{S}(t))_{t\geq 0}$ the C_0-semigroups generated by A and $\left(\begin{smallmatrix} P_0 & S \\ R & Q \end{smallmatrix}\right)$, respectively, the relation $\tilde{S}(t) = U^{-*} S(t) U^*$ holds for all $t \geq 0$. Consequently, $(S(t))_{t\geq 0}$ and $(\tilde{S}(t))_{t\geq 0}$ have the same growth bounds. In that way, let us have a look at the sign of the growth bound of the semigroup $(S(t))_{t\geq 0}$. The latter is negative according to Propositions 2.1.7 and 2.1.5. As the growth bound of the semigroup generated by P_0 is negative too, the growth bound of the semigroup generated by Q is also negative, showing that this semigroup is exponentially stable. Thanks to Proposition 7.3.4 the system $\Sigma_{y\to\eta}$ is BISBO stable in the sense of Assumption 7.3.4. As a consequence, funnel control is feasible for the sine-Gordon equation (7.4.11) with the output given in (7.4.14).

We shall now illustrate the feasability of funnel control on (7.4.11) with some numerical simulations. Let us consider the following set of parameters: $\alpha = \pi + \frac{1}{6}, \nu = -1$. The initial conditions for the variables x and $\partial_t x$ have been chosen as

$$x(z,0) = \frac{1}{6}\sin\left(\frac{\pi}{2}z\right), \partial_t x(z,0) = \frac{1}{5}(2z^2 - z^4),$$

while the reference signal $y_{\text{ref}}(t) = \frac{1}{5}\cos(e^{-\frac{t}{4}}) \in W^{1,\infty}(\mathbb{R}^+,\mathbb{R})$. The function $\phi(t)$, whose inverse determines the funnel boundaries, is fixed to

$$\phi(t) = \frac{1}{e^{-2t} + 0.0025}.$$

The relation $\phi \in \Phi$ holds true. Moreover, it can be shown easily that the vector of initial conditions $(x(z,0) \quad \partial_t x(z,0))^T$ lie in $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$.

The output trajectory (7.4.14) with the reference signal $y_{\text{ref}}(t)$ are represented in Figure 7.7 whereas the corresponding funnel control is given in Figure 7.8. It can be seen that the output tracks the reference quite well. The tracking error is depicted in Figure 7.9 wherein one observes that it remains within the funnel boundaries, as was to be expected. The state trajectories corresponding to $x(z,t)$ and $\partial_t x(z,t)$ are shown in Figures 7.10 and 7.11, respectively.

Remark 7.4.2 *The method that has been used to compute numerically the state trajectories is based on a finite-dimensional approximation of the operators A and F by means of finite differences. The spatial coordinate has been discretized into n equal pieces, n* $= 50$. *Based on this, a finite-dimensional approximation of the operator A has been obtained, let us denote it by* $A_n \in \mathbb{R}^{2n\times 2n}$. *Let us also denote by* $X_n \in \mathbb{R}^{2n}$ *the approximation of the state vector X. Its components are given by*

$$X_n^i(t) = x((i-1)h,t), X_n^{i+n}(t) = \frac{\partial x}{\partial t}((i-1)h,t), i = 1,\ldots,n,$$

where h stands for the discretization step ($h = \frac{1}{n-1}$*). The nonlinear term, F, has been discretized in order to obtain a vector,* F_n*, whose action on the state* X_n *is given by*

$$F_n^i(X_n) = 0, F_n^{i+n}(X_n) = \nu \sin(X_n^i), i = 1,\ldots,n.$$

The control operator b_N *has been discretized too, yielding a control vector,* $b_{N,n} \in \mathbb{R}^{2n}$*, whose components are expressed as*

$$b_{N,n}^i = 0, b_{N,i+n} = 1_N((i-1)h), i = 1,\ldots,n.$$

The corresponding output function has also been computed numerically via the routine `trapz` *of Matlab©. Let us denote by* $y_n(t)$ *the approximation of the output function. Then, the linear finite-dimensional approximation of (7.4.11),* $\dot{X}_n(t) = A_n X_n(t) + F_n(X_n(t)) + b_{N,n} u_n(t)$*, has been numerically integrated via the routine* `ode15s` *of Matlab©, where the approximated funnel controller is given by*

$$u_n(t) = \frac{-e_n(t)}{1-\phi^2(t)e_n^2(t)}, e_n(t) = y_n(t) - y_{ref}(t).$$

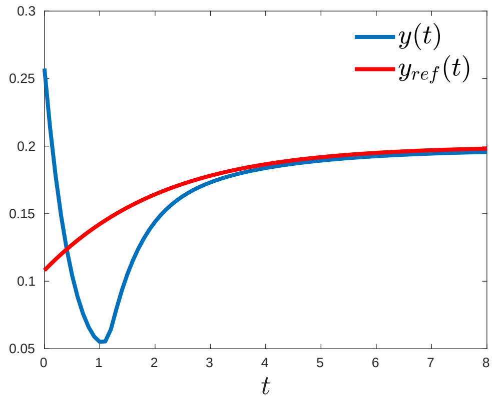

Figure 7.7 – Output trajectory (7.4.14) with the reference signal $y_{\mathrm{ref}}(t)$.

