

RESEARCH OUTPUTS / RÉSULTATS DE RECHERCHE

Truncated Lévy distributions in an inelastic gas

Lambiotte, R.; Brenig, L.

Published in:
Physics Letters. A

DOI:
[10.1016/j.physleta.2005.07.030](https://doi.org/10.1016/j.physleta.2005.07.030)

Publication date:
2005

Document Version
Publisher's PDF, also known as Version of record

[Link to publication](#)

Citation for published version (HARVARD):

Lambiotte, R & Brenig, L 2005, 'Truncated Lévy distributions in an inelastic gas', *Physics Letters. A*, vol. 345, no. 4-6, pp. 309-313. <https://doi.org/10.1016/j.physleta.2005.07.030>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal ?

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.



Truncated Lévy distributions in an inelastic gas

R. Lambiotte^{a,b,*}, L. Brenig^a

^a *Physique Statistique, Plasmas et Optique Non-linéaire, Université Libre de Bruxelles, Campus Plaine, Boulevard du Triomphe, Code Postal 231, 1050 Bruxelles, Belgium*

^b *SUPRATECS, Sart-Tilman, Université de Liège, B5, B-4000 Liège, Belgium*

Received 30 June 2005; accepted 12 July 2005

Available online 21 July 2005

Communicated by A.R. Bishop

Abstract

We study a one-dimensional model for granular gases, the so-called inelastic Maxwell model. We show theoretically the existence of stationary solutions of the unforced case, that are characterized by an infinite average energy per particle. Moreover, we verify the quasi-stationarity of these states by performing numerical simulations with a finite number of particles, thereby highlighting truncated Lévy distributions for the velocities.

© 2005 Elsevier B.V. All rights reserved.

PACS: 45.70.Vn; 05.40.Fb; 45.20.Dd

Keywords: Granular models of complex systems; Random walks and Lévy flights; Kinetic theory

1. Introduction

High energy tails are generic for the velocity distributions of fluidized granular media, i.e., low density systems of macroscopic particles performing inelastic interactions. Contrary to the specific shape of the tail which depends on the details of the grains, overpopulation seems to be a general feature of inelastic

systems, which expresses their deep non-equilibrium nature. Recently, classes of simplified models, the so-called inelastic Maxwell models (IMM) have been introduced. These models derive from a mathematical simplification of the non-linear Boltzmann equation for inelastic hard spheres, assuming that the collision rate may be replaced by an average rate, independent of the relative velocity of the colliding particles. This model introduces mathematical simplifications which may help to highlight the origin of the anomalous velocity distributions and serve as a simple tool to study far-from-equilibrium systems. Quoting M.H. Ernst, one can consider that “what harmonic oscillators are

* Corresponding author.

E-mail addresses: renaud.lambiotte@ulg.ac.be (R. Lambiotte), lbrenig@ulb.ac.be (L. Brenig).

for quantum mechanics and dumb-bells for polymer physics, is what elastic and inelastic Maxwell models are for kinetic theory”, (Ref. [1]).

Usually, people focus on two kinds of solutions of the inelastic Boltzmann equation, namely scaling solutions describing the approach towards total rest state and heated stationary solutions. In the following, however, we focus on another class of solutions of inelastic gases, which are stationary states of the unforced case. Before going further, let us point the recent exhaustive study of Ben-Naim and Machta (Ref. [2]), that focuses independently on the same issue. Our analysis is done in the context of the one-dimensional IMM, where we show that stable Lévy distributions play a central role. The reasons for studying these stationary Lévy velocity distributions are multiple. First, they correspond to exact solutions for an inelastic gas, and this fact is very rare. By discussing the physical relevance of these states, whose mean energy diverges, we show that there exists physical solutions which are very close to these stationary states and which may be considered *quasi stationary*. This Letter is also motivated by the revived interest regarding generalized thermodynamical equilibrium theories. Indeed, long tail distributions have been found in diverse fields of science and are the object of a growing number of researches. From the point of view of statistical physics, several attempts have been made in order to show physical mechanisms underlying these anomalous behaviours, such as Tsallis’s non-extensive entropy-based statistics (Ref. [3]). The study of the above Lévy distributions could help to build a thermodynamic-like foundation to processes such as anomalous diffusion, fractal dynamics. . . . This problem, which has first been questioned by Montroll and Schlesinger (Ref. [4]), has recently lead to study simple kinetic models in order to show equilibrium-like features of Lévy distributions (Refs. [5,6]). In this work, we will also discuss the range of validity of such Lévy states, and show that they are limited to the context of Maxwell-like kinetic models.