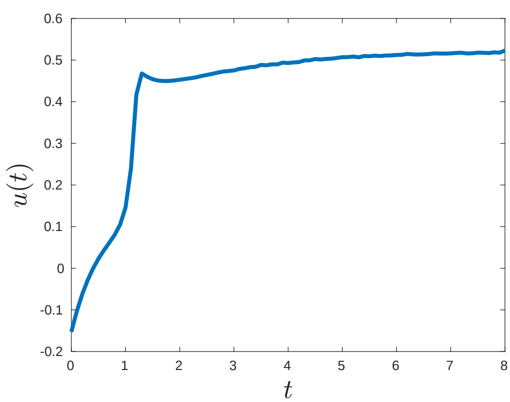

Figure 7.8 – Input trajectory.

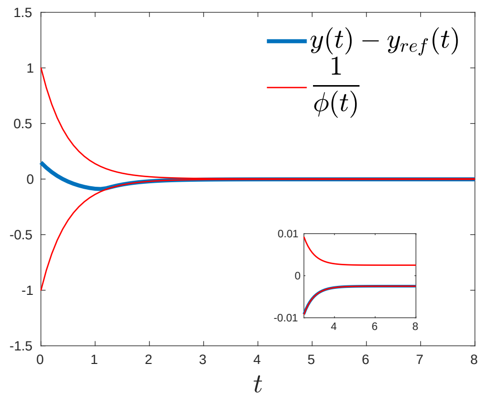

Figure 7.9 – Output error trajectory $y(t) - y_{\text{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

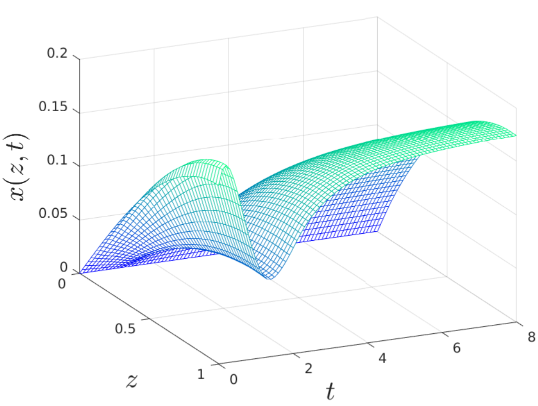

Figure 7.10 – Closed-loop state trajectory $x(t,z)$.

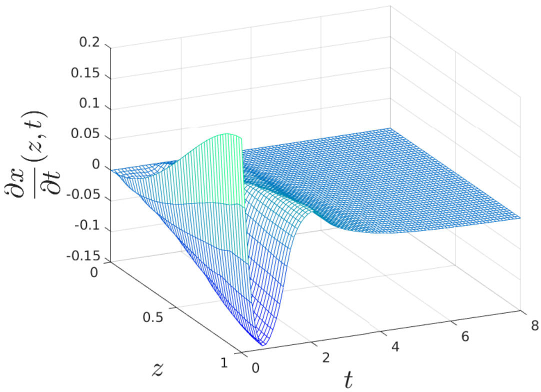

Figure 7.11 – Closed-loop state trajectory $\partial_t x(t,z)$.

7.4.2.2 Dirichlet boundary conditions

In contrast to the model introduced in the previous section, the dynamics that are of interest here are expressed as

$$\begin{cases} \partial_{tt}^2 x = \partial_{zz}^2 x - \alpha \partial_t x + \nu \sin(x) + b(z)u(t) \\ x(0,t) = 0, x(1,t) = 0. \end{cases} \tag{7.4.16}$$

Obviously that model admits the same abstract representation as (7.4.11), i.e. $\dot{\zeta}(t) = A\zeta(t) + F(\zeta(t)) + Bu(t), \zeta(0) = \zeta_0$, where the state vector $\zeta = \left(\begin{smallmatrix} x \\ \partial_t x \end{smallmatrix}\right)$ is considered on the state space $Z = D(A_0^{\frac{1}{2}}) \times X$, while the operator A_0 is defined as $A_0 = -\frac{d^2}{d\zeta^2}$ on the domain

$$D(A_0) = \left\{ x \in H^2([0,1];\mathbb{R}), x(0) = 0 = x(1) \right\}.$$

It can be seen that the operator A_0 is self-adjoint and coercive. In particular, thanks to the Poincaré's inequality there holds

$$\langle A_0 x, x \rangle_X = \int_0^1 \left(\frac{dx}{dz} \right)^2 dz \geq \pi^2 \|x\|_X^2,$$

which means that A_0 is coercive. Hence defining the square root of A_0 makes sense in this context, see (Curtain and Zwart, 2020, Lemma A.3.82). In particular, its domain is given by

$$D(A_0^{\frac{1}{2}}) = \left\{ x \in H^1([0,1];\mathbb{R}), x(0) = 0 = x(1) \right\} = H_0^1([0,1];\mathbb{R}).$$

The operator $A : D(A) = D(A_0) \times D(A_0^{\frac{1}{2}}) \subset Z \to Z$ is still given by $A = \left(\begin{smallmatrix} 0 & I \\ -A_0 & -\alpha I \end{smallmatrix}\right)$. Thanks to (Curtain and Zwart, 2020, Example 2.3.5) the newly defined operator A is the infinitesimal generator of a C_0-semigroup on the Hilbert space Z. The nonlinear operator $F : Z \to Z$ is the same operator as the one defined in Section 7.4.2.1, meaning that Assumptions 7.3.1 and 7.3.2 are satisfied here. As operator B, we consider the operator that sends the scalar input u into Z as $Bu = b(z)u = \begin{pmatrix} 0 \\ b_N(z) \end{pmatrix} u$ with b_N being expressed as $b_N(z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{4}{n\pi} \sin(n\pi z)$ for a fixed natural number N. Is is easy to see that this definition of b is such that $b \in D(A)$. As function c we shall consider the following

$$c(z) = \begin{pmatrix} \frac{\alpha+\sqrt{\alpha^2-4\pi^2}}{2\pi^2} \sin(\pi z) \\ \sin(\pi z) \end{pmatrix},$$

which is such that the output trajectory is expressed as

$$\begin{aligned} y(t) &= \langle c, \zeta(t) \rangle_Z \\ &= \frac{\alpha + \sqrt{\alpha^2 - 4\pi^2}}{2\pi} \int_0^1 \cos(\pi z) \partial_z x(z,t) dz + \int_0^1 \sin(\pi z) \partial_t x(z,t) dz. \end{aligned} \tag{7.4.17}$$

As it has already been explained in the previous example, again this choice of function c does not have particular physical meanings when interpreted as measurements but it is used in order to get a desired form of the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$. This function c lies in $D(A^*)$ since both the first and the second components are in $H^2([0,1];\mathbb{R})$ and vanishes for $z = 0$ and $z = 1$. Moreover, it is such that the $Z-$inner product between b and c is given as $\langle b, c \rangle_Z = \frac{2}{\pi} > 0$. This has the consequence that Assumption 7.3.3 is satisfied. As it is been explained in Section 7.4.2.1, the state space Z admits the decomposition $Z = \mathscr{C} \oplus \mathscr{I}$ where $\mathscr{I}$ is given by (7.4.15). These facts imply that (7.4.11) may be written as (7.3.11)–(7.3.12). In order to show that funnel control is feasible for (7.4.16), it remains to prove that Assumption 7.3.4 is satisfied in the present example. According to the definition of the operator F, it holds that $\|F \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \|_Z \leq |\nu|$ for any $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in Z$. As a consequence, showing that the operator Q defined in (7.3.10) is the infinitesimal generator of an exponentially stable C_0-semigroup is a sufficient condition for BISBO stability to hold, see Proposition 7.3.4.

Therefore, let us expand the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$. Observe that the operator $P_0 : \mathbb{R} \to \mathbb{R}, P_0 y = p_0 y$ where $p_0 = \frac{\langle A^* c, b \rangle_Z}{\langle c, b \rangle_Z}$. By computing the explicit expression of p_0, one gets the following

$$\begin{aligned} p_0 = \frac{\langle A^* c, b \rangle_Z}{\langle c, b \rangle_Z} &= \frac{\left\langle \begin{pmatrix} -\sin(\pi z) \\ \left(\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2}\right) \sin(\pi z) - \alpha \sin(\pi z) \end{pmatrix}, \begin{pmatrix} 0 \\ b_N \end{pmatrix} \right\rangle_Z}{\langle \sin(\pi z), b_N \rangle_X} \\ &= -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4\pi^2}}{2}, \end{aligned}$$

which means that p_0 is the generator of the exponentially stable C_0-semigroup

$$\left(e^{(-\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2})t} \right)_{t \geq 0}.$$