2. Lévy velocity distribution

The one-dimensional IMM has been first introduced by Ben-Naim and Krapivsky (Ref. [7]), and derives from a simplification of the Boltzmann equation

for inelastic hard rods. The collision rule between grains 1 and 2 reads

$$v'_1 = v_1 - \frac{(1+\alpha)}{2\alpha} v_{12} v'_2 = v_2 + \frac{(1+\alpha)}{2\alpha} v_{12}, \quad (1)$$

where the prime velocities are the pre-collisional velocities. The inelasticity is accounted through the so-called normal restitution coefficient α restricted to the interval $[0:1]$. Standard procedures, such as molecular chaos hypothesis, lead to the one-dimensional inelastic Boltzmann equation. The IMM follows after the replacement of the collision rate by an average rate and a rescaling of the time scale

$$\frac{\partial f(v_1; t)}{\partial t} + f(v_1; t) = \int_{-\infty}^{\infty} dv_2 \left(\frac{1}{\alpha} \right) f(v'_1; t) f(v'_2; t), \quad (2)$$

where one has furthermore assumed that the system is and remain homogeneous in the course of time. Finite energy solutions of (2) have a natural tendency to reach the asymptotic solution $\delta(v)$, expressing the natural cooling down due to the dissipative interactions. These solutions have been proven to exhibit multiscaling (Ref. [7]), i.e., the system is characterized by an infinite set of independent cooling rates. Moreover, as shown by Baldassarri et al. (Ref. [8]), multiscaling leads to a power law scaling velocity solution.

It is well known that the Maxwell like collision operator of (2) is very similar to a convolution process (Ref. [9]). This property is clear after using the Bobylev Fourier transform method to the kinetic equation

$$\begin{aligned} \frac{\partial \phi(k; t)}{\partial t} + \phi(k; t) \\ = \phi\left(\left(\frac{1+\alpha}{2}\right)k; t\right) \phi\left(\left(\frac{\alpha-1}{2}\right)k; t\right), \end{aligned} \quad (3)$$

where $\phi(k; t)$ is the characteristic function of the velocity distribution, defined by: $\phi(k; t) = \frac{1}{2\pi} \int dv \times f(v, t) e^{ikv}$. This expression reveals the mechanisms which dominate the long time dynamics of the kinetic model. Indeed, the evolution of $\phi(k^*; t)$ is determined by the values of $\phi(k; t)$ for $|k| \leq |k^*|$ (this comes directly from the inequalities $|(\frac{1+\alpha}{2})k^*| \leq |k^*|$ and $|(\frac{1-\alpha}{2})k^*| \leq |k^*|$). Consequently, as time increases, $\phi(k^*; t)$ is determined by the initial condition $\phi(k; 0)$ for smaller and smaller values of $|k|$ and, in the long time limit, the characteristic functions are dominated

by their initial behaviour at $k \rightarrow 0$. If the initial energy diverges, and if the small k development writes $\phi(k) = 1 + a(t)|k|$, one easily shows that $\partial_t a(t) = 0$, i.e., the quantity a describing the small k singularity of $\phi(k)$ is a constant of motion. In that case, the stationary solution of (3) is the Lévy distribution

$$\phi(k) = e^{-a|k|}, \quad (4)$$

which, in velocity space rewrites

$$f(v) = \frac{a}{\pi} \frac{1}{(a^2 + v^2)}. \quad (5)$$

Let us stress that this class of states is characterized by an infinite energy, a condition which is a priori necessary in order to make compatible the inelasticity of the interactions and the stationarity of the velocity distribution. Moreover, analogies with the central-limit theorem suggest that (5) is the asymptotic stationary solution for a large class of initial distributions, thereby drawing a parallel with the Maxwell–Boltzmann equilibrium distribution (Ref. [10]). One should also remark that the quantity a also shares some properties with an equilibrium temperature, such as the equipartition property in the case of mixtures (Ref. [5]).

3. Physical relevance and simulations of Lévy distributions

In this section, we would like to discuss the physical relevance of the stationary Lévy velocity distributions derived in the previous section. At first sight, the only physical stationary solution should be the total rest state, $\phi(k) = 1$ ($e^{-a|k|}$, with $a = 0$), because an infinite average energy for the velocity distribution sounds unphysical. It is nonetheless interesting to point out that, despite this infinite average behaviour, each realization of the statistical process is finite, i.e., each particle carries a finite but unbounded energy. Moreover, physical systems are composed by a finite number of constituents. Combining these two remarks, one expects that the average energy of a physical system is finite and decreases because of dissipation according to Haff law: $T(t) = T_0 e^{-\frac{1-\alpha^2}{2}t}$, even if its velocities are initially Lévy distributed. In order to understand this paradox, we have performed computer simulations of the inelastic Maxwell model. This