Going one step further, the computation of the operator $S : \mathscr{I} \to \mathbb{R}$ reveals that for any $\eta \in \mathscr{I}$ there holds

$$
\begin{aligned}
S\eta &= \langle \eta, P_{\mathscr{I}} A^* c \rangle_Z = \left\langle \eta, \begin{pmatrix} -1 \\ -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2} \end{pmatrix} \sin(\pi z) - p_0 c \right\rangle_Z \\
&= \left\langle \eta, \begin{pmatrix} -1 \\ -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2} \end{pmatrix} \sin(\pi z) - \left(-\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2} \right) \begin{pmatrix} \frac{\alpha+\sqrt{\alpha^2-4\pi^2}}{2\pi^2} \\ 1 \end{pmatrix} \sin(\pi z) \right\rangle_Z \\
&= \left\langle \eta, \sin(\pi z) \begin{pmatrix} -1 + \frac{1}{4\pi^2}\left(\alpha - \sqrt{\alpha^2-4\pi^2}\right)\left(\alpha + \sqrt{\alpha^2-4\pi^2}\right) \\ -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2} + \frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\pi^2}}{2} \end{pmatrix} \right\rangle_Z \\
&= \left\langle \eta, \sin(\pi z) \begin{pmatrix} -1 + \frac{1}{4\pi^2}\left(\alpha^2 - \alpha^2 + 4\pi^2\right) \\ 0 \end{pmatrix} \right\rangle_Z = 0,
\end{aligned}
$$

which means that

$$
\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\pi^2}}{2} & 0 \\ R & Q \end{pmatrix}.
$$

Thanks to the same arguments used in Section 7.4.2.1, the growth bound of the C_0- semigroup generated by the operator Q is negative provided that the same conclusion holds for the growth bound of the C_0- semigroup generated by $\left(\begin{smallmatrix} P_0 & 0 \\ R & Q \end{smallmatrix}\right)$. According to the similarity transformation $\left(\begin{smallmatrix} P_0 & 0 \\ R & Q \end{smallmatrix}\right) = U^{-*} A U^*$ and thanks to Proposition 2.1.7, we conclude that the C_0- semigroup generated by the operator Q is exponentially stable, which implies that funnel control is feasible for (7.4.16).

Let us now illustrate the feasability of funnel control on (7.4.16) with numerical simulations. We consider the following set of parameters: $\alpha = 2\pi + \frac{1}{16}, \nu = -1$. Note that this choice of the damping parameter α entails that the condition $\alpha^2 - 4\pi^2 > 0$ is satisfied, which guarantees that the output is real. The initial conditions for the variables x and $\partial_t x$ have been chosen as

$$
x(z,0) = 2z^3 - 3z^2 + z, \partial_t x(z,0) = z^2 - z^4,
$$

while the reference signal $y_{\text{ref}}(t) = \sin(4\pi t)e^{-2t} + 0.2$ which lies in $W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. The function $\phi(t)$, whose inverse determines the funnel boundaries, is fixed to

$$
\phi(t) = \frac{4}{e^{-2t} + 0.005}.
$$

Thanks to this definition of the function ϕ, it is easy to see that ϕ belongs to the class Φ introduced in (7.2.7). Furthermore, it can be shown easily that the vector of initial conditions $(x(z,0) \quad \partial_t x(z,0))^T$ lies in $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$. Moreover, the initial conditions together with the definition of the reference signal and the funnel boundaries are such that $\phi(0)|e(0)| = 0.4812 < 1$. Note also that the number of basis functions in the approximation of the indicator function, N, has been fixed to 50.

The output trajectory (7.4.17) with the reference signal $y_{\text{ref}}(t)$ are represented in Figure 7.12 whereas the corresponding funnel control is given in Figure 7.13. The tracking error is depicted in Figure 7.14 wherein one observes that it remains within the funnel boundaries, as was to be expected. The state trajectories corresponding to $x(z,t)$ and $\partial_t x(z,t)$ are shown in Figures 7.15 and 7.16, respectively.

For the numerical method that has been used, we refer to Remark 7.4.2. The same tools have been set, the only difference is in the approximation of the operator A, which has yield another matrix, $\tilde{A}_n$, due to the other type of boundary conditions.

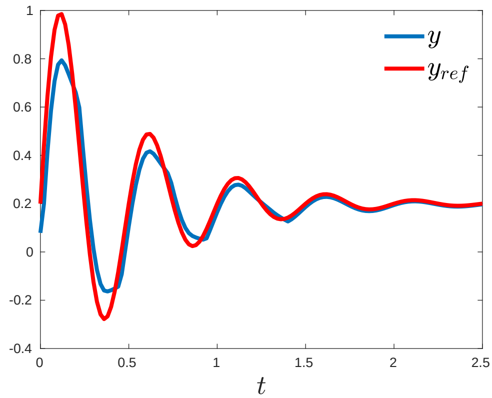

Figure 7.12 – Output trajectory (7.4.14) with the reference signal $y_{\text{ref}}(t)$.

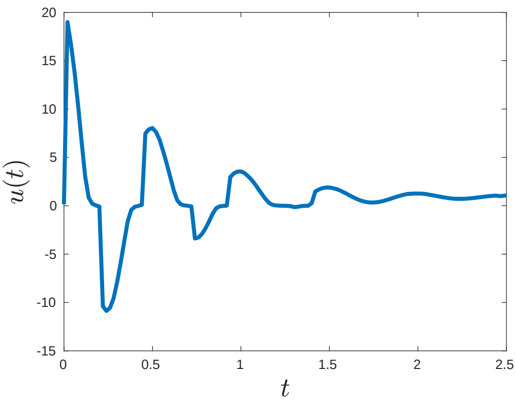

Figure 7.13 – Input trajectory.

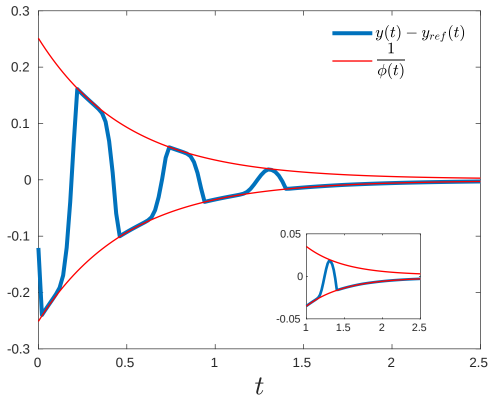

Figure 7.14 – Output error trajectory $y(t) - y_{\text{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

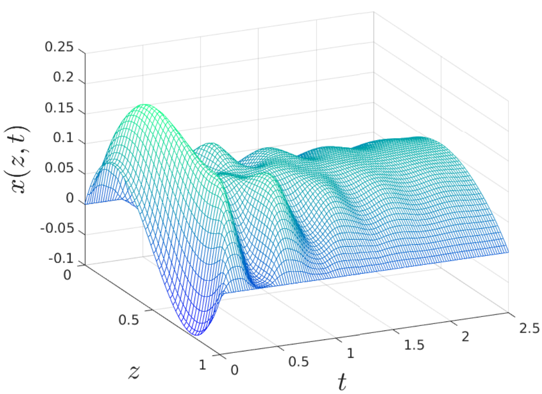

Figure 7.15 – Closed-loop state trajectory $x(z,t)$.

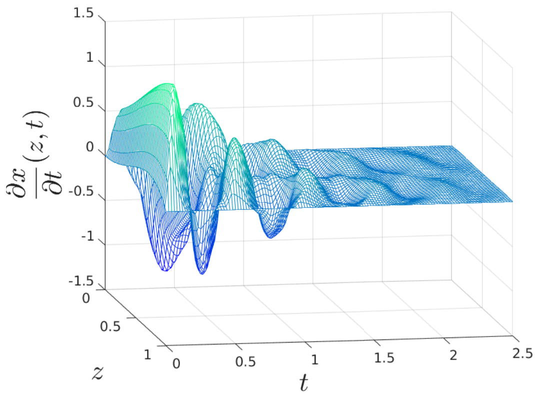

Figure 7.16 – Closed-loop state trajectory $\partial_t x(z,t)$.

Conclusion and perspectives

Conclusion

In this thesis, nonlinear infinite-dimensional systems are studied. A particular attention to the illustration of the theoretical results is taken into account, while the manuscript is relatively more theory oriented.

As a first door to enter the world of nonlinear distributed parameter systems, Chapter 1 is dedicated to the presentation of different types of chemical reactor models. As it is highlighted, the plug-flow tubular reactor with axial dispersion is the conducting application on which most of the theoretical concepts are applied all along the thesis.

Chapter 2 is dedicated to the presentation of the most needed concepts for the understanding of the technicalities of this thesis, going from the well-posedness of abstract Cauchy problems to the characterization of the stability of strongly-continuous semigroups. Therein we pay attention to the presentation of pertinent tools for the rest of the manuscript.

The way that goes from equilibria to the control aspects starts in Chapter 3. There, the equilibria of the plug-flow tubular reactor with axial dispersion are characterized in terms of existence and multiplicity. Moreover, approximated solutions of the equilibria are given and a linear stability analysis is performed by considering equal Peclet numbers. To this end, perturbation theory is used and a large diffusion phenomenon is considered.