can be achieved by using the usual stochastic interpretation of the Boltzmann equation (Ref. [13]). To do so, we initially distribute velocities according to a Lorentz distribution, and check that the energy of the system decreases like the exponential law prescribed by theory. The velocity distribution function is also shown to converge towards the Dirac delta distribution in the long time limit, as expected. In contrast, a careful study of the evolution of the core of the velocity distribution shows that it remains stationary during a non-negligible time interval (Fig. 1). This property, together with the finiteness of the number of particles, allows one to make a link between the IMM and truncated Lévy distributions (TLD). This family of distributions has been introduced by Mantegna and Stanley (Refs. [11,12] in order to justify the use of Lévy-like distributions in finite-size systems. They are defined by:

$$\begin{aligned} f_{\text{trunc}}^\beta(x) &= f_L^\beta(x) \quad \text{if } |x| < l, \\ f_{\text{trunc}}^\beta(x) &= 0 \quad \text{if } |x| > l, \end{aligned} \quad (6)$$

where l is the cutoff length and f_L^β is the Lévy distribution of order β defined to be the inverse Fourier transform of $e^{-a|k|^\beta}$. In a simulation of the IMM, the cutoff length is naturally imposed by the upper boundary value of the energy of the particles. In order to evaluate the stationarity of the Lévy core distribution, we have measured the evolution of $f(0; t)$. Asymptotically, this variable should behave like $f(0; t) \sim e^{\frac{1-\alpha^2}{4}t}$, which is predicted from the scaling solution of Baldassarri (Ref. [8]). In the simulations, we observe the transition between two linear regimes in logarithmic scale (Fig. 2): An initial regime during which $f(0; t)$ is constant which corresponds to the quasi stationary regime of the core of the velocity distribution, and a linearly increasing regime, which comes from the energy decrease in the system. Amazingly, while $f(0; t)$ increased of only ten percent during ten mean collision times, the energy decreased by a factor 100 during the same time period.

4. Conclusions

As a conclusion, we would like to discuss the range of validity of the stationary velocity distributions described above. Indeed, the inelastic Maxwell model

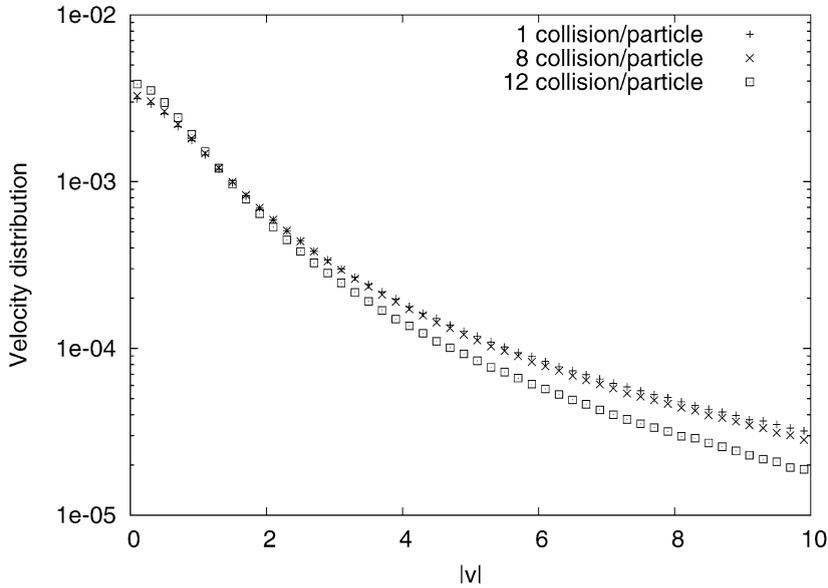


Fig. 1. Evolution of the core of the velocity distribution in log scale. The cutoff length of the Lévy distribution is 10000. The inelasticity is $\alpha = 0.5$ and the number of particles 10000000. We plot the velocity distribution after 1, 8 and 12 collisions per particle.