The next concept that has been studied is how to make the link between the stability properties of a linearized model with the corresponding nonlinear model. This is located in Chapter 4 in which we build a completely new framework to tackle this question, thanks to a newly defined concept of differentiability for nonlinear operators. Based on the work of Al Jamal and Morris (2018) and on the difficulty of checking Fréchet differentiability for nonlinear operators defined on Hilbert spaces, a continuously embedded space often chosen as a multiplicative algebra allows to simplify the problem in order to get satisfactory stability results in the norm of the original state space. As a consequence a new notion of local exponential stability is also introduced. The theory that is developed here is applied to fill the gap between the conclusion obtained on the linear stability of the equilibria for the tubular reactor in Chapter 3 and their stability with respect to the corresponding nonlinear system.

Moving to the field of control, Chapter 5 is the direct extension of the results presented in Chapter 4. Control inputs expressed as state feedbacks are considered and perturbation based results are obtained. In particular, it is shown how the Fréchet differentiability (in the adapted sense) of the nonlinear operator semigroup is useful to deduce Fréchet differentiability of the nonlinear closed-loop semigroup provided that the latter is continuously dependent on the initial condition at 0. A particular class of linear quadratic optimally controlled systems is shown to fulfill the required assumptions provided a series convergence result holds.

The last topic that is considered in this thesis is related to adaptive control and is studied in Chapters 6 and 7. Under the assumption of a completely known plug-flow tubular reactor model, an adaptive nonlinear integral controller is developed in Chapter 6. This extends the classical proportional integral control action since we take into account the nonlinear nature of the studied infinite-dimensional model. This is done by incorporating an additional term whose objective is to manage the nonlinear Arrhenius law and to make the derivative of a particular Lyapunov functional the same as it should have been if the system was linear. This is performed with the aim of tracking constant reference signals by the scalar output which consitutes an average temperature in the reactor.

Chapter 7 aims at continuing the work of Chapter 6 but by enlarging the feasability of funnel control, which reveals to be an appropriate tracking output control method for model-free dynamical systems whose input - output differential relation has to fulfill some quite smooth assumptions. This chapter has been considered both from theoretical and application points of view. In particular, the new theoretical framework is applied to an output tracking problem for a nonlinear plug-flow tubular reactor and for two models of damped sine-Gordon equations.

This thesis aims at answering some questions related to nonlinear functional analysis coupled together with systems and control theory. Some fundamental questions are addressed and new developments and/or points of view are considered, enlarging the applicability of some analysis tools or control approaches. However, a lot of work remains to be done in this very rich field of mathematics. Some ideas, suggestions and improvements are discussed in the following section.

Suggestions for future research

A first perspective we want to mention concerns the study of the equilibria of the nonlinear plug-flow tubular reactor with axial dispersion, and especially their stability. The linearized stability analysis that has been performed considers the assumption that the Peclet numbers are equal, which is, as already mentioned, not really meaningful from an application point of view. The extension of these results to the case of different Peclet numbers is a very interesting open question and can be really challenging since the change of variables $\chi(z,t) = \zeta_1(z,t) - \delta\zeta_2(z,t)$ does not decouple the system (3.2.15) anymore. This complicates a lot the stability analysis. A Lyapunov based approach could be an idea to investigate, which should rely on an extension of a Lyapunov function used for equal Peclet numbers and which should contain an additional term that should vanish when $Pe_h = Pe_m$.

As second further research topic we move to Chapter 4 where a more systematic way of choosing the alternative space Y could be interesting. The advantage of being a degree of freedom in the analysis could be even better if some canonical choice was available. For instance, an a priori good choice for this space could be the domain of the linear operator A equipped with the graph norm $\|\cdot\|_A$. Checking the assumptions for this choice in the case of a general operator A could be the first way of getting an extension to that theory. Another choice could also be the domain of A^α, for $\alpha \in \mathbb{R}^+$, which is invariant under the semigroup generated by A. What is also imagined as further research is to try to recover the classical local stability result with the new framework of this thesis. In other words, one should be interested in considering initial conditions small in $X-$norm that are not necessarily small in $Y-$norm (or that are even not in Y) for which exponential stability holds. This should enlarge the class of admissible initial conditions. Also studying the basin of attraction of the equilibria should be an interesting topic to look at.

Another interesting question to look at is related to Chapter 5 and, in particular, to the class of LQ-optimally controlled systems that we introduce. There, the assumption that the linearized operator dynamics $\mathscr{A} := A + df(x^e)$ is of Riesz-spectral type is made. This could be a great progress to tackle other classes of systems. A first idea could be to consider operators which are the generators of holomorphic semigroups, see (Curtain and Zwart, 2020, Definition 2.5.1). Checkable conditions ensuring that the resulting computed stabilizing state feedback K stabilizes the linearized system on Y should be investigated. In the same chapter, other perspectives aim also at considering other types of control inputs than state feedbacks to stabilize locally the nonlinear system around an equilibrium. If state feedbacks are maintained, other manners of stabilizing the linearized system than using a LQ-optimal control approach could be interesting to investigate. We think for instance at the positive stabilization approach developed in Abouzaïd et al. (2021). The condition $\inf_{n,m\in\mathbb{N},n\neq m} |\lambda_n - \lambda_m| = \mu > 0$ could also probably be relaxed. In that way, ideas related on how to preserve the spectrum determined growth assumption in closed-loop without assuming a minimal spectral gap should be envisaged.

Other perspectives related to Chapter 7 are numerous. We notice that considering nonlinear infinite-dimensional systems with higher relative degrees than 1 should enlarge considerably the applicability of funnel control. Therefore, the extension of the Byrnes-Isidori form should be a good way to start with, finding inspiration in Ilchmann et al. (2016). The consideration of unbounded control or observation operators should also lead to a very difficult but interesting control problem. Since the funnel controller is a model-free control approach that is based on the input-output description of a dynamical system, the difficulties related to an unbounded control or observation operator should be a bit easier than expected by looking at the transfer function of the system. In that way, the transcription of the Assumptions 7.2.1 - 7.2.2 in the temporal domain to the frequency domain should also be made in order to be able to work directly on the transfer function of the system. The assumption of global

Lipschitz continuity of the operator dynamics is also an interesting question to look at. Especially, a result considering only local Lipschitz continuity would be appreciable. As another perspective we think of an extension to the case of multi-input multi-output systems. This should lead to require the invertibility of some matrix related to the input and the output operators, which should extend the condition $\langle b, c\rangle_X > 0$ in the one-dimensional case.

The perspectives we want to emphasize now are more general and not necessarily related to particular chapters of this thesis. As it has been mentioned in Chapter 1, investigating other types of chemical reactions could be valuable and of primor importance. For instance, one could take a closer look at the Van der Vusse or the Williams-Otto reactions, whose are of interest in Hudon et al. (2008) and Williams and Otto (1960) or Hudon et al. (2005), respectively. The most complicated scheme of the Van der Vusse reaction is given by

$$\begin{aligned} A &\to B \to C \\ 2A &\to D, \end{aligned}$$

and the dynamics that should be considered are expressed as

$$\begin{cases} \partial_\tau T(\zeta,\tau) = -v\partial_\zeta T(\zeta,\tau) - \frac{1}{\rho C_p}\Delta H^T R(C,T) + \frac{4h}{\rho C_p d}(1_{[0,L]}(\zeta)T_w(\tau) - T(\zeta,\tau)), \\ \partial_\tau C_A(\zeta,\tau) = -v\partial_\zeta C_A(\zeta,\tau) + (-1 \quad 0 \quad -1)R(C,T), \\ \partial_\tau C_B(\zeta,\tau) = -v\partial_\zeta C_B(\zeta,\tau) + (1 \quad -1 \quad 0)R(C,T), \end{cases} \tag{7.4.18}$$

where $\Delta H = [\Delta H_1 \quad \Delta H_2 \quad \Delta H_3]^T$ contains the enthalpies of each reaction and the vector of reaction rates $R(C,T)$ is explained with the Arhenius law, i.e.

$$R(C,T) = \begin{pmatrix} k_1 C_A e^{\frac{-E_1}{T}} & k_2 C_B e^{\frac{-E_2}{T}} & k_3 C_A^2 e^{\frac{-E_3}{T}} \end{pmatrix}^T,$$

with $C = (C_A \quad C_B)^T$ containing the concentrations of A and B, respectively. The variable T stands for the temperature whereas $\zeta \in [0,1]$ and $t \geq 0$ are the spatial and temporal variables, respectively. We should associate to the PDEs (7.4.18) the following boundary conditions

$$T(0,\tau) = T_{in}, C_A(0,\tau) = C_{A,in}, C_B(0,\tau) = C_{B,in}, \tag{7.4.19}$$

where $T_{in}, C_{A,in}$ and $C_{B,in}$ are the inlet temperature and inlet concentrations, respectively. The values of the different parameters involved in the Van der Vusse reaction are presented in Hudon et al. (2008). The scalar variable T_w plays the role of a control variable. It should be designed for instance in order to maximize or minimize the outlet concentration of A or B while regulating the temperature inside the reactor. This objective has already been taken into account in Hudon et al. (2008) where an extremum-seeking control approach has been considered. A comparison of this control technique with a funnel controller could be interesting to look at. Steps like well-posedness or open-loop stability of the homogeneous dynamics should be also investigated. Existence and multiplicity of equilibria, their linear and nonlinear stability are open questions that should be studied for further research thanks to the theories developed in this thesis.