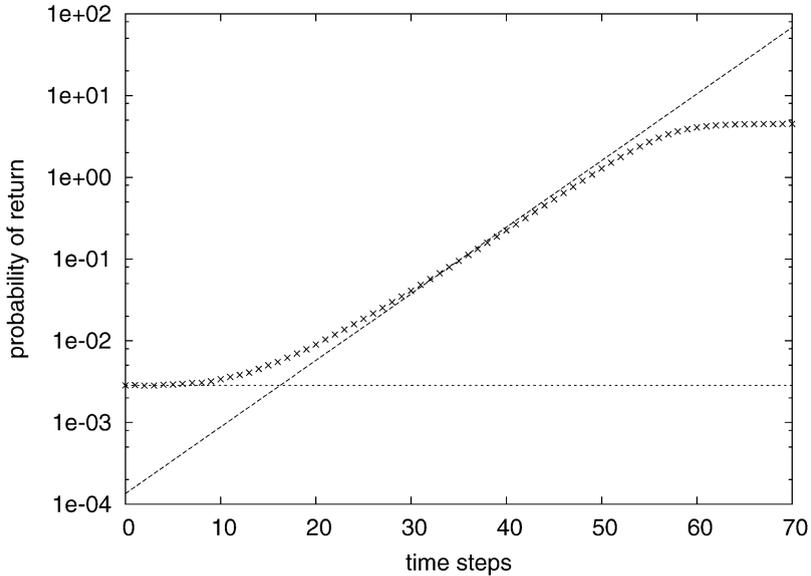


Fig. 2. Evolution of the probability of return $f(0; t)$ for the same system as in Fig. 1. The lines correspond to the theoretical predictions of the two regimes. The asymptotic saturation is due to the finite size of the measurement cells.

studied in this Letter is exceptional because of its strong similarities with random walk theory, which do not apply in general to more complex kinetic models. Therefore, it is important to recall a qualitative difference between elastic and inelastic systems, namely

the fact that the asymptotic solutions of inelastic gases depend on their modeling while elastic systems always converge towards the Maxwell–Boltzmann distribution. This universality relies on the conservation of energy and on the time symmetric dynamics of elastic

systems. Altogether, these properties imply the emergence of detailed balance at equilibrium. In contrast, it is straightforward to prove that stationary Lévy distributions do not exhibit detailed balance (Ref. [10]), thereby showing the dependence of the stationary solution on the collisional kernel. This result confirms that Lévy distributions are associated to a summation of random variables, while the Gaussian also arises from more general principles.

Nevertheless, Maxwell models constitute an ideal framework, due to their mathematical simplicity, in order to study non-trivial features of the Boltzmann equation, such as non-equipartition of energy in inelastic mixtures, or infinite energy solutions of the elastic case (Ref. [14]). Moreover, they lead to a natural derivation of fractional Fokker–Planck equations (Ref. [5]), and to the emergence of Lévy distributions, whose interest in physics is therefore enhanced. In this Letter, we also discussed the physical relevance of such infinite mean energy solutions by performing computer simulations of the model. Given the finiteness of the number of particles in the system, we showed that these solutions are equivalent to truncated Lévy velocity distributions, i.e., to a finite mean energy solution. It is important to stress that such a solution emerges from the dynamics itself, and not from an ad hoc assumption for the truncature. Moreover, computer simulations indicate that, while the total energy decays exponentially fast, as predicted by theory, the core of the distribution remains stationary during an arbitrary long period of time. Finally, we would like to point the recent work by Ben-Naim and Machta (Ref. [2]), who propose an experimental realization

of infinite energy steady states by injecting energy at large velocities. This experiment should open new perspectives in kinetic theory, and motivate detailed studies of this family of solutions.

Acknowledgements

We would like to thank Dr. Figueiredo for fruitful discussions. R.L. would also like to thank especially M. Ausloos and M. Mareschal for invitations at the Université de Liège and at CECAM, respectively. This work has been done partly thanks to the financial support of FRIA, and ARC.

References

- [1] M.H. Ernst, R. Brito, *Europhys. Lett.* 58 (2002) 182.
- [2] E. Ben-Naim, J. Machta, *Phys. Rev. Lett.* 94 (2005) 138001.
- [3] C. Tsallis, *J. Stat. Phys.* 52 (1988) 479.
- [4] E.W. Montroll, M.F. Schlesinger, *J. Stat. Phys.* 32 (1983) 209.
- [5] E. Barkai, *Phys. Rev. E* 68 (2003) 055104(R).
- [6] D.H. Zanette, M.H. Montemurro, *Phys. Lett. A* 316 (2003) 184.
- [7] E. Ben-Naim, P.L. Krapivsky, *Phys. Rev. E* 61 (2000) R5.
- [8] A. Baldassarri, U.M.B. Marconi, A. Puglisi, *Europhys. Lett.* 58 (2002) 14.
- [9] M.H. Ernst, *Phys. Rep.* 78 (1981) 1.
- [10] R. Lambiotte, PhD thesis, Université Libre de Bruxelles.
- [11] R.N. Mantegna, H.E. Stanley, *Phys. Rev. Lett.* 73 (1994) 2946.
- [12] M.F. Schlesinger, *Phys. Rev. Lett.* 74 (1995) 4959.
- [13] G.A. Bird, *Molecular Gas Dynamics*, Clarendon Press, 1976.
- [14] A.V. Bobylev, C. Cercignani, *J. Stat. Phys.* 110 (2003) 333.