By invoking the adaptive extremum-seeking control (ESC) approach, we think of possible perspectives too. This field has attracted a lot of attention, with contributions by Krstic et al. (1995b), Ariyur and Krstic (2003), Hudon et al. (2005), Hudon et al. (2008) and Dochain et al. (2011). As studied in these two last references and more recently in Oliveira and Krstic (2021), it can be observed that ESC has captured more and more attention for infinite-dimensional systems. Despite this fact, no rigourous framework has been built yet for ESC applied to distributed parameter systems. This could be a challenging step to investigate from a control point of view.

When speaking of applications, one thinks automatically of state observers. This should be an important question to explore when dealing with real-life systems for which all the state is not available. A lot of literature is already available for such questions. However, this is different when working with nonlinear infinite - dimensional systems. A first way to start with such a topic could be the natural extension of the Luenberger-type observer for the class of systems (4.1.2). In that way, the dynamics of the estimated state should take the form

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L(\hat{y}(t) - y(t)), \\ \hat{x}(0) = \hat{x}_0, \end{cases}$$

where $\hat{y}(t)$ denotes an estimated output given by $\hat{y}(t) = C\hat{x}(t)$ with $C \in \mathscr{L}(X, \mathbb{R})$ and $L \in \mathscr{L}(\mathbb{R}, X)$, X being the state space. As a consequence, the dynamics of the state error trajectory $e := x - \hat{x}$ should be expressed as

$$\begin{cases} \dot{e}(t) = Ae(t) + f(e(t) + \hat{x}(t)) - f(\hat{x}(t)) - LCe(t), \\ e(0) = e_0 := x_0 - \hat{x}_0. \end{cases} \tag{7.4.20}$$

The objective here should be the design of an appropriate operator L such that the $X-$norm of the error dynamics e converges to 0 as t tends to ∞. This should be particularly challenging as the system is nonlinear and infinite-dimensional. A possible approach to start with should consist in linearizing the dynamics (7.4.20) around the null state and then designing the operator C on the linearized dynamics. Then by using the framework developed in Chapter 5 based on the adapted Fréchet differentiability of the nonlinear operator f, one should try to deduce conclusions on the local exponential stability of (7.4.20). Note that Luenberger-type state observers have already been considered for semilinear systems of the form (4.1.2) by Meurer (2013). Other references that are more focused on state observer design in the field of process engineering are Dochain et al. (1992), Bastin (2013), Dochain (2003) or Mohd Ali et al. (2015).

Going still one step further, extending what has been done in this thesis to the case where the control and/or the observation operators are unbounded should lead to a great perspective. A lot of work should be done to consider properly the framework of boundary controlled and observed nonlinear infinite-dimensional systems, to characterize the well-posedness, the stability and the control of such a class of systems. Preliminary answers to these kinds of questions are already available in Schwenninger (2020) or in Hastir et al. (2019). This should be continued to questions like stability and stabilization.

The question of nonautonomous nonlinear dynamical systems on infinite dimensional spaces could also be envisaged as perspective, that is the consideration of operators A and f in (2.2.1) that depend explicitely on the time. The different ideas of this thesis should then be extended appropriately.

One last thing we would like to mention is related to the Koopman operator. For a finite-dimensional dynamical system

$$\dot{x}(t) = F(x(t)), x(0) = x_0 \in \mathbb{R}^n, \tag{7.4.21}$$

the semigroup of Koopman operators $(\mathscr{T}(t))_{t\geq 0}$ is defined on an infinite-dimensional space $\mathscr{X}$ by $\mathscr{T}(t)f = f \circ \phi^t$, where $f \in \mathscr{X}$ and $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ represents the flow of the nonlinear dynamical system (7.4.21). This powerful tool is useful to transform a nonlinear finite-dimensional system into a linear infinite-dimensional one. A lot of properties of the infinitesimal generator of $(\mathscr{T}(t))_{t\geq 0}$ are able to give informations on the nonlinear system (7.4.21) such as the asymptotic behavior of the state trajectories for instance. A comprehensive overview of what has been done with the Koopman operator is available in Mauroy et al. (2020). More recently, the question of exact linearization of a nonlinear system thanks to the Koopman operator has been considered for nonlinear PDEs in Mauroy (2021). Such an extension could be a possible perspective of this thesis, seeing which machinery needs to be built in order to work on a linear system instead of a nonlinear one without losing any information. Another interesting question thanks to the Koopman operator and the ideas of Mauroy (2021) should be the transcription of the results of this thesis to a linear setting in some appropriate functional space.

